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A WITT'S THEOREM FOR UNIMODULAR LATTICES

Y. C. LEE

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A WITT'S THEOREM FOR UNIMODULAR LATTICES

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Let K be a dyadic local field, \circ its ring of integers, L a regular unimodular lattice over \circ . If x and y are vectors in L, we ask for necessary and sufficient conditions to map x isometrically to y. Trojan and James obtain conditions via a T-invariant when \circ is 2-adic. Hsia uses characteristic sets and G-invariants for vectors and he solves the problem when \circ is dyadic in general. We define here a new numerical invariant, the degree of a vector, which reflects more on the structure of L and the relationship between x, y and L. The Witt conditions will be stated in terms of this degree invariant.

1. Introduction. Let π be a prime element generating the maximal ideal of \circ and let e be such that $2\circ = \pi^{e}\circ$. Let Q be a quadratic form on L, B its associated symmetric bilinear form. Then Q and B are connected by

$$Q(x + y) = Q(x) + Q(y) - B(x, y)$$
.

The lattice L is unimodular simply means $B(L, L) = \mathfrak{o}$ and det L is a unit. The structure of unimodular lattices is well-known and can be found in O'Meara [7]. A vector v is primitive if $v \notin \pi L$. Hence v is primitive if and only if $B(v, L) = \mathfrak{o}$.

PROPOSITION 1. Let v be a vector in a unimodular lattice L. Then $v \in \pi^k L$ if and only if $B(v, L) \subseteq \pi^k \mathfrak{o}$.

Proof. The necessity is trivial. Assume $B(v, L) \subseteq \pi^k o$ and h is the highest power of π that divides v, that is, $v = \pi^k w$ for some primitive vector w. Hence B(w, L) = o and $B(v, L) = B(\pi^k w, L) = \pi^k o$ and $h \ge k$.

If $z \in L$ satisfies ord $Q(z) \leq$ ord B(z, L), then the map σ_z given by

$$\sigma_z(v) = v - B(v, z)z/Q(z)$$

is an integral isometry known as the reflection of z. The group of integral isometries is denoted by O(L). O'Meara and Pollak [8], [9] have shown that O(L) is generated by reflections except in a few cases when the residue field $o/\pi o$ contains only two elements, and that in the exceptional cases one extra generator given by an Eichler transform is needed. If *i* is a nonzero isotropic vector and z satisfies B(i, z) = 0 then an Eichler transform E_z^i is defined by:

$$E_z^{i}(v) = v + B(v, i)z - B(v, z)i - Q(z)B(v, i)i$$
 .

We say two vectors x and y are associated, denoted $x \sim y$ if there is a $\phi \in O(L)$ such that $\phi(x) = y$. For each k such that $e \ge k \ge 0$, let

$$L^{(-k)} = \{v \in L \colon Q(v) \in \pi^{-k} \mathfrak{O}\}$$
 .

Each $L^{(-k)}$ is invariant under the action of O(L) and

$$L = L^{(-e)} \supseteq \cdots \supseteq L^{(-1)} \supseteq L^{(0)}$$
.

DEFINITION. The lattice L is said to have degree k if

$$L=\cdots=L^{\scriptscriptstyle (-k)}
eq L^{\scriptscriptstyle (-k+1)}$$
 ,

The sublattice $L^{(0)}$ is called the even sublattice of L. A degree 0 lattice is simply called an even lattice.

2. The degree invariant.

DEFINITION. Let v be a primitive vector. The degree of v, m(v), is given by $m(v) = \text{ord } B(v, L^{(0)})$.

If d is the degree of L, then clearly $\pi^{[d/2]}L \subseteq L^{(0)}$, where [d/2] denotes the smallest integer greater than d/2. Consequently, $m(v) \leq [d/2]$.

Furthermore, if v is a primitive vector with degree m and w is another primitive vector with $w - v \in \pi^m L$, then the degree of w is also m. For we have $w = v + \pi^m z$ and

$$egin{aligned} B(w,\,L^{\scriptscriptstyle(0)}) &= B(v\,+\,\pi^{m}z,\,L^{\scriptscriptstyle(0)}) \ &= B(v,\,L^{\scriptscriptstyle(0)}) + \pi^{m}B(z,\,L^{\scriptscriptstyle(0)}) \ &= \pi^{m}\mathfrak{o} \;. \end{aligned}$$

3. Witt's theorem.

THEOREM. Let x and y be primitive vectors such that Q(x) = Q(y). Then x is associated to y if and only if m(x) = m(y) = m and $y - x \in \pi^m L$.

