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Several properties of the Chebyshev method of summability, defined by G. G. Bilodeau, are investigated. Specifically, it is shown that the Chebyshev method is translative and is a Gronwall method. It is shown that the de Vallee Poussin method is stronger than the Chebyshev method, and that the Chebyshev method is not stronger than the $(C, 1)$ method. The final result shows that the Chebyshev method exhibits the Gibbs phenomenon.

Let $\sum(-1)^i u_i$ be an alternating series with partial sums $s_n = \sum_{i=0}^n (-1)^i u_i$. Define a sequence of polynomials $\{P_n(t)\}$ by $P_n(t) = \sum_{k=0}^n a_{nk} t^k$, $P_n(1) = 1$, $n = 0, 1, 2, \dots$. The series $\sum(-1)^i u_i$ will be called summable (P_n) to the value s if $\lim \sigma(P_n) = s$, where $\sigma(P_n) = \sum_{k=0}^n a_{nk} s_k$. Bilodeau [1] considered the following question. What are sufficient conditions on P_n for $\sigma(P_n)$ to speed up the rate of convergence of a convergent sequence $\{s_n\}$? For sequences $\{u_n\}$ which are moment sequences, i.e., u_n has the representation $u_n = \int_0^1 t^n d\alpha(t)$, where $\alpha(t) \in BV[0, 1]$, he obtains the estimate $|\sigma(P_n) - s|/|r_n| \leq (\mu_n/|r_n|) \int_0^1 t(1+t)^{-1} |d\alpha(t)|$, where $s = \sum_{i=0}^{\infty} (-1)^i u_i$, $r_n = s_n - s$, and $\mu_n = \max_{0 \leq t \leq 1} |P_n(-t)|$. Adopting μ_n as a measure of the value of the method $\sigma(P_n)$, the most desirable sequence of polynomials will be those for which μ_n is a minimum, subject to the constraint $P_n(1) = 1$ for each n . The Chebyshev polynomials, defined by $T_n(x) = \cos nx$, $n = 0, 1, 2, \dots$, $x = \cos \theta$, form the best approximation to the zero function over the interval $[-1, 1]$. When translated to $[0, 1]$ they give $P_n(t) = T_n(1 + 2t)/T_n(3)$ as the best polynomials to minimize μ_n , where

$$(1) \quad T_n(x) = [(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]/2,$$

and

$$T_n(3) = (\alpha^n + \alpha^{-n})/2, \quad \alpha = 3 + \sqrt{8} \approx 5.828.$$

The infinite matrix $A = (a_{nk})$, associated with these polynomials, has entries

$$(2) \quad a_{nk} = \begin{cases} 1/T_n(3), & k = 0 \\ \frac{2^{2k-1}}{T_n(3)} \left[2 \binom{n+k}{n} - \binom{n+k-1}{n-k} \right], & 0 < k \leq n \\ 0, & k > n. \end{cases}$$

Bilodeau calls the associated summability method the Chebyshev or σ -method.

We begin by establishing some properties of the maximal entry in each row of σ .

LEMMA 1. *For each positive integer $n > 2$, there exists an integer p such that*

$$\begin{aligned} a_{nk} &< a_{n,k+1} \quad \text{for } 0 \leq k < p \\ a_{nk} &\geq a_{n,k+1} \quad \text{for } p \leq k < n. \end{aligned}$$

Proof. For $0 < k \leq n$ we may write

$$(3) \quad a_{nk} = \frac{2^{2k-1}n}{kT_n(3)} \binom{n+k-1}{n-k},$$

so that $a_{nk}/a_{n,k+1} = (k+1)(2k+1)/2(n^2-k^2)$. Treating k as a continuous variable and differentiating with respect to k , it follows that $a_{nk}/a_{n,k+1}$ is increasing in k . The proof is completed by noting that $a_{n0} < a_{n1} < a_{n2}$ and $a_{n,n-1} > a_{nn}$ for each $n > 2$.

LEMMA 2. *For each $n, p = [x_0]$ where $x_0 = (-3 + (32n^2 - 7)^{1/2})/8$.*

Proof. Since $a_{n1} < a_{n2}$ and $a_{n,n-1} > a_{nn}$, there exists a real positive number x_0 such that $a_{nx_0} = a_{n,x_0+1}$ which implies $2x_0^2 + 3x_0 + 1 = 2n^2 - 2x_0^2$. Since x_0 is positive, $x_0 = (-3 + (32n^2 - 7)^{1/2})/8$.

