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In his investigations of the "natural domain of validity" for all Siegel Formulae over number fields, Igusa was lead to a certain class of representations, which however, make sense over any field, not just number fields. Calling these representations absolutely admissible, Igusa analyzed their arithmetic nature in "Geometry of absolutely admissible representations" [4], to find the ring of invariants, the stabilizers of various points etc. The objective of [2] and of this paper is to show that for function fields, the absolutely admissible representations arise from the same arithmetic questions concerning the Siegel Formula, as was the case for number fields. In [2] we obtained a list of the composition factors of the representations that arise in this manner. In the present paper we show that these representations are in fact completely reducible, whence for the characteristic of the function field sufficiently large (a bound given explicitly for each group) the arithmetic of invariants discussed in [4] hold for function fields, exactly as for number fields.

The method of proof is cohomological, using the structure theory of semi-simple groups to find a sufficient condition, which will guarantee that the extensions split. A case by case examination shows that this condition is satisfied in every case save for SL_2 and E_6, where further arguments are needed.

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1. A sufficient condition for extensions to split.

Let G be a linear algebraic group defined over a field k and fix a universal domain $\Omega \supset k$. Let V, W be two rational G-modules that are finite dimensional vector spaces over Ω and set $W^* = \mathrm{Hom}_\Omega(W, \Omega)$. By the theory of derived functors we can identify $\mathrm{Ext}_G(V, W)$ with $H(G, V \otimes_\Omega W^*)$, since the category of G-modules has enough injectives [3]. In particular $\mathrm{Ext}^1_G(V, W) \cong H^1(G, V \otimes_\Omega W^*)$, whence the extension of W by V splits if $H^1(G, V \otimes_\Omega W^*) = 0$.

PROPOSITION 1. *Let G be a semi-simple algebraic group, V a rational G-module, which is a finite dimensional vector space over Ω. Fix a maximal torus T of G and a Borel subgroup $B \supset T$,*

which will serve to order the roots of G. If no weight of G in V is of the form $\pm a\ \alpha$, where $a \geqq 1$, integer, α a simple root of G, then $H^1(G, V) = 0$.

Proof. Let P_α be the one parameter subgroup associated with the root α, i.e., the image of the morphism $x_\alpha: G_\alpha \to G$, where $tx_\alpha(z)t^{-1} = x_\alpha(t^\alpha z)$. Here $t \in T$, α is a root of G (with respect to T), $z \in \Omega$ and $t^\alpha = \alpha(t)$. Then G is generated by T, P_α, $P_{-\alpha}$ for $\alpha \in \Delta$, the simple roots for the ordering determined by B. Let $f: G \to V$ be a 1-cocycle. Since the cohomology groups of tori are trivial, we can modify f by a suitable coboundary to have $f(t) = 0$, for all $t \in T$. Now $V = \sum_s V_s$, where $V_s = \{v \in V \mid t.v = t^s v, \text{ all } t \in T\}$. So, writing $f(x_\alpha(z)) = \sum_s l_s(z)$, where $z \to l_s(z)$ is a regular function $G_a \to V_s$, since it is given by a polynomial function, we have $f(x_\alpha(z)) = \sum_{i \geqq 0, s} z^i l_{s,i}$ with $l_{s,i} \in V_s$. But $f(id) = 0$, so in fact $f(x_\alpha(z)) = \sum_{i \geqq 1, s} z^i l_{s,i}$.

Now, using the cocycle relation $f(gg') = f(g) + g \cdot f(g')$ and the conjugation relation $tx_\alpha(z)t^{-1} = x_\alpha(t^\alpha z)$ we see upon comparing the coefficients of z^i that if $l_{s,i} \neq 0$, then $t^{\alpha i} = t^s$, so if the weights of G in V are not of the form $\pm i\alpha$, $i \geqq 1$ $\alpha \in \Delta$ we must have $f = 0$, whence $H^1(G, V) = 0$.

COROLLARY. *If V, W are G-modules, then the extension of W by V splits if the weights of $V \otimes W^*$ are not of the form $\pm i\alpha$ for some $i \geqq 1$, $\alpha \in \Delta$.*

Further, every G-module all of whose composition factors have the same highest weight λ, is completely reducible, for G semi-simple. This is a consequence of the fact that for an irreducible representation there is a unique Borel stable line, the multiplicity of λ is one and every other weight is of the form $s = \lambda - \sum_{\alpha \in \Delta} m_\alpha(s)\alpha$, with $m_\alpha(s) \geqq 0$ an integer [1]. We shall give a proof of this known result because of its importance in our work and for lack of a convenient reference.

LEMMA (Humphreys). *G a semi-simple group, V a rational G-module all of whose composition factors have the same highest weight. Then V is completely reducible.*

Proof. For $y \in V_s$, $x_\alpha(z) \cdot y = y +$ vectors of weights $s + i\alpha$ $(i > 0)$. This can be seen exactly as in Proposition 1; hence if λ is the highest weight that occurs in V, every $v \in V_\lambda$ gives rise to a Borel stable line, since vectors of weight $\lambda + i\alpha (i > 0, \alpha > 0)$ cannot occur. Also span $(G.\ v) =$ span $(U^-.\ v)$ for such $v \in V_\lambda$, so this has a

unique line of weight λ, whence span $(G.\, v)$ is an irreducible G-module, since any G-submodule must have a Borel stable line, while the only highest weights in V are λ, thus such a line must contain v, by the uniqueness just remarked. Let W be the G-submodule of V generated by the vectors of weight λ. Then $W = \langle G.\, v_1\rangle + \cdots + \langle G.\, v_r\rangle$ is completely reducible. If $W \neq V$, then $V \cong W + V/W$ as T-module, the highest weight that can occur in V/W is less than λ, contrary to V having only λ as highest weight of the composition factors.

The irreducible representations of G are characterized by their highest weight and for a given λ, the irreducible G-module (in characteristic P) having highest weight λ is a quotient of the corresponding module (in characteristic 0). Thus, to obtain complete reducibility, it suffices to show that for the modules in characteristic 0 the weights are not of the form $\pm a\alpha$. By the above lemma, we need only consider the composition factors corresponding to different weights.

2. **Complete reducibility of admissible representations.** From [2] and the previous remarks, we need to investigate the following representations.

Type A_n. $\rho_1 + \rho_n$, $\rho_1 + \rho_2$, $\rho_1 + \rho_{n-1}$, $\rho_2 + \rho_n$, $\rho_{n-1} + \rho_n$, $\rho_P a_{A_1} + {}_P b_{A_n}$. By the outer automorphism of the Dynkin diagram we have $\check{\rho}_i = \rho_{n-i+1}$ where $\check{\rho}$ denotes the contragredient representation.

Case $\rho_i + \rho_n$. $\rho_1 \otimes \check{\rho}_n = \rho_1 \otimes \rho_1$. The simple roots are $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3, \cdots, \lambda_n - \lambda_{n+1}$, while the weights of λ_1 are $\lambda_1, \lambda_2, \cdots, \lambda_n$, $\lambda_{n+1} = (\lambda_1 + \cdots + \lambda_n)$, whence the weights $\rho_1 \otimes \rho_1$, being $\lambda_i + \lambda_j$ are never of the form $\pm a(\lambda_i - \lambda_{i+1})$ if $n \geqq 2$.

For $n = 1$, $\alpha = 2\lambda_1$ and the weights of $\rho_1 \otimes \rho_1$ are $\pm 2\lambda_1$, 0, so a further argument is needed. But for f: $SL_2 \to V \otimes V$ a 1-cocycle, modified so that $f|_T = 0$, we have $t.f(x_\alpha(z)) = f(x_\alpha(t^\alpha z))$, so if $V = \langle v_1, v_2\rangle$, $f(x_\alpha(z)) = l_1(z)v_1 \otimes v_1 + l_2(z)v_1 \otimes v_2 + l_3(z)v_2 \otimes v_1 + l_4(z)v_2 \otimes v_2$. Then the above relation implies that $f(x_\alpha(z)) = az(v_1 \otimes v_1)$, $f(x_{-\alpha}(z)) = bz(v_2 \otimes v_2)$ for constants a, b.

The cocycle relation applied to the element $x_\alpha(z)\ x_{-\alpha}(z')$ when $zz' + 1 \neq 0$, now implies that $a = b = 0$, whence $H^1(SL_n, \rho_1 \otimes \rho_1) = 0$ for all n.

Case $\rho_1 + \rho_2$. The analysis is similar to the above, since the weights of ρ_2 are known to be $\lambda_i + \lambda_j (i \neq j)$, $\check{\rho}_2 = \rho_{n-1}$.

Case $\rho_{p^a\Lambda_1+p^b\Lambda_n}$. $\rho_{p^a\Lambda_1} \otimes \check{\rho}_{p^b\Lambda_n}$ has weights p^{a+b} (weights of $\rho_1 \otimes \rho_1$), whence for $n \neq 1$, we have complete reducibility. But by [2], $n = 1$ cannot occur, hence this case by case examination shows that every one of the representations known to be admissible for SL_n is completely reducible. As we know this listing is exhaustive for $p > 3/2n(n+1)$.

Type B_n. For $G = \mathrm{Spin}_{2n+1}$, we must show $\rho_1 + \rho_n$ is completely reducible, for $n = 2, 3, 4$. The point now is to obtain the weights of ρ_n, the spin representation. By Chevalley's theorem [6, p. 14], if W is the Weyl group acting on E, with F a subset of E, then W_F, the subgroup of W which keeps F pointwise fixed is generated by the reflections W_{α_i} which keeps F pointwise fixed.

LEMMA. *The weights of* ρ_n *are* $\{1/2(\pm\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n)\}$.

Proof. In characteristic zero, ρ_n is a representation of degree 2^n. For Λ_n the highest weight of ρ_n, W the Weyl group, $W.\Lambda_n$, the orbit of Λ_n has $[W: W_{\Lambda_n}]$ elements. By Chevalley's theorem, we have $W_{\Lambda_n} \subset W(A_{n-1})$, the Weyl group of type A_{n-1} whence

$$|W.\Lambda_n| \geqq |W: W(A_{n-1})| = 2^n n!/n! = 2^n ,$$

so that W acts transitively on the weights of ρ_n. In particular ρ_n is of degree 2^n in characteristic p with the weights $\{W.\Lambda_n\} = \{1/2(\pm\lambda_1 \pm \cdots \pm \lambda_n)\}$. Note that ρ_n is self contragredient.

For B_n, the simple roots are $\{\lambda_1 - \lambda_2, \cdots, \lambda_{n-1} - \lambda_n, \lambda_n\}$ with the weights of ρ_1 being $\pm\lambda_i$. Thus $\pm\lambda_i + 1/2(\lambda_1 \pm \cdots \pm \lambda_n)$ is never of the form $\pm a\alpha$, $\alpha \in \Delta$, whence $H^1(\mathrm{Spin}_{2n+1}, \rho_1 \otimes \rho_n) = 0$. Therefore all the representations of a group of type B_n listed in [2] are completely reducible.

Type C_n. Now $G = \mathrm{Sp}_{2n}$ and we must show $\rho_1 + \rho_2$ to be completely reducible where ρ_1 is the standard realization of G on $M_{2n,1}$ and ρ_2 is the representation on $M_{2n,1} \wedge M_{2n,1}$. Thus the weights of ρ_1 are $\pm\lambda_i$, for λ_2 they are $\pm\lambda_i, \lambda_j, 0$. But the simple roots are $\{\lambda_i - \lambda_{i+1} 1 \leqq i < n, 2\lambda_n\}$ so we see that $\rho_1 + \rho_2$ is completely reducible.

Type D_n. Now $G = \mathrm{Spin}_{2n}$ and we must show that the following are completely reducible.

$n = 4$: $\rho_1 + \rho_3 + \rho_4$, $\rho_1 + \rho_3$, $\rho_1 + \rho_4$, $\rho_3 + \rho_4$

$n = 5$: $\rho_1 + \rho_2$, $\rho_1 + \rho_4$, $\rho_1 + \rho_5$

$n = 6$: $\rho_1 + \rho_5$, $\rho_1 + \rho_6$ in general and if characteristic of $k = 2$, for

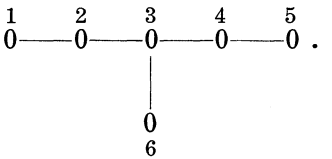
$n = 5$: $\rho_{2\Lambda_1} + \rho_4$, $\rho_{2\Lambda_1} + \rho_5$. Exactly as was the case for type B_n, we see that the Weyl group operates transitively on the weights of ρ_{n-1}, ρ_n (resp.) and as $W(D_n) = S_n \cdot (\boldsymbol{Z}/2\boldsymbol{Z})^{n-1}$ with $\lambda_i \to \pm\lambda_i$ with an even number of negative signs, so that the weights are, for ρ_{n-1}: $\{1/2(\pm\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n) | \text{even number of negative signs}\}$, for ρ_n: $\{1/2(\pm\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n) | \text{odd number of negative signs}\}$.

Therefore $\rho_1 \otimes \check{\rho}_{n-1}$, $\rho_1 \otimes \check{\rho}_n$ do not have weights of the form $\pm a\alpha$, $\alpha \in \{\lambda_i - \lambda_{i+1}\ 1 \leqq i < n, \lambda_{n-1} + \lambda_n\}$, whence $\rho_1 + \rho_{n-1}$, $\rho_1 + \rho_n$ are completely reducible for all n. Also by the above, for $n = 4$, $\rho_3 \otimes \rho_4$ has weights $1/2(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4) + 1/2(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)$ where in first parenthesis have an even, in second parenthesis an odd number of negative signs, thus again $\rho_3 + \rho_4$ is completely reducible. Finally, $\rho_1 + \rho_3 + \rho_4$ is completely reducible, since if V is the module for ρ_1, V' the module for $\rho_3 + \rho_4$, then $V' = \check{V}'$ and the weights of $V \otimes V'$ are the weights of $\rho_1 \otimes \rho_3$, $\rho_1 \otimes \rho_4$, whence by above, never of the form $\pm a\alpha$, $\alpha \in \Delta$.

For $n = 5$ the weights of ρ_2 are $\pm\lambda_i \pm \lambda_j (i \neq j)$ so it is self contragredient, whence $\rho_1 + \rho_5$ seen to be completely reducible.

If the characteristic is two, $n = 5$, $\rho_{2\Lambda_1}$ has weights 2 {weights of ρ_1}, so weights of $\rho_{2\Lambda_1} + \rho_4$ (resp. $\rho_{2\Lambda_1} + \rho_5$) are $\pm 2\lambda_i + 1/2(\pm\lambda_1 \pm \cdots \pm \lambda_5)$ odd (resp. even) number of negative signs, so once again it is completely reducible.

Type E_6. Here our notation differs from that of Bourbaki, with the labelling of the Dynkin diagram being



We need to investigate $\rho_1 + \rho_5$.

The method used to find the weights of the spin and half spin representations cannot be used now, since there is no convenient description of the Weyl group for E_6. However, we can apply Springer's criterion [5], which involves only the highest weights Λ_1, Λ_5. We find $3(\Lambda_5 + \rho, \Lambda_5 + \rho) = 135$ where $2\rho =$ sum of the positive roots, hence for $p \neq 2$ $\rho_1 + \rho_5$ is completely reducible.

Thus we have shown, by the above case by case examination that every representation listed in [2] is completely reducible, which if the characteristic is sufficiently large, shows all absolutely admissible representations to be completely reducible.

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