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Triangulability of p-algebras of a Lie p-algebra L is discussed. Necessary and sufficient conditions are determined that the maximal triangulable subalgebras of the p-subalgebras of L be the normalizers of their maximal nilsubalgebras. The maximal triangulable ideals of L are located within a specific interval of a canonical ascending chain of ideal terminating in the solvable radical of L.

1. **Preliminaries.** We follow [2], [4], [6], [7], [8] for terminology and background.

Throughout the paper, k is a field of characteristic $p > 0$, K is the algebraic closure of k, L is a *Lie p-algebra* over k and L_K is the corresponding Lie p-algebra $L \otimes_k K$ over K.

The theorem of Ado-Iwasawa-Jacobson [2, p. 10] assures that L has faithful p-representations. We are concerned here with the conditions p-representations of L or of p-subalgebras of L be diagonalizable or triangulable over k or over K.

The condition that L be a *torus* is that L be abelian and L_K contain no nonzero *nilpotent element* or, equivalently, that every p-representation of L be diagonalizable over K. An element of L is *semisimple* if it is contained in some torus of L. Using the notation $\langle y \rangle$ for the p-subalgebra of L generated by y, an element x of L is semisimple if and only if $x \in \langle x^p \rangle$. (E.g., see [7, Prop. 2.5]).

PROPOSITION 1.1. *x is semisimple if and only if $x - x^p$ is semisimple.*

Proof. If $x - x^p$ is semisimple, then $x - x^p$ is contained in $\langle (x - x^p)^p \rangle$ and therefore in $\langle x^p \rangle$. Thus, $x \in \langle x^p \rangle$ and x is semisimple. The other direction is obvious.

The condition that L be a *split* torus is that L be a torus such that every (respectively some faithful) p-representation of L is diagonalizable over k. For a torus L to be split, it is necessary and sufficient that L be the k-span kL_π of $L_\pi = \{x \in L \mid x^p = x\}$. (See [2], [7, pp. 127-128].)

The condition that L be *nil* is that L consist of nilpotent elements or, equivalently, that every (respectively some faithful) p-representation of L be *nil triangulable* over k. There is a unique maximal nil ideal of L, which is a p-ideal of L called the *nilradical* Nil L of L. One proves this by observing that if A and B are nil ideals, the $A+B$ is a nil ideal. For $(A+B)/B$ and B are nil, whence $A+B$ is nil.

Some of the following material is closely related to work of Seligman [5] and Schue [3]-notably Theorem 1.2 and Corollaries 1.5, 2.4, 2.5. We therefore give only a short sketch of a proof for Theorem 1.2, which we base on Lemma 1.3 whose proof is of interest in its own right. The proofs of 1.5, 2.4, 2.5 are then quite short in the present development, and are included since they give new insight into the results.

THEOREM 1.2 (Seligman [5], Schue [3]). *Every (respectively some faithful p-representation of L is triangulable over k and only if the Lie p-algebra $L/\mathrm{Nil}\ L$ is a split torus).*

Sketch of Proof. Let ρ be a representation of L on V. If $L/\mathrm{Nil}\ L$ is a split torus, one constructs a sequence $V = V_n \supset V_{n-1} \supset \cdots \supset V_1 = 0$ of $\rho(L)$-stable subspaces such that $\mathrm{Nil}\, L V_j \subset V_{i-1}$ for $1 < i \leqq n$, then refines it, using the split torus $L/\mathrm{Nil}\, L$, to a sequence with one-dimensional quotients to show that $\rho(L)$ is triangulable. Conversely, if $\rho(L)$ is a faithful triangulable representation, one gets a $\rho(L)$-stable sequence $V_n \supset \cdots \supset V_1 = 0$ with one-dimensional quotients and finds that $\mathrm{Nil}\, L = \{x \in L \mid x V_{i+1} \subset V_i$ for $1 \leqq i \leqq n-1\}$. Since $L/\mathrm{Nil}\, L$ is then abelian with faithful triangulable representation $V_n/V_{n-1} \oplus \cdots V_2/V_1$, $L/\mathrm{Nil}\, L$ is a split torus by Lemma 1.3 below.

LEMMA 1.3. *Let L be an abelian Lie p-algebra having no nilpotent elements and suppose that L has a faithful triangulable p-representation over k. Then L is a split torus.*

Proof. We first show that every element of L is semisimple. To simplify the notation, we take a faithful triangulation over k and assume that L is a Lie p-algebra of upper triangular matrices. Suppose that some $x \in L$ is not semisimple and take such an x with minimal rank. Letting a be an nonzero eigenvalue of x, $a^{-1}x$ is triangular with eigenvalue 1, so that $a^{-1}x - (a^{-1}x)^p$ has lower rank than does x. Thus, $a^{-1}x - (a^{-1}x)^p$ is semisimple. But then Proposition 1.1 implies that $a^{-1}x$ is semisimple, so thas x is semisimple a contradiction. Thus, every element of L is semisimple, and L is a torus. Thus, the faithful triangulable representation over k is actually diagonalizable over k, so that L is a split torus.