LEMMA 3. *For each $n > 6, p = [x_0] > n/2$.*

It is sufficient to show that $x_0 - 1 \geq n/2$; i.e., $8(2n^2 - 11n - 16) \geq 0$, for $n > 6$. With $g(n) = 2n^2 - 11n - 16$ we have $g'(n) > 0$ for $n > 11/4$, hence g is increasing for $n > 11/4$, and g is positive for $n > 6$ and n an integer.

LEMMA 4. *With p and a_{np} as defined in Lemmas 2 and 3, $\lim_n a_{np} = 0$.*

From (3), and Stirling's formula,

$$\begin{aligned} a_{np} &= \frac{n2^{2p-1}\Gamma(n+p)}{PT_n(3)\Gamma(n-p+1)\Gamma(2p)} \\ (4) \quad &\sim \frac{n2^{2p-1}(n+p-1)^{n+p-1}e^{-(n+p-1)}(2\pi(n+p-1))^{1/2}}{p\alpha^n(n-p)^{n-p}e^{-(n-p)}(2\pi(n-p))^{1/2}(2p-1)^{2p-1}e^{-(2p-1)}(2\pi(2p-1))^{1/2}} \\ &= \frac{1}{2\sqrt{\pi}} \frac{\left(p - \frac{1}{2}\right)^{1/2}}{p} \frac{n}{((n+p-1)(n-p))^{1/2}} \left(\frac{n+p-1}{\alpha(n-p)}\right)^{n-p} \left(\frac{n+p-1}{\sqrt{\alpha}\left(p - \frac{1}{2}\right)}\right)^{2p}. \end{aligned}$$

Both $((n + p - 1)/\alpha(n - p)^{n-p})$ and $((n + p - 1)/\sqrt{\alpha}(p - 1/2)^{2p})$ are bounded above. Therefore $\lim_n a_{np} = 0$.

Cooke [3, p. 119] shows that a necessary and sufficient condition for a regular matrix to be absolutely translative for all bounded sequences $\{z_n\}$ is that the matrix A satisfies $\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0$.

THEOREM 1. *The σ -method is absolutely translative for all bounded sequences.*

Proof. Bilodeau [1, p. 296] has shown that the σ -method is regular. From Lemma 1,

$$\begin{aligned} \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| &= \sum_{k=1}^{p-1} (a_{n,k+1} - a_{nk}) + \sum_{k=p}^n (a_{nk} - a_{n,k+1}) \\ &= 2a_{np} - a_{n0}. \end{aligned}$$

The regularity of A implies that $\lim_n a_{n0} = 0$, and the result follows from Lemma 4.

For unbounded sequences, we consider the class of sequences $\{z_n\}$ satisfying $|z_k| \leq \theta_k$ (θ_k real, positive, and increasing), where $\sum_{k=0}^{\infty} a_{nk}\theta_{k+1}$, $\sum_{k=0}^{\infty} a_{n,k+1}\theta_{k+1}$, and $\rho_n = \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}|\theta_{k+1}$ exist for each n . Cooke [3, p. 119] shows that a necessary and sufficient condition for a regular matrix to be absolutely translative for all (unbounded) $\{z_n\}$ satisfying $|z_k| \leq \theta_k$ together with conditions stated above, is that $\lim_n \rho_n = 0$.

THEOREM 2. *The σ -method is absolutely translative for all (unbounded) sequences $\{z_n\}$ such that $z_k = o(\sqrt{k})$. This result is best possible.*

With $|z_n| = \theta_n$, and using Lemma 1,

$$\begin{aligned} \rho_n &= \sum_{k=0}^{p-1} (a_{n,k+1} - a_{nk})\theta_{k+1} + \sum_{k=p}^n (a_{nk} - a_{n,k+1})\theta_{k+1} \\ (5) \quad &\leq \theta_{p-1} \sum_{k=1}^{p-1} (a_{n,k+1} - a_{nk}) + \theta_n \sum_{k=p}^n (a_{nk} - a_{n,k+1}) \\ &\leq \theta_n (a_{np} - a_{n0} + a_{np} - 0) = 0(\sqrt{n})(2a_{np} - a_{n0}). \end{aligned}$$

It will be sufficient to show that $\overline{\lim}_n 2\sqrt{n}a_{np}$ is finite. But this follows immediately from (4), since $\lim_n (n(p - 1/2))^{1/2}/p = 2^{1/4}$, and the remaining limits have already been shown to be finite.

To show that the result is best possible we shall replace $o(\sqrt{k})$

by \sqrt{k} and verify that ρ_n does not tend to zero.

From (5), $\rho_n \geq \sqrt{p} \sum_{k=p}^n (a_{nk} - a_{n,k+1}) = \sqrt{p} a_{np}$, which does not tend to zero.

Direct calculations verify that σ is not a weighted mean, Nörlund, Hausdorff, or generalized Hausdorff method.

