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THE LEBESGUE CONSTANTS FOR (f, d_n) -SUMMABILITY

RICHARD ARTHUR SHOOP

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It is well-known that the Fourier series of a continuous periodic function need not be pointwise convergent. This fact is a consequence of the unboundedness of the Lebesgue constants, which are the norms of the partial sum operators. It is equally-known that the Fourier series of a continuous function is uniformly $(C, 1)$ -summable to the value of the function. Thus, the question naturally arises as to which summability matrices are effective in the limitation of Fourier series of continuous functions. In this paper we consider a very general class of matrices, the (f, d_n) means, and show that their Lebesgue constants are unbounded. An interesting corollary is that the Fourier series of a continuous periodic function need not be everywhere almost convergent.

If $A = (a_{nk})$ is a regular summability matrix, the n th Lebesgue constant corresponding to A is defined by

$$(1.1) \quad L_n(A) = \frac{2}{\pi} \int_0^{\pi/2} \left| \frac{\sum_{k=0}^{\infty} a_{nk} \sin(2k+1)t}{\sin t} \right| dt .$$

The sequence $\{L_n(A)\}$ is of considerable importance in the theory of Fourier series in that the unboundedness of this sequence implies the existence of a continuous function whose Fourier series fails to be A -summable at a specified point [1, pp. 58-60]. Conversely, if

$$\sum_{k=0}^{\infty} k |a_{nk}| < \infty \quad (n = 0, 1, \dots)$$

and if the sequence $\{L_n(A)\}$ is bounded, then the Fourier series of each function continuous on an interval $[a, b]$ is uniformly A -summable to the value of the function on $[a, b]$. Extensive study of the Lebesgue constants has been made by a number of authors including Livingston [4] for the Euler means, Ishiguro [2] and Newman [7] for Taylor summability, Lorch [5] for the Borel exponential and integral methods, Sledd [10] for Sonnenschein matrices, and Lorch and Newman for $[F, d_n]$ means [6], and for Hausdorff means [8].

The (f, d_n) means are defined as follows: Let f be a nonconstant function, analytic on the disc $|z| < R$ for some $R > 1$, and let $\{d_n\}$ be a sequence of complex numbers, such that for all n , $d_n \neq -f(1)$. The elements of the matrix A are then given by the relations

$$\alpha_{00} = 1, \quad \alpha_{0k} = 0 \quad (k \geq 1)$$

$$(1.2) \quad \prod_{j=1}^n \frac{f(z) + d_j}{f(1) + d_j} = \sum_{k=0}^{\infty} a_{nk} z^k .$$

This family of matrices was introduced by Smith [11] as a generalization of the $[F, d_n]$ means of Jakimovski [3], to which they reduce if $f(z) = z$. In case $d_j = 0$ for all j , (1.2) becomes

$$[f(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k ,$$

and A is called the Sonnenschein matrix generated by f [12]. The purpose of the paper shall be to derive an asymptotic expansion for the sequence $\{L_n(A)\}$ for a class of regular (f, d_n) matrices. In the final section, we shall demonstrate that the unboundedness of the Lebesgue constants for a particular (f, d_n) mean implies the existence of continuous functions whose Fourier series fail to be everywhere almost convergent.

2. Preliminaries. In addition to the assumptions made regarding the function f and the sequence $\{d_n\}$ we further assume that

(2.1) the Maclaurin coefficients of f are real and nonnegative;

$$(2.2) \quad |f(z)| < 1 \quad \text{for} \quad |z| \leq 1 (z \neq 1) ;$$

$$(2.3) \quad f(1) = f'(1) = 1 , \quad \text{while} \quad f''(1) \neq 0 ;$$

$$(2.4) \quad d_n \geq 0 \quad \text{for all} \quad n ;$$

$$(2.5) \quad \sum_n (1 + d_n)^{-1} = \infty .$$

Condition (2.5) is necessary for regularity of A , as is condition (2.2) in case $d_n = 0$ for all n . Moreover, conditions (2.1), (2.4), and (2.5) are sufficient for regularity [11]. The following two lemmas will be useful in §3. The first of these is due to Lorch and Newman [6].

LEMMA 2.1. Let $|a_k| \leq 1$ and $|b_k| \leq 1$ for $k = 1, \dots, n$, and let A be a positive constant. If $|a_k - b_k| \leq A c_k$ for $k = 1, \dots, n$ then

$$\left| \prod_{k=1}^n a_k - \prod_{k=1}^n b_k \right| \leq A \sum_{k=1}^n c_k .$$

LEMMA 2.2. Let K be a positive constant and let $\alpha, \beta \in [0, \pi/2]$. If $|e^{i\alpha} - e^{i\beta}| \leq K$, then $|\alpha - \beta| \leq K\pi/2$.

Proof. $e^{i\alpha} - e^{i\beta} = 2i \exp [i(\alpha + \beta)/2] \sin [(\alpha - \beta)/2]$, so

$$|e^{i\alpha} - e^{i\beta}| = 2|\sin [(\alpha - \beta)/2]| \geq \frac{2}{\pi}|\alpha - \beta|,$$

and the lemma follows.

3. The asymptotic behavior of $\{L_n(A)\}$. According to (1.1),

$$L_n(A) = \frac{2}{\pi} \int_0^{\pi/2} \frac{|K_n|}{\sin t} dt,$$

where

$$K_n = \sum_{k=0}^{\infty} a_{nk} \sin(2k+1)t.$$

From (1.2) and (2.3) it follows that

$$K_n = \frac{1}{2i} \left\{ e^{it} \prod_{j=1}^n \frac{f(e^{2it}) + d_j}{1 + d_j} \right\} - e^{-it} \prod_{j=1}^n \frac{f(e^{-2it}) + d_j}{1 + d_j} \right\}.$$

Define

$$\begin{aligned} R_j e^{i\theta_j} &= f(e^{2it}) + d_j \\ \rho_j e^{i\varphi_j} &= f(e^{-2it}) + d_j. \end{aligned}$$

The assumptions made about f cause its Taylor expansion about $z_0 = 1$ to be of the form

$$(3.1) \quad f(z) = z + a_2(z-1)^2 + O(z-1)^3,$$

where $a_2 = f''(1)/2 > 0$ by (2.1). It follows that

$$(3.2) \quad R_j e^{i\theta_j} = e^{2it} + d_j - 4a_2 t^2 + O(t^3)$$

$$(3.3) \quad \rho_j e^{i\varphi_j} = e^{-2it} + d_j - 4a_2 t^2 + O(t^3).$$

Since $d_j \geq 0$ for all j , these relations imply that

$$(3.4) \quad \rho_j = R_j + O(t^3)$$

and

$$(3.5) \quad \varphi_j = -\theta_j + O\left(\frac{t^2}{1+d_j}\right).$$

Now (3.4) implies that

$$\left| \frac{R_j}{1+d_j} - \frac{\rho_j}{1+d_j} \right| \leq \frac{Kt^3}{1+d_j},$$

so that

$$\left| \prod_{j=1}^n \frac{R_j}{1+d_j} - \prod_{j=1}^n \frac{\rho_j}{1+d_j} \right| \leq Kt^3 \sum_{j=1}^n (1+d_j)^{-1} \equiv KH_n t^3,$$

by Lemma 2.1. It follows that

$$\begin{aligned}
 K_n &= \frac{1}{2i} \left\{ \left(\prod_{j=1}^n \frac{R_j}{1+d_j} \right) \exp \left[i \left(t + \sum_{j=1}^n \theta_j \right) \right] \right. \\
 &\quad \left. - \left(\prod_{j=1}^n \frac{\rho_j}{1+d_j} \right) \exp \left[i \left(-t + \sum_{j=1}^n \varphi_j \right) \right] \right\} \\
 &= \frac{1}{2i} \prod_{j=1}^n \frac{R_j}{1+d_j} \left\{ \exp \left[i \left(t + \sum_{j=1}^n \theta_j \right) \right] \right. \\
 &\quad \left. - \exp \left[i \left(-t + \sum_{j=1}^n \varphi_j \right) \right] \right\} + O(H_n t^3) \\
 &= \left(\prod_{j=1}^n \frac{R_j}{1+d_j} \right) \sin \left[\frac{1}{2} \sum_{j=1}^n (\theta_j - \varphi_j) + t \right] \\
 &\quad \times \exp \left[\frac{i}{2} \sum_{j=1}^n (\theta_j + \varphi_j) \right] + O(H_n t^3).
 \end{aligned}$$

Hence,

$$|K_n| = \left(\prod_{j=1}^n \frac{R_j}{1+d_j} \right) \left| \sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) + t \right] \right| + O(H_n t^3).$$

Suppose that $0 < \xi < \pi/2$. Then

$$\begin{aligned}
 &\int_0^\xi \frac{|K_n|}{\sin t} dt \\
 (3.6) \quad &= \int_0^\xi \prod_{j=1}^n \frac{R_j}{1+d_j} \frac{\left| \sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) + t \right] \right|}{\sin t} dt + O(H_n \xi^3).
 \end{aligned}$$

We may replace $\sin t$ by t in the integral on the right of (3.6), introducing an error of $O(\xi)$.

Thus,

$$\begin{aligned}
 &\int_0^\xi \frac{|K_n|}{\sin t} dt \\
 &= \int_0^\xi \prod_{j=1}^n \frac{R_j}{1+d_j} \frac{\left| \sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) + t \right] \right|}{t} dt + O(\xi) + O(H_n \xi^3).
 \end{aligned}$$

Using the expansion

$$\begin{aligned}
 &\sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) + t \right] \\
 &= \sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) \right] \cos t + \cos \left[\frac{1}{2} \sum (\theta_j - \varphi_j) \right] \sin t,
 \end{aligned}$$

we obtain

$$\int_0^\xi \prod_{j=1}^n \frac{R_j}{1+d_j} \cdot \frac{1}{t} \left| \sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) + t \right] - \sin \left[\frac{1}{2} \sum (\theta_j - \varphi_j) \right] \right| dt \\ \leq \int_0^\xi \frac{1 - \cos t}{t} dt + \int_0^\xi \frac{\sin t}{t} dt = O(\xi).$$

Therefore,

$$(3.7) \quad \int_0^\xi \frac{|K_n|}{\sin t} dt \\ = \int_0^\xi \prod_{j=1}^n \frac{R_j}{1+d_j} \cdot \frac{\left| \sin \frac{1}{2} \sum (\theta_j - \varphi_j) \right|}{t} dt + O(\xi) + O(H_n \xi^2).$$

We now estimate $\left(\int_\xi^{\pi/2} |K_n|/\sin t \right) dt$. To this end, define $\operatorname{Re}^{i\theta} = f(e^{2it})$. From (3.1) it follows that

$$R = 1 - 4a_2 t^2 + O(t^3),$$

or

$$(3.8) \quad R = 1 - 4a_2 t^2 (1 + t\psi(t)),$$

where ψ is bounded in a neighborhood of $t = 0$. Now

$$R_j^2 = |f(e^{2it}) + d_j|^2 \\ = R^2 + 2Rd_j \cos \theta + d_j^2 \\ \leq R^2 + 2Rd_j + d_j^2.$$

Substitution of right-hand side of (3.8) for R yields

$$(3.9) \quad R_j^2 \leq 1 + 16a_2^2 t^4 (1 + t\psi(t))^2 - 8a_2 t^2 (1 + t\psi(t)) \\ + d_j^2 + 2d_j - 8a_2 d_j t^2 (1 + t\psi(t)).$$

If t is sufficiently close to zero, then $|t\psi(t)| < 1/2$, and the right-hand side of (3.9) is dominated by

$$(3.10) \quad (1 + d_j)^2 - (1 + d_j)[4a_2 t^2 - 36a_2^2 t^4].$$

If we further insist that t be less than $(18a_2)^{-1/2}$, then it follows that $36a_2^2 t^4 < 2a_2 t^2$, so (3.9) and (3.10) combine to yield

$$(3.11) \quad R_j^2 \leq (1 + d_j)^2 - 2a_2 t^2 (1 + d_j).$$

Using (3.11) and the familiar inequality $1 + x < e^x$, valid for real x , we obtain

$$\left(\frac{R_j}{1+d_j} \right)^2 \leq 1 - \frac{2a_2 t^2}{1+d_j} \leq \exp \left(-\frac{2a_2 t^2}{1+d_j} \right),$$

so that

$$\prod_{j=1}^n \frac{R_j}{1 + d_j} \leq \exp [-a_2 H_n t^2].$$

Analogously, one shows that

$$\prod_{j=1}^n \frac{\rho_j}{1 + d_j} \leq \exp [-a_2 H_n t^2].$$

Hence,

$$|K_n| \leq \frac{1}{2} \left[\prod \frac{R_j}{1 + d_j} + \prod \frac{\rho_j}{1 + d_j} \right] \leq \exp [-a_2 H_n t^2],$$

and

$$\int_{\xi}^{\pi/2} \frac{|K_n|}{\sin t} dt \leq \frac{\pi}{2} \int_{\xi}^t \frac{\exp[-a_2 H_n t^2]}{t} dt = O[\xi^{-1} \exp(-H_n \xi^2)].$$

This estimate, together with (3.7) gives

$$(3.12) \quad \int_0^{\pi/2} \frac{|K_n|}{\sin t} dt = \int_0^{\xi} \prod_{j=1}^n \frac{R_j}{1 + d_j} \frac{\left| \sin \frac{1}{2} \sum (\theta_j - \varphi_j) \right|}{t} dt + O(\xi) + O(H_n \xi^3) + O[\xi^{-1} \exp(-H_n \xi^2)].$$

We now replace the product appearing in the integral on the right of (3.12) by a more manageable expression. By equation (3.2)

$$f(e^{2it}) + d_j = 1 + d_j + 2it - (2 + 4a_2)t^2 + O(t^3),$$

so that

$$\begin{aligned} \frac{f(e^{2it}) + d_j}{1 + d_j} &= 1 + \frac{2it}{1 + d_j} - \frac{2 + 4a_2 t^2}{1 + d_j} + O\left(\frac{t^3}{1 + d_j}\right) \\ &= \exp \left\{ \frac{2it}{1 + d_j} - \left[\frac{4a_2}{1 + d_j} + \frac{2d_j}{(1 + d_j)^2} \right] t^2 \right\} + O\left(\frac{t^3}{1 + d_j}\right). \end{aligned}$$

By Lemma 2.1, it follows that

$$(3.13) \quad \prod_{j=1}^n \frac{f(e^{2it}) + d_j}{1 + d_j} = \exp \{2iH_n t - 4a_2 H_n t^2 - S_n t^2\} + O(H_n t^3),$$

where $S_n = 2 \sum_{j=1}^n d_j (1 + d_j)^{-2}$. Hence,

$$\prod_{j=1}^n \frac{R_j}{1 + d_j} = \exp \{-(4a_2 H_n + S_n) t^2\} + O(H_n t^3),$$

and (3.12) becomes

$$\int_0^{\pi/2} \frac{|K_n|}{\sin t} dt = \int_0^{\xi} \exp[-(4a_2 H_n + S_n)t^2] \frac{\left| \sin \frac{1}{2} \sum (\theta_j - \varphi_j) \right|}{t} dt \\ + O(\xi) + O(H_n \xi^3) + O[\xi^{-1} \exp(-H_n \xi^2)].$$

From (3.13) it follows that

$$\exp(i \sum \theta_j) = \exp(2i H_n t) + O(H_n t^3).$$

Lemma 2.2 now implies that

$$\sum \theta_j = 2H_n t + O(H_n t^3).$$

In similar fashion it is shown that

$$\sum \varphi_j = -2H_n t + O(H_n t^3).$$

Hence, $\sin 1/2 \sum (\theta_j - \varphi_j) = \sin 2H_n t + O(H_n t^3)$, and

$$(3.14) \quad \int_0^{\pi/2} \frac{|K_n|}{\sin t} dt = \int_0^{\xi} \exp[-(4a_2 H_n + S_n)t^2] \frac{|\sin 2H_n t|}{t} dt \\ + O(\xi) + O(H_n \xi^3) + O[\xi^{-1} \exp(-H_n \xi^2)].$$

Here, the interval of integration may be extended from $[0, \xi]$ to $[0, \pi/2]$ with an error of $O[\xi^{-1} \exp(-H_n \xi^2)]$. This having been done, we now let $\xi = H_n^{-3/8}$. Since $H_n \rightarrow \infty$ as $n \rightarrow \infty$, all of the error terms in (3.14) become $o(1)$. We now make the substitution $u_n = 2H_n$ and $s_n = 4a_2 H_n + S_n$. Thus

$$L_n(A) = \frac{2}{\pi} \int_0^{\pi/2} \exp(-s_n t^2) \frac{|\sin u_n t|}{t} dt + o(1).$$

Since, for our choice of u_n and s_n , $s_n \rightarrow \infty$ and $u_n^2/s_n \rightarrow \infty$, the derivation of [6; §5] may be applied, yielding

$$(3.15) \quad L_n(A) = \frac{2}{\pi^2} \log \frac{u_n^2}{s_n} + \alpha + o(1),$$

where

$$\alpha = -\frac{2}{\pi^2} C + \frac{2}{\pi} \int_0^1 \frac{\sin t}{t} dt - \frac{2}{\pi} \int_1^\infty \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt,$$

and C denotes Euler's constant. In terms of H_n and S_n , our expansion takes the form

$$L_n(A) = \frac{2}{\pi^2} \log \left(\frac{4H_n^2}{4a_2 H_n + S_n} \right) + \alpha + O(1),$$

from which it is clear that $\{L_n(A)\}$ is an unbounded sequence. We

note in conclusion that if $d_j = 0$ for all j , then $S_n = 0$ and $H_n = n$, so that for Sonnenschein methods we have

$$L_n(A) = \frac{2}{\pi^2} \log \frac{n}{a_2} + \alpha + o(1),$$

which is the result obtained by Sledd [10].

4. A special case. A regular matrix A is said to be strongly regular provided every almost convergent sequence is A -summable. Lorentz [9] has shown that a necessary and sufficient condition for strong regularity of a regular matrix $A = (a_{nk})$ is that

$$(4.1) \quad \lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0.$$

If we take $f(z) = e^{z-1}$ and $d_n = 0$ for all n , then the resulting (f, d_n) mean is the Borel matrix:

$$a_{nk} = e^{-n} \frac{n^k}{k!}.$$

Now

$$\begin{aligned} \sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| &= e^{-n} \left\{ \sum_{k=1}^{n-1} \frac{n^k}{(k+1)!} [n - (k+1)] + \sum_{k=n}^{\infty} \frac{n^k}{(k+1)!} [(k+1) - n] \right\} \\ &= e^{-n} \left[\sum_{k=1}^{n-1} \frac{n^{k+1}}{(k+1)!} - \sum_{k=1}^{n-1} \frac{n^k}{k!} + \sum_{k=n}^{\infty} \frac{n^{k+1}}{(k+1)!} \right] \\ &= e^{-n} \left[2 \binom{n}{n!} - n \right]. \end{aligned}$$

By Stirlings formula, $n^n/n!e^n = O(n^{-1/2})$, so that $e^{-n}[2(n^n/n!) - n] \rightarrow 0$ as $n \rightarrow \infty$, and A is strongly regular. It follows that there exist continuous functions whose Fourier series fail to be almost convergent; for if this were not the case, then the Borel matrix would sum the Fourier series of each continuous function, contrary to the unboundedness of the Borel-Lebesgue constants.

REFERENCES

1. G. H. Hardy and W. W. Rogosinski, *Fourier Series*, Cambridge (1944).
2. K. Ishiguro, *The Lebesgue constants for (i, r) summation of Fourier series*, Proc. Japan Acad., **36** (1960), 470-474.
3. A. Jakimovski, *A generalization of the Latotsky method*, Michigan Math. J., **6** (1959), 270-290.
4. A. E. Livingston, *The Lebesgue constants for Euler (E, p) summation of Fourier series*, Duke Math. J., **21** (1954), 309-314.
5. L. Lorch, *The Lebesgue constants for Borel summability*, Duke Math. J., **11** (1944), 459-467.

6. L. Lorch and D. J. Newman, *On the $[F, d_n]$ summation of Fourier series*, Comm. on Pure and Appl. Math., **15** (1962), 109-118.
7. ———, *The Lebesgue constants for (\tilde{r}, r) summation of Fourier series*, Canad. Math. Bull., **6**, no. 2 (1963), 179-182.
8. ———, *The Lebesgue constants for regular Hausdorff methods*, Canad. J. Math., **13** (1961), 283-298.
9. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math., **80** (1948), 167-190.
10. W. T. Sledd, *The Gibbs phenomenon and Lebesgue constants for regular Son-nenschein matrices*, Canad. J. Math., **14** (1962), 723-728.
11. G. Smith, *On the (f, d_n) method of summability*, Canad. J. Math., **17** (1965), 506-526.
12. J. Sonnenschein, *Sur les séries divergents*, Bull. Acad. Royale de Belgique, **35** (1949), 594-601.

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