THEOREM 1.4. *Every (respectively some faithful) p-representation of L is triangulable over K if and only if $L/\mathrm{Nil}\, L$ is abelian.*

Proof. This is proved just as was Theorem 1.2 except that, over K, we invoke elementary linear algebra in noting that every representation of an abelian Lie algebra such as $L/\mathrm{Nil}\, L$ is triangulable over K.

COROLLARY 1.5. *Let k be perfect. Then every (respectively some faithful) p-representation of L is triangulable over K if and only if $L/\mathrm{Nil}\, L$ is a torus.*

Proof. If $L/\mathrm{Nil}\, L$ is abelian, it has no nilpotent elements and, since k is perfect, it is therefore a torus by [7, Prop. 2.5]. Thus, $L/\mathrm{Nil}\, L$ is abelian if and only if it is a torus.

THEOREM 1.6. *Every (respectively some faithful) p-representation of L is triangulable over some separable algebraic extension of k if and only if $L/\mathrm{Nil}\, L$ is a torus.*

Proof. One direction follows from Theorem 1.2 and the fact that every torus splits over some separable algebraic extension. (See [8, p. 127].) Suppose, conversely, that L has a faithful triangulable p-representation over a separable algebraic extension k' of k. One shows easily from the finite dimensionality of L that k' can be taken to be a finite dimensional Galois extension, with no loss of generality. Let $L' = L_{k'}$. Then $L'/\mathrm{Nil}\, L'$ is a split torus over k'. Since $\mathrm{Nil}\, L'$ is stable under the Galois group G of k'/k, $k'\,\mathrm{Nil}\, L \subset (\mathrm{Nil}\, L') = k'(\mathrm{Nil}\, L')^G \subset k'\,\mathrm{Nil}\, L$. (See [6, §1.3]). Thus, $\mathrm{Nil}\, L' = k'\,\mathrm{Nil}\, L$. It follows that $(L/\mathrm{Nil}\, L)_{k'}$ and $L'/\mathrm{Nil}\, L'$ are p-isomorphic, hence that $L/\mathrm{Nil}\, L$ is a torus.

2. Triangulable p-subalgebras of L. The observation in §1 motivate the following definition.

DEFINITION 2.1. L is *triangulable* if $L/\mathrm{Nil}\, L$ is abelian. And L is *separably triangulable* if $L/\mathrm{Nil}\, L$ is a torus.

We now restate Corollary 1.5 as follows.

THEOREM 2.2. *If k is perfect, L is triangulable if and only if L is separably triangulable.*

THEOREM 2.3. *L is separably triangurable if and only if $L = T \oplus \operatorname{Nil} L$ (direct sum of subspaces) for each maximal torus T of L.*

Proof. One direction is clear. Suppose conversely that $L/\operatorname{Nil} L$ is a torus and let T be a maximal torus of L. Then $(T + \operatorname{Nil} L)/\operatorname{Nil} L$ is a maximal torus of $L/\operatorname{Nil} L$ by [7, Theorem 2.16], so that $(T + \operatorname{Nil} L)/\operatorname{Nil} L = L/\operatorname{Nil} L$ and $T + \operatorname{Nil} L = L$.

COROLLARY 2.4 (*Seligman* [5], *Schue* [3]). *Let k be perfect. Then $L/\operatorname{Nil} L$ is abelian if and only if $L = T \oplus \operatorname{Nil} L$ for each maximal torus T of L.*

COROLLARY 2.5 (*Seligman* [5], *Schue* [3]). *Let k be perfect. Then each $x \in L$ can be written uniquely as $x = x_s + x_n$ where x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. Furthermore, x_s and x_n are contained in the p-subalgebra $\langle x \rangle$ generated by x.*

Proof. Let $A = \langle x \rangle$. Since A is abelian, $A = T \oplus \operatorname{Nil} A$ by Theorems 2.2 and 2.3, and $x = x_s + x_n$ with $x_s \in T$, $x_n \in \operatorname{Nil} A$. The unicity of the x_s, x_n is proved as in the case of the classical Jordan decomposition of a linear transformation.

The decomposition $x = x_s + x_n$ is called the *Jordan decomposition* of x.

3. Maximal triangulable subalgebras of L.

Throughout the remainder of the paper, we assume that k is algebraically closed.