Gronwall [4, p. 102] defined a general class of summability methods, each member of which involves a pair of analytic functions f and g . Specifically, the (f, g) -transform of a given series $\sum_{k=0}^{\infty} u_k$ is the sequence $\{U_n\}$ defined implicitly by the formal power series identity

$$(6) \quad g(w) \sum_{n=0}^{\infty} u_n [f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n,$$

where f and g satisfy the following properties. Let $\Delta = \{w \mid |w| < 1\}$. The function $z = f(w)$ is analytic in $\bar{\Delta} - \{1\}$, continuous and $1 - 1$ in $\bar{\Delta}$, with $f(0) = 0$, $f(1) = 1$, and $|f(w)| < 1$ for $w \in \Delta$. Moreover, $w = f^{-1}(z)$ has the representation $w = 1 - (1 - z)^\lambda [a + a_1(1 - z) + \dots]$, where $\lambda \geq 1$, $a > 0$, and the quantity in brackets is a power series in $1 - z$ with a positive radius of convergence. The function g satisfies $g(w) \neq 0$ for $w \in \Delta$ and has the form $g(w) = (1 - w)^{-\delta} + \gamma(w)$ for some $\delta > 0$, where $\gamma(w)$ is analytic in $\bar{\Delta}$. Also $g(w) = \sum_{n=0}^{\infty} b_n w^n$, with $b_n \neq 0$ for each n . The series $\sum_{k=0}^{\infty} u_k$ is said to be (f, g) -summable to s if $\lim U_n = s$.

Examples of (f, g) -methods are the Cesàro methods of order k , (C, k) , for k real and greater than -1 ; (E, β) (Euler-Knopp) for $0 < \beta \leq 1$; de la Vallée Poussin summability (V) ; a generalized (V) -summability (Vk) , introduced by Gronwall; and a method of summation of Obrechhoff. We will now show that the Chebyshev method is also a Gronwall method.

Writing $s_n = \sum_{k=0}^n u_k$, the (f, g) -method can be expressed as a sequence to sequence method by rewriting (6) in the form

$$(7) \quad g(w)[1 - f(w)] \sum_{n=0}^{\infty} s_n [f(w)]^n = \sum_{n=0}^{\infty} b_n U_n w^n.$$

Using (7), (f, g) can be expressed as a triangular matrix transformation of the form $U_n = \sum_{k=0}^n a_{nk} s_k$, with $a_{nk} = \gamma_{nk}/b_n$, where γ_{nk} is defined by

$$(8) \quad [1 - f(w)]g(w)[f(w)]^k = \sum_{n=k}^{\infty} \gamma_{nk} w^k.$$

(See, for example, the discussion on page 40 of [2], where the roles of γ_{nk} and a_{nk} have been interchanged.) From (8) it follows that

$$(9) \quad a_{nn} = [f'(0)]^n / b_n, \quad n \geq 0.$$

THEOREM 3. *The Chebyshev method is a Gronwall method with $f(w) = w(\alpha - 1)^2 / (\alpha - w)^2$, $g(w) = (1 - w)^{-1} + \gamma(w)$, and $\gamma(w) = w / (\alpha^2 - w)$, where $\alpha = 3 + \sqrt{8}$.*

Proof. If (6) is a Gronwall method, then, from (8) with $k = 0$ and (2),

$$[1 - f(w)]g(w) = \sum_{n=0}^{\infty} b_n a_{n0} w^n = \sum_{n=0}^{\infty} b_n w^n / T_n(3).$$

Thus

$$(10) \quad \begin{aligned} f(w) &= 1 - [g(w)]^{-1} \sum_{n=0}^{\infty} b_n w^n / T_n(3), \\ f'(w) &= [g'(w)/g^2(w)] \sum_{n=0}^{\infty} b_n w^n / T_n(3) - [g(w)]^{-1} \sum_{n=1}^{\infty} n b_n w^{n-1} / T_n(3) \end{aligned}$$

and $f'(0) = [g'(0)/g^2(0)](b_0/T_0(3)) - b_1/g(0)T_1(3) = 2b_1/3b_0$, since $T_0(3) = 1$ and $T_1(3) = 3$.

From (9) and (3),

$$(11) \quad b_n = (2b_1/3b_0)^n T_n(3) / 2^{2n-1} = (b_1/6b_0)^n (\alpha^n + \alpha^{-n}).$$

In particular, $b_1 = b_1/b_0$, which implies $b_0 = 1$, since each $b_n \neq 0$. One can also deduce that $b_0 = 1$ from (9), since $a_{00} = 1$.

Thus

$$\begin{aligned} g(w) &= 1 + \sum_{n=1}^{\infty} b_n w^n \\ &= 1 + \sum_{n=1}^{\infty} [(b_1 \alpha w / 6)^n + (b_1 w / 6 \alpha)^n] \\ &= 1 + \frac{b_1 \alpha w}{6 - b_1 \alpha w} + \frac{b_1 w}{6 \alpha - b_1 w} \\ &= \frac{6}{6 - b_1 \alpha w} + \frac{b_1 w}{6 \alpha - b_1 w}. \end{aligned}$$

For g to have the required form choose $b_1 = 6/\alpha$.

From (10), and (11), with $b_1 = 6/\alpha$,

$$\begin{aligned} f(w) &= 1 - [g(w)]^{-1} \left[1 + \sum_{n=1}^{\infty} 2(w/\alpha)^n \right] \\ &= 1 - [g(w)]^{-1} \left[1 + \frac{2w}{\alpha - w} \right] \\ &= 1 - \frac{(\alpha + w)}{\alpha - w} \cdot \frac{(1 - w)(\alpha^2 - w)}{(\alpha^2 - w^2)} \\ &= 1 - \frac{(1 - w)(\alpha^2 - w)}{(\alpha - w)^2} = \frac{w(\alpha - 1)^2}{(\alpha - w)^2}. \end{aligned}$$

We now show that f is a 1-1 selfmapping of Δ . If $f(w_1) = f(w_2)$, i.e.,

$$\frac{w_1(\alpha - 1)^2}{(\alpha - w_1)^2} = \frac{w_2(\alpha - 2)^2}{(\alpha - w_2)^2},$$

then $(w_1 - w_2)(\alpha^2 - w_1 w_2) = 0$. Since $w_1, w_2 \in \Delta$, $w_1 w_2 \neq \alpha^2$, so $w_1 = w_2$. By the Maximum Modules Theorem, it is sufficient to show that $|f(w)| \leq 1$ for $w = e^{i\theta}$. $|f(e^{i\theta})| = (\alpha - 1)^2(\alpha^2 - 2 \cos \theta + 1) \leq 1$.

We now verify that $w = f^{-1}(z)$ is regular on $\bar{\Delta} - \Delta$, except possibly at $z = 1$, and that $0 \in \Delta$. f^{-1} is regular except at $z = 0$, so now we must show

$$\min_{0 \leq \theta < 2\pi} |f(e^{i\theta})| \geq \delta > 0.$$

$|f(e^{i\theta})| = (\alpha - 1)^2/T(\theta)$, where $T(\theta) = (\alpha + 1)^2 - 4\alpha \cos^2 \theta/2$. A direct calculation certifies that the maximum of $T(\theta)$ occurs at $\theta = \pi$, and $T(\pi) = [(\alpha - 1)/(\alpha + 1)]^2 > 0$.

It remains to show that at $z = 1$, $1 - w = (1 - z)^2[a + a_1(1 - z) + \dots]$, $\lambda \geq 1$, $a > 0$. $z = f(w) = (\alpha - 1)^2 w/(\alpha - w)^2$. From the equation $z = f(w)$ we obtain $1 - z = (1 - w)(\alpha^2 - w)/(\alpha - w)^2$, which when solved for $1 - w$ yields

$$1 - w = \frac{-(\alpha - 1)(1 - 2z - \alpha) \pm (\alpha - 1)(\alpha + 1)\sqrt{1 - 4\alpha(1 - z)/(\alpha + 1)^2}}{-2z}.$$

Now divide the numerator and the denominator by -2 and write z in the denominator as $1 - (1 - z)$.

$$1 - w = \left\{ \frac{(\alpha - 1)}{2} [2(1 - z) - (\alpha + 1)] \pm \frac{(\alpha^2 - 1)}{-2} \left[1 - \frac{4\alpha}{2(\alpha + 1)^2} (1 - z) + -\frac{1}{8} \frac{16\alpha^2}{(\alpha + 1)^4} (1 - z)^2 + \dots \right] \right\} \cdot \sum_{k=0}^{\infty} (1 - z)^k.$$

Using the negative branch,

$$\begin{aligned} 1 - w &= \left\{ (\alpha - 1)(1 - z) - \frac{\alpha(\alpha^2 - 1)}{(\alpha + 1)^2} (1 - z) - \frac{1}{8} \frac{(\alpha^2 - 1)}{2} \frac{16\alpha^2}{(\alpha + 1)^4} (1 - z)^2 \right. \\ &\quad \left. + \dots \right\} \cdot \{1 + (1 - z) + (1 - z)^2 + \dots\}. \\ &= (1 - z) \left\{ (\alpha - 1) - \frac{\alpha(\alpha - 1)}{\alpha + 1} + \sum_{k=1}^{\infty} b_k (1 - z)^k \right\} \end{aligned}$$

Thefore $1 - w = (1 - z)^2[a + a_1(1 - z) + \dots]$ where $\lambda = 1$ and $a = (\alpha - 1)/(\alpha + 1) > 0$.

Theorem 3, along with Theorems 1 and 2 of [2] show that the Chebyshev method is neither an $[F, d_n]$ nor a Sonnenschein method.

One of the important properties of (f, g) -summability is the following [5, p. 267]:

Let (f, g) , (f_1, g_1) be two Gronwall means which map regions D, D_1 and with exponents λ, λ_1 . If $\lambda > \lambda_1$, and D is interior to D_1 , then (f, g) is stronger than (f_1, g_1) ; i.e., $(f, g) \supset (f_1, g_1)$.

The de la Vallee Poussin method (V) [4, p. 103] is a Gronwall method with $\delta = 2^{-1}$, $f(w) = (1 - \sqrt{1-w})/(1 + \sqrt{1-w})$, $g(w) = (1-w)^{-1/2}$ and $\lambda = 2$.

THEOREM 4. $(V) \supset (\sigma)$.

Proof. Since $\lambda_{(V)} = 2$, $\lambda_{(\sigma)} = 1$, it is enough to show that $D(V)$ is interior to $D(\sigma)$, that is,

$$\left| \frac{1 - \sqrt{1-w}}{1 + \sqrt{1-w}} \right| \leq \left| \frac{(\alpha - 1)^2 w}{(\alpha - w)^2} \right|.$$

It suffices to consider $|w| = 1$; thus we need to show

$$(12) \quad \frac{1}{|(1 + \sqrt{1-w^2})|} \leq \frac{(\alpha - 1)^2}{|(\alpha - w)^2|}.$$

Writing $1 - w = \rho e^{i\phi}$, where $-\pi < \phi < \pi$, (12) becomes

$$|\alpha - 1 + \rho e^{i\phi}|^2 \leq (\alpha - 1)^2 |1 + \rho^{1/2} e^{i\phi/2}|^2,$$

i.e.,

$$2(\alpha - 1) \cos \phi + \rho \leq 4\alpha(2\rho^{-1/2} \cos \phi/2 + 1).$$

Since $\cos \phi/2 > 0$, it is sufficient to show that $2(\alpha - 1) \cos \phi + \rho \leq 4\alpha$, which is readily verified.

THEOREM 5. $\sigma \not\supseteq (C, 1)$.

We shall make use of the well-known result that if A and B are regular summability methods, and B is a triangle, then $(A) \supseteq (B)$ if and only if AB^{-1} is regular.

Consider $D = AC^{-1}$, where A is the Chebyshev method and C is $(C, 1)$. C^{-1} has entries

$$c_{nk}^{-1} = \begin{cases} -n, & k = n - 1 \\ n + 1, & k = n \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$d_{nk} = \begin{cases} (k+1)a_{nk} - (k+1)a_{n,k+1}, & k < n \\ (n+1)a_{nn}, & k = n \\ 0, & \text{elsewhere.} \end{cases}$$

We shall show that D has infinite norm.

$$\sum_{k=0}^n |d_{nk}| = \sum_{k=0}^{p-1} (k+1)(a_{n,k+1} - a_{nk}) + \sum_{k=p}^{n-1} (k+1)(a_{nk} - a_{n,k+1}) + a_{nn}(n+1).$$

Now,

$$\begin{aligned} \sum_{k=0}^{p-1} (k+1)(a_{n,k+1} - a_{nk}) &= \sum_{k=0}^{p-1} (k+1)a_{n,k+1} - \sum_{k=0}^{p-1} ka_{nk} - \sum_{k=0}^{p-1} a_{nk} \\ &= \sum_{j=1}^p ja_{nj} - \sum_{k=0}^{p-1} ka_{nk} - \sum_{k=0}^{p-1} a_{nk} \\ &= pa_{np} - \sum_{k=0}^{p-1} a_{nk}. \end{aligned}$$

$$\begin{aligned} \sum_{k=p}^{n-1} (k+1)(a_{nk} - a_{n,k+1}) &= \sum_{k=p}^{n-1} ka_{nk} + \sum_{k=p}^{n-1} a_{nk} - \sum_{k=p}^{n-1} (k+1)a_{n,k+1} \\ &= \sum_{k=p}^{n-1} ka_{nk} + \sum_{k=p}^{n-1} a_{nk} - \sum_{j=p+1}^n ja_{nj} \\ &= pa_{np} - na_{nn} + \sum_{k=p}^{n-1} a_{nk}. \end{aligned}$$

Therefore,

$$\sum_{k=0}^n |d_{nk}| = pa_{np} - \sum_{k=0}^{p-1} a_{nk} + pa_{np} - na_{nn} + \sum_{k=p}^{n-1} a_{nk} + a_{nn}(n+1).$$

Since the Chebyshev method has row sums equal to 1,

$$\sum_{k=p}^{n-1} a_{nk} = 1 - \sum_{k=0}^{p-1} a_{nk} - a_{nn}.$$

Thus

$$\sum_{k=0}^n d_{nk} = 2pa_{np} - 2 \sum_{k=0}^{p-1} a_{nk} + 1.$$

But $\sum_{k=0}^{p-1} a_{nk} \leq 1$, so it is sufficient to show $pa_{np} \rightarrow \infty$. This follows immediately from (2), since $\lim \sqrt{n} = \infty$ and the remaining limits have already been shown to be finite and nonzero.

The Fourier series

$$\sum_{k=1}^{\infty} \sin kt/k = (\pi - t)/2, \quad 0 < t \leq \pi,$$

converges for all t , and the function has a jump at $t = 0$. Hence

the convergence is nonuniform at $t = 0$; that is, the sequence $\{s_n(t_n)\}$, where $\{t_n\}$ is a positive null sequence and

$$(13) \quad s_n(t) = \sum_{k=1}^n \sin kt/k, \quad s_0 = 0,$$

has several limit points, depending on the manner in which t_n approaches 0.

If $\lim nt_n = \tau \geq 0$, then $\lim s_n(t_n) = \int_0^\tau (\sin t/t)dt$, and the maximal limit is attained when $\tau = \pi$, in which case

$$(14) \quad \lim s_n(t_n) = \int_0^\pi \frac{\sin t}{t} dt = \frac{\pi}{2} \times 1.17897 \dots$$

On the other hand, $(\pi - t)/2 \rightarrow \pi/2$ as $t \downarrow 0$. Thus the limit points of $\{s_n(t_n)\}$ cover an interval which extends beyond $f(0+)$ if $f(0+) \neq 0$. This situation is called the Gibbs phenomenon relative to the partial sums.

We shall now show that the corresponding phenomenon occurs for the Chebyshev means.

THEOREM 6. *The Chebyshev means of (13) satisfy*

$$(15) \quad \lim \sigma_n(t_n) = \int_0^{\tau/\sqrt{2}} \frac{\sin y}{y} dy \text{ as } nt_n \rightarrow \tau \text{ and } nt_n^2 \rightarrow 0,$$

and

$$\limsup \sigma_n(t_n) \leq \int_0^\pi \frac{\sin t}{t} dt.$$

The lim sup inequality is an immediate consequence of (14) and the well-known fact that, for any totally regular matrix A , and any sequence $x = \{x_n\}$, $\limsup A_n(x) \leq \limsup x_n$.

The proof of the theorem is similar to that of [6]. One may write $s_n(t)$ in the form

$$s_n(t) = -t/2 + \int_0^t \frac{\sin(n + 1/2)x}{2 \sin(x/2)} dx.$$

Since $\sin(k + 1/2)x = \mathcal{F}(\exp(i(k + 1/2)x))$,

$$\sigma_n(t) = -t/2 + \mathcal{F} \left[\int_0^t \frac{1}{2 \sin(x/2)} \sum_{k=0}^n a_{nk} e^{ikx} e^{ix/2} dx \right].$$

From [1, p. 297], $\sum_{k=0}^n a_{nk} e^{ikx} = T_n(1 + 2e^{ix})/T_n(3)$, where $T_n(x)$ is defined by (1).

Define

$$\begin{aligned} \rho e^{i\beta} &= 1 + 2e^{ix} + [(1 + 2e^{ix})^2 - 1]^{1/2} \\ &= 1 + 2e^{ix} + 2e^{ix/2}e^{ix/4}(2 \cos x/2)^{1/2}. \end{aligned}$$

Let $a = (2 \cos x/2)^{1/2}$. Then $\rho \cos \beta = 1 + 2(\cos x + a \cos(3x/4))$,

$$(16) \quad \rho \sin \beta = 2(\sin x + a \sin(3x/4)),$$

and

$$(17) \quad \rho^2 = 5 + 4(\cos x + a \cos(3x/4)) + 8(\cos(x/2) + a \cos(x/4)).$$

Therefore $1 + 2e^{ix} - [(1 + 2e^{ix})^2 - 1]^{1/2} = \rho^{-1}e^{-i\beta}$, and assume $0 < x \leq t \leq \pi/2$.

$$\begin{aligned} \sigma_n(t) + t/2 &= \frac{1}{2T_n(3)} \int_0^t \frac{1}{2 \sin(x/2)} [\rho^n \sin(n\beta + x/2) \\ &\quad - \rho^{-n} \sin(n\beta - x/2)] dx = \frac{1}{4T_n(3)} \left\{ \int_0^t \rho^n \cot(x/2) \sin n\beta dx \right. \\ &\quad \left. + \int_0^t \rho^n \cos n\beta dx + - \int_0^t \rho^{-n} \cot(x/2) \sin n\beta dx + \int_0^t \rho^{-n} \cos n\beta dx \right\}. \end{aligned}$$

From (17), ρ is monotone decreasing in x for $0 < x \leq \pi/2$. Therefore for $0 < x \leq \pi/2$, $\rho < \alpha$. Thus

$$\left| \frac{1}{2T_n(3)} \int_0^t \rho^n \cos n\beta dx \right| < \int_0^t (\rho/\alpha)^n dx < t,$$

so that there exists an η satisfying $|\eta| < 1$ such that

$$\frac{1}{2T_n(3)} \int_0^t \rho^n \cos n\beta dx = \eta t.$$

Now assume that $t = t_n$, $nt_n \rightarrow \tau$, $0 \leq \tau \leq \infty$, and $nt_n^2 \rightarrow 0$.

Since, from (17), $\rho \geq \sqrt{5}$,

$$\left| \frac{1}{4T_n(3)} \int_0^t \rho^{-n} \cos n\beta dx \right| < \pi/4(\alpha\sqrt{5})^n = o(1).$$

$$(18) \quad \left| \frac{1}{4T_n(3)} \int_0^t \rho^{-n} \cot(x/2) \sin n\beta dx \right| < \frac{1}{2(\alpha\sqrt{5})^n} \int_0^t n\beta \cot(x/2) dx.$$

We wish to show that $\beta < x$. For $0 < x \leq \pi/2$, from (16), $\rho \sin \beta < 2(1 + a) \sin x$. From (17), if $\cos(3x/4) + 2 \cos(x/4) \geq 2$, then $\rho > 2(a + 1)$. In the interval $[0, \pi/2]$,

$$\begin{aligned} \cos(3x/4) + 2 \cos(x/4) &\geq \cos(3\pi/8) + \cos(\pi/8) \\ &= \cos(\pi/8)(4 \cos^2(\pi/8) - 1). \end{aligned}$$

Since $\cos(\pi/8) = \sqrt{2 + \sqrt{2}}/2$, it is sufficient to show that

$$\frac{\sqrt{2 + \sqrt{2}}}{2} \left(\frac{4}{4} (2 + \sqrt{2}) - 1 \right) \geq 2,$$

which is easily verified. Therefore $0 < \sin < \beta(\rho/2(1 + a)) \sin \beta < \sin x$, and $\beta < x$.

For $0 < x \leq \pi/2$, $2 \leq x/\sin(x/2) \leq \pi/\sqrt{2}$. Substituting in (18) we have

$$\begin{aligned} \left| \frac{1}{4T_n(3)} \int_0^t \rho^{-n} \cot(x/2) \sin n\beta dx \right| &< \frac{n}{2(\alpha\sqrt{5})^n} \int_0^{\pi/2} \cos(x/2) \cdot \frac{x}{\sin(x/2)} dx \\ &< \frac{n\pi^2}{4\sqrt{2}(\alpha\sqrt{5})^n} = o(1), \end{aligned}$$

and

$$\sigma_n(t) + (1 - \eta)t/2 = \frac{1}{4T_n(3)} \int_0^t \rho^n \cot(x/2) \sin n\beta dx + o(1).$$

Using (17), and the values of a and α ,

$$\begin{aligned} 1 - (\rho/\alpha)^2 &= [17 + 12\sqrt{2} - 5 - 4(\cos x + a \cos(3x/4)) \\ &\quad - 8(\cos(x/2) + a \cos(x/4))]/\alpha^2 \\ &= \frac{4}{\alpha^2} [1 - \cos x + 2(1 - \cos(x/2)) + \sqrt{2}(1 - \cos(3x/4)\sqrt{\cos(x/2)}) \\ &\quad + 2\sqrt{2}(1 - \cos(x/4)\sqrt{\cos(x/2)})]. \end{aligned}$$

Since $0 < \cos(x/2) < 1$,

$$\begin{aligned} 1 - \cos(x/4)\sqrt{\cos(x/2)} &\leq 1 - \cos(x/4) \cos(x/2) \\ &= 1 - (\cos(3x/4) + \cos(x/4))/2. \end{aligned}$$

Similarly, $1 - \cos(3x/4)\sqrt{\cos(x/2)} \leq 1 - (\cos(5x/4) + \cos(x/4))/2$. Therefore,

$$\begin{aligned} 1 - (\rho/\alpha)^2 &\leq \frac{4}{\alpha^2} [2 \sin^2(x/2) + 4 \sin^2(x/4) + \sqrt{2}(2 \sin^2(5x/8) \\ &\quad + 2 \sin^2(x/8))/2 + \sqrt{2}(2 \sin^2(3x/8) + 2 \sin^2(x/8))] \\ &\leq \frac{4}{\alpha^2} [2(x/2)^2 + 4(x/4)^2 + \sqrt{2}((5x/8)^2 + (x/8)^2) \\ &\quad + 2\sqrt{2}((3x/8)^2 + (x/8)^2)] \\ &= \frac{4}{\alpha^2} \left(3/4 + \frac{46\sqrt{2}}{64} \right) x^2 < \frac{x^2}{4}. \end{aligned}$$

Since $0 < \rho/\alpha < 1$, $1 - \rho/\alpha \leq 1 - (\rho/\alpha)^2$, so that $1 - \rho/\alpha < x^2/4$. $0 <$

$1 - (\rho/\alpha)^n = (1 - \rho/\alpha) \sum_{k=0}^{n-1} (\rho/\alpha)^k < n(1 - \rho/\alpha) < nx^2/4$. Therefore $1 - (\rho/\alpha)^n = \lambda nx^2$ for some λ satisfying $0 < \lambda < 1/4$, so that we may write

$$\begin{aligned} \frac{1}{2T_n(3)} \int_0^t \rho^n \cot(x/2) \sin n\beta dx &= \frac{\alpha^n}{2T_n(3)} \left[\int_0^t \cot(x/2) \sin n\beta dx \right. \\ &\quad \left. - n \int_0^t \lambda x^2 \cot(x/2) \sin n\beta dx \right] \\ &\times n \left| \int_0^t \lambda x^2 \cot(x/2) \sin n\beta dx \right| < n \int_0^t x^2 \cot(x/2) dx \\ &\leq \frac{nt\pi}{\sqrt{2}} \int_0^t dx < nt^2 = o(1), \end{aligned}$$

since $\lim nt_n^2 = 0$. Note that $\lim \alpha^n/2T_n(3) = 1$.

Using (17),

$$\begin{aligned} \frac{\rho\beta}{2} - \frac{2}{\alpha} \left(1 + \frac{3\sqrt{2}}{4} \right) x &= \frac{\rho}{\alpha} (\beta - \sin \beta) - \frac{2}{\alpha} (x - \sin x) \\ &\quad - \frac{2\sqrt{2}}{\alpha} \left(\frac{3x}{4} - \sin(3x/4) \sqrt{\cos(x/2)} \right), \end{aligned}$$

so that

$$\begin{aligned} |\rho\beta/\alpha - rx| &\leq \frac{\rho}{\alpha} |\beta - \sin \beta| + \frac{2}{\alpha} |x - \sin x| \\ &\quad + \frac{2\sqrt{2}}{\alpha} \left| \frac{3x}{4} - \sin(3x/4) \sqrt{\cos(x/2)} \right|, \end{aligned}$$

where $r = 2(1 + 3\sqrt{2}/4)/\alpha = (4 + 3\sqrt{2})/2\alpha = (4 + 3\sqrt{2})(3 - 2\sqrt{2})/2 = 1/\sqrt{2}$.

But $0 \leq 3x/4 - \sin(3x/4) \sqrt{\cos(x/2)} \leq 3x/4 - \sin(3x/4) \cos(x/2)$, $\sin(3x/4) \geq 3x/4 - (3x/4)^3/3!$, and $\cos(x/2) \geq 1 - x^2/4$, so that

$$\begin{aligned} |3x/4 - \sin(3x/4) \sqrt{\cos(x/2)}| &\leq 3x/4 - (3x/4 - (3x/4)^3/6)(1 - x^2/4) \\ &= 33x^3/128. \end{aligned}$$

Since $0 < x - \sin x < x^3$ and $\beta < x$,

$$|\rho\beta/\alpha - x/\sqrt{2}| \leq (\rho\beta^3 + 2x^3 + 33\sqrt{2}x^3/64)/\alpha < 2x^3.$$

Also, $|\beta - x/\sqrt{2}| \leq |\rho\beta/\alpha - x/\sqrt{2}| + (1 - \rho/\alpha)\beta < 2x^3 + x^3 = x^3$, so that $\beta = x/\sqrt{2} + \mu x^3$, where $|\mu| < 3$.

The remainder of the proof of (15) is the same as that of [6], beginning with formula (2.7), and will therefore be omitted.

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