We remark that the condition $y - x \in \pi^m L$ expresses how close the vectors x and y must be. With the upperbounds calculated for m, this condition becomes quite appealing. Before proving the theorem we first set up an invariant which has its own importance.

DEFINITION. If a is an element of K, the quotient field of o, let

$$S_a = \{u \in L \colon Q(u) \equiv a \mod \mathfrak{o}\}$$
.

DEFINITION. If v is a primitive vector of degree m, let

$$S_a(v) = B(v, u) \mod \pi^m \mathfrak{o}$$

where u is a vector in S_a .

Since $Q(\phi(u)) = Q(u)$ for any isometry ϕ , it is clear that S_a is invariant under O(L). To show that $S_a(v)$ is well-defined, let u, u' be vectors in S_a . Then $Q(u) \equiv Q(u') \equiv a \mod o$, and

$$Q(u - u') = Q(u) - Q(u') - B(u', u - u')$$

= 0 mod o.

Hence $u - u' \in L^{(0)}$ and

$$B(v, u) - B(v, u') = B(v, u - u')$$

= 0 mod $\pi^m o$.

Since S_a is invariant under O(L), we immediately obtain that if x and y are associated, then $S_a(x) = S_a(y)$.

Proof of theorem. Let x and y be associated vectors. Clearly m(x) = m(y). For each nonempty S_a , we have

$$S_a(x) \equiv S_a(y) \mod \pi^m \mathfrak{o}$$
 .

Therefore,

$$B(y-x, u) \equiv 0 \mod \pi^m \mathfrak{o}$$

for any $u \in S_a$. Since the collection $\{S_a\}$ partitions L, this means

$$B(y - x, u) \equiv 0 \mod \pi^m \mathfrak{o}$$

for all $u \in L$. Proposition 1 shows that $y - x \in \pi^m L$.

It is convenient to collect that following two results.

PROPOSITION 2. Let x and y be primitive vectors such that Q(x) = Q(y) and m(x) = m(y) = 0. Then $x \sim y$.

Proof. This is a direct application of Kneser's theorem [6].

PROPOSITION 3. Let x and y be primitive vectors such that Q(x) = Q(y) and m(x) = m(y) = m. Then $x \sim y$ provided one of the following holds:

(i) $y - x \in \pi^m L$ and ord Q(y - x) = 2m;

(ii) $y - x \in \pi^m L$ and there is a vector $u \in L^{(0)}$ with $\operatorname{ord} Q(u) = 0$ and $\operatorname{ord} B(x, u) = \operatorname{ord} B(y, u) = m$;

(iii) $y - x \in \pi^{m+1}L$ and there is a vector $u \in L^{(-1)} - L^{(0)}$ with ord B(x, u) = ord B(y, u) = m.

Proof. Let $z = \pi^{-m}(y - x)$.

(i) Since Q(z) is a unit, the reflection σ_z is integral and sends x to y.

(ii) We may assume ord Q(z) > 0. Let

$$z' = z + B(x, u)u/\pi^m Q(u)$$
.

Then it is easily shown that Q(z') is a unit. Hence $\sigma_{z'}$ and σ_u are integral reflections and $\sigma_{z,i}$, $\sigma_u(x) = y$.

(iii) Again assume ord Q(z) > 0. Let

$$z' = z + B(x, u) u / \pi^{m+1} Q(u) \; .$$

Then ord Q(z') = -1. Hence $\sigma_{z'}$ is integral and $\sigma_{z'}\sigma_u(x) = y$.

Proof of theorem (continued). Let x and y be primitive vectors satisfying the conditions Q(x) = Q(y), m(x) = m(y) = m and $y - x \in \pi^m L$.

If m = 0, Proposition 2 settles the problem. Let $m \ge 1$. We may further assume that $\operatorname{ord} Q(y - x) > 2m$, otherwise Proposition 3 (ii) already provides the necessary isometry. We proceed with the proof in a series of lemmas.

LEMMA 1. If
$$B(x, L^{(-1)}) = B(y, L^{(-1)}) = \pi^{m-1}\mathfrak{o}$$
, then $x \sim y$.

Proof. Since $B(x, L^{(0)}) = B(y, L^{(0)}) = \pi^m \mathfrak{o}$, we know that $L^{(-1)} - L^{(0)}$ is a nonempty set. Choose v and w from this set so that ord $B(x, v) = \operatorname{ord} B(y, w) = m - 1$. Then one of the vectors v, w, v + w, which we denote by u, will satisfy $\operatorname{ord} B(x, u) = \operatorname{ord} B(y, u) = m - 1$. This vector u also lies in $L^{(-1)} - L^{(0)}$. Let $z = \pi^{-m}(y - x)$ and $z' = z + B(x, u)u/\pi^{m-1}Q(u)$. Then $\sigma_{z'}$ is an integral isometry and $\sigma_{z'}\sigma_u(x) = y$.

LEMMA 2. If $B(x, L^{(-2)}) = B(y, L^{(-2)}) = \pi^{m-1}\mathfrak{o}$ and $B(x, L^{(-1)}) = B(y, L^{(-1)}) = \pi^m \mathfrak{o}$, then $x \sim y$.

Proof. As in Lemma 1, we can choose a vector z from $L^{(-2)}$ -

 $L^{(-1)}$ such that ord B(x, z) = ord B(y, z) = m - 1. Then Proposition 3 (ii) can be applied with $u = \pi z$.

LEMMA 3. Assume $B(x, L^{(-2)}) = B(y, L^{(-2)}) = \pi^m \mathfrak{o}$. If there is a vector $z \in L^{(-2)} - L^{(-1)}$, then $x \sim y$.

Proof. There are vectors v and w in $L^{(0)}$ such that ord B(x, v) = ord B(y, w) = m. One of the three vectors v, w, v + w, which will be denoted by u, must satisfy ord B(x, u) = ord B(y, u) = m. If ord Q(u) = 0, Proposition 3 (ii) can be used. Otherwise let $u' = u + \pi z$. Then ord Q(u') = 0 and ord B(x, u') = ord B(y, u') = m. Hence Proposition 3 (ii) can again be used.

From here on we may assume that $B(x, L^{(-2)}) = B(y, L^{(-2)}) = \pi^m \mathfrak{o}$, and that there are no vectors u in L with $\operatorname{ord} Q(u) = -2$. This further means that there are no vectors u in L with Q(u) having negative even orders. Hence the degree of L equals -2h + 1 for some positive integer h. By an earlier remark and Proposition 2, we may assume that h > m > 1.

LEMMA 4. Under the above assumptions, the lattice L has one of the following decompositions:

(i) $L = \mathfrak{o}v^{\perp}M$ if L is odd-dimensional,

(ii) $L = (\mathfrak{o}v \oplus \mathfrak{o}w)^{\perp}M$ if L is even-dimensional,

where ord Q(v) = -2h + 1, ord $Q(w) \ge 1$, B(v, w) = 1 and M is an even sublattice.

Proof. (i) We can write $L = \mathfrak{ov}_1 \perp M_1$, where $\operatorname{ord} Q(v_1) = -2h + 1$. If M_1 is not even, then M_1 contains vectors u with $\operatorname{ord} Q(u)$ being some negative odd integer. By adding appropriate vectors av_1 , $a \in \mathfrak{o}$, to these vectors, we can form a new decomposition $L = \mathfrak{ov}_2 \perp M_2$ with the degree of M_2 less than the degree of M_1 . By induction we can obtain the desired decomposition.

(ii) Starting with a decomposition $L = (\mathfrak{o}v_1 \bigoplus \mathfrak{o}w_1) \perp M_1$, we can use v_1 to change M_1 until $L = (\mathfrak{o}v_2 \bigoplus \mathfrak{o}w_2) \perp M$, where M is even. Finally, since ord $Q(w_2)$ is odd or greater than 0, we can use v_2 to change w_2 to obtain the desired decomposition.

LEMMA 5. Let L be odd-dimensional. Then there exists an isometry ϕ such that $\phi(x) - y \in \pi^{m+1}L$. Hence $x \sim y$.

Proof. Let $L = \mathfrak{ov} \perp M$ be given by Lemma 4. Write

$$x = av + \pi^m z$$
, $y = bv + \pi^m z'$

where z, z' are primitive vectors of M. Since

$$Q(x) - Q(y) = (a^2 - b^2)Q(v) + \pi^{2m}(Q(z) - Q(z')) = 0$$
,

we have

$$(a^2-b^2)Q(v)\equiv 0 \mod \pi^{2m} \mathfrak{o}$$

so that

$$a^{\scriptscriptstyle 2}-b^{\scriptscriptstyle 2}\equiv 0 \mod \pi^{\scriptscriptstyle 2m+2h-1}\mathfrak{o}$$
 .

and

$$a-b\equiv 0 \,\, \operatorname{mod} \pi^{m+h} \mathfrak{o}$$
 .

Hence $\pi^{2m}(Q(z) - Q(z')) \equiv 0 \mod \pi^{2m+1}\mathfrak{o}$ and

$$Q(z) - Q(z') \equiv 0 \mod \pi o$$
.

There exists a vector $w \in M$ with B(w, z') = 1. Let $u = z' + c\pi w$, so that $Q(u) = Q(z') + c\pi + c^2\pi^2 Q(w)$. The equation

$$Q(oldsymbol{z}') + c \pi + c^2 \pi^2 Q(w) = Q(oldsymbol{z})$$

can be solved for c by Hensel's lemma. Since Q(u) = Q(z), m(u) = m(z) = 0, by Proposition 2 there is an isometry ϕ in O(M) such that $\phi(z) = u$. Now $z - u \in \pi^{m+1}L$. By Proposition 3 (iii), $\phi(x) \sim y$ and so $x \sim y$.

LEMMA 6. Let L be even-dimensional. Then $x \sim y$.

Proof. Let $L = (ov \oplus ow) \perp M$ be given by Lemma 4. Write

$$x=\pi^{m}av+w+\pi^{m}z$$

 $y=\pi^{m}bv+arepsilon w+\pi^{m}z'$

where z and z' are in M and ε is a unit. Then

$$egin{array}{lll} 0 = Q(x) - Q(y) = \pi^{2m}(a^2 - b^2)Q(v) + \pi^m(a - barepsilon) \ &+ (1 - arepsilon^2)Q(w) + \pi^{2m}(Q(z) - Q(z')) \;. \end{array}$$

Using an argument similar to that used in Lemma 5, we show that $a - b \in \pi^{h} \mathfrak{o}$ and $1 - \varepsilon \in \pi^{m} \mathfrak{o}$. Hence for some $c \in \mathfrak{o}$,

$$egin{array}{ll} \pi^{m}(a-barepsilon)&=\pi^{m}(a-(a+\pi^{h}c)arepsilon)\ &\equiv\pi^{m}a(1-arepsilon)\,\,\mathrm{mod}\,\,\pi^{2m+1}\mathfrak{o}\;. \end{array}$$

If ord $(1 - \varepsilon) \ge m + 1$, then $\pi^m a(1 - \varepsilon) \equiv 0 \mod \pi^{2m+1} o$. Hence $Q(z) - Q(z') = 0 \mod \pi o$ and there is an isometry $\phi \in O(M)$ such that $\phi(z) - z \in \pi M$. Hence $\phi(x) - y \in \pi^{m+1}L$ and $\phi(x) \sim y$ by Proposition 3 (iii).

If ord $(1 - \varepsilon) = m$ we note that a and b must be simultaneously units or nonunits.

(1) Both a and b are units. Then $\operatorname{ord} \pi^m(a - b\varepsilon) = 2m$. Hence ord (Q(z) - Q(z')) = 0, and at least one of Q(z), Q(z') is a unit. Without loss of generality, let Q(z) be a unit. If $\operatorname{ord} B(z, z') \ge 1$, then the vector u = z + w fulfills the hypothesis of Proposition 3 (ii), hence $x \sim y$. Now let $\operatorname{ord} B(z, z') = 0$.

(i) ord $Q(z') \ge 1$. There exists a vector $z'' = z + \zeta z'$ such that ζ is a unit and Q(z'') = Q(z'). For this z'', we have $B(z'', M) = B(z', M) = \mathfrak{o}$. Hence there is an isometry $\phi \in O(M)$ with $\phi(z') = z''$. Proposition 3 (ii) can now be used on $\phi(x)$ and y, with u = z.

(ii) ord $Q(z) = \operatorname{ord} Q(z') = 0$. Since Q(z) - Q(z') is not zero, the residue field $\mathfrak{o}/\pi\mathfrak{o}$ must possess more than two elements. And since ord $B(x, w + \zeta z) = 0$ for all units ζ , we can choose a unit ζ such that ord $B(y, w + \zeta z) = 0$ as well. Now Proposition 3 (ii) can be used with $u = w + \zeta z$.

(2) Both a and b are nonunits. Then $\operatorname{ord} a(1-\varepsilon) \geq 2m+1$. Here $Q(z) - Q(z') \equiv 0 \mod \pi o$. Hence there is an isometry $\phi \in O(M)$ such that $\phi(z) \equiv z' \mod \pi M$. Now we can rewrite

$$egin{array}{lll} x=\pi^{m+1}a'v+w+\pi^m z\ y=\pi^{m+1}b'v+arepsilon w+\pi^m z+\pi^{m+1}\overline{z}\ , \end{array}$$

where $\overline{z} \in M$. Since x and y are primitive, z must also be primitive. Hence there exists a primitive vector $z'' \in M$ which decomposes M as:

$$M = (\mathfrak{o} z \oplus \mathfrak{o} z'') \perp M'$$
 .

If ord $Q(z) \ge 1$, we may choose z'' so that ord Q(z'') = 0. Hence the hypothesis of Proposition 3 (ii) is satisfied with u = z'' + w. Assume now ord Q(z) = 0. If we can choose a vector z'' with ord Q(z'') = 0, we are again done. Otherwise we can choose a vector z'' with Q(z'') = 0. Let $\gamma = (\varepsilon - 1)/\pi^m$. Consider the Eichler transform $E_{iw}^{z'}$ on x:

$$E_{\tau w}^{z''}(x) = x + B(x, z'')\gamma w - B(x, \gamma w)z'' - Q(\gamma w)B(x, z'')z''.$$

An easy calculation shows that

$$x - E^{z^{\prime\prime}}_{rw}(x) \equiv arepsilon w mod \pi^{m+1}L$$
 .

Hence

$$y-E_{ au w}^{z^{\prime\prime}}(x)\equiv 0 \ \operatorname{mod} \pi^{m+1}L$$
 .

Proposition 3 (iii) can be applied to y and $E_{fw}^{z''}(x)$, with $u = \pi^{k} v$.

Y. C. LEE

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Pacific Journal of MathematicsVol. 80, No. 2October, 1979

K. Adachi, On the multiplicative Cousin problems for $N^p(D)$	297
Howard Banilower, <i>Isomorphisms and simultaneous extensions in</i> $C(S)$	305
B. R. Bhonsle and R. A. Prabhu, An inversion formula for a distributional	
finite-Hankel-Laplace transformation	313
Douglas S. Bridges, <i>Connectivity properties of metric spaces</i>	325
John Patton Burgess, A selection theorem for group actions	333
Carl Claudius Cowen, Commutants and the operator equations	
$AX = \lambda XA$	337
Thomas Curtis Craven, <i>Characterizing reduced Witt rings. II</i>	341
J. Csima, Embedding partial idempotent d-ary quasigroups	351
Sheldon Davis, A cushioning-type weak covering property	359
Micheal Neal Dyer, Nonminimal roots in homotopy trees	371
John Erik Fornaess, <i>Plurisubharmonic defining functions</i>	381
John Fuelberth and James J. Kuzmanovich, On the structure of finitely	
generated splitting rings	389
Irving Leonard Glicksberg, Boundary continuity of some holomorphic	
functions	425
Frank Harary and Robert William Robinson, Generalized Ramsey theory.	
IX. Isomorphic factorizations. IV. Isomorphic Ramsey numbers	435
Frank Harary and Allen John Carl Schwenk, The spectral approach to	
determining the number of walks in a graph	443
David Kent Harrison, <i>Double coset and orbit spaces</i>	451
Shiro Ishikawa, Common fixed points and iteration of commuting	
nonexpansive mappings	493
Philip G. Laird, On characterizations of exponential polynomials	503
Y. C. Lee, A Witt's theorem for unimodular lattices	509
Teck Cheong Lim, On common fixed point sets of commutative	
mappings	517
R. S. Pathak, On the Meijer transform of generalized functions	523
T. S. Ravisankar and U. S. Shukla, <i>Structure of</i> Γ <i>-rings</i>	537
Olaf von Grudzinski, <i>Examples of solvable and nonsolvable convolution</i>	
equations in $\mathscr{K}'_p, p \geq 1$	561
Roy Westwick, Irreducible lengths of trivectors of rank seven and eight	575