THEOREM 3.1 (*Chwe* [1]). *Suppose that L consists of semisimple elements. Then L is a torus.*

Proof. Suppose that x is a noncentral element of L. Then there is a nonzero element $y \in L$ with $[y, x] = \lambda y$ for some nonzero scalar λ. But then $(\operatorname{ad} y)^2 x = 0$, so that $(\operatorname{ad} y)x = 0$ by the semisimplicity of $\operatorname{ad} y$, a contradiction. Thus, every element of L is central, so that L is a torus*.

The reader may now easily prove Theorem 3.1 for any k if L is solvable or has a split maximal torus. Some of the following material can be generalized accordingly.

THEOREM 3.2. *Let U be a maximal nil p-subalgebra of L. Then the normalizer $N(U) = \{x \in L \mid [x, u] \subset U\}$ of U is a maximal triangulable p-subalgebra of L.*

* The author wishes to thank the referee for suggesting this nice simple proof.

Proof. One easily verifies that $N(U)$ is a Lie p-subalgebra of L and U is a p-ideal of $N(U)$. Thus, the quotient $N(U)/U$ is a Lie p-algebra. Take $x \in N(U)$ and write $x = x_s + x_n$. Then the p-subalgebra $\langle x_n, U \rangle$ generated by x_n and U is nil since x_n normalizes U. But the maximality of U, it follows that $x_n \in U$. But then the element $x + U = x_s + U$ of $N(U)/U$ is semisimple. Thus, $N(U)/U$ consists of semisimple elements. It follows that $U = \mathrm{Nil}\, N(U)$. Furthermore, $N(U)/\mathrm{Nil}\, N(U) = N(U)/U$ is a torus, by Theorem 3.1. Consequently, $N(U)$ is triangulable. Suppose that B is a triangulable p-subalgebra of L containing $N(U)$. Then $B = T \oplus \mathrm{Nil}\, B$ with $\mathrm{Nil}\, B \supset U$. By the maximality of U, $\mathrm{Nil}\, B = U$. But then $B \subset N(U)$. Since $N(U)$ is triangulable and B is maximal triangulable, it follows that $B = N(U)$. Thus, $N(U)$ is maximal triangulable p-subalgebra of L.

The "converse" of Theorem 3.2 is that every maximal triangulable p-subalgebra B of L can be expressed as the normalizer $B = N(U)$ of some maximal nil p-subalgebra U of L. This "converse" does not hold for any of the *bad* Lie p-algebras which we now define.

DEFINITION 3.3. A Lie p-algebra L is *bad* if it has the form $L = kt \oplus T_0 \oplus U \oplus kx$ where T_0 is a central torus, U is a central nil p-subalgebra, $t^p = t$, $[t, x] = x$ and x^{p^e} is a nontrivial element of the center of L for all $e \geqq 1$.

Note for any such bad L that $T = kt \oplus T_0$ is a maximal torus, $T_0 \oplus U$ is the center of L and the Cartan subalgebra $kt \oplus T_0 \oplus U = T \oplus U = L_0$ is a maximal triangulable p-subalgebra of $L = L_0 \oplus kx = L_0 \oplus L_1$. Moreover, $V = U \oplus k(x - x_s)$ is a maximal nil p-subalgebra of L and the corresponding maximal triangulable subalgebra is $B = N(V) = T_0 \oplus V$. Since L_0/U is a torus, U is the nil radical of L_0. Since U is not a maximal nil subalgebra of L, L_0 cannot be the normalizer of a maximal nil subalgebra of L. Thus the "converse" of Theorem 3.2 is false for all bad Lie p-algebras.

As an explicit example of a bad Lie p-algebra L, let $L = kt \oplus T_0 \oplus U \oplus kx$ where $T_0 = ks$, U has basis $\{x_i - s \mid 1 \leqq i \leqq e - 1\}$, s and the x_i are central, $[t, x] = x$, $t^p = t$, $s^p = s$, $x^{p^i} = x_i$ for $1 \leqq i \leqq e - 1$ and $x^{p^e} = s$. This example is p-represented by $p^e \times p^e$ matrices where s is the identity matrix, t is the diagonal matrix with diagonal entires $1, 2, \cdots, p^e (\mathrm{mod}\, p)$, x is the cyclic permutation matrix

$$x = \begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ \vdots & & \ddots \quad 1 \\ 1 & 0 \cdots\cdots & 0 \end{pmatrix}$$

and $x_i = x^{p^i}$ for $1 \leqq i \leqq e - 1$.

DEFINITION 3.4. A Lie p-algebra L is *regular* if for every p-subalgebra M of L, every maximal triangulable subalgebra of M is the normalizer in M of a maximal nil subalgebra of M.

THEOREM 3.5. *A Lie p-algebra L is regular if and only if L contains no bad p-subalgebras.*

Proof. One direction follows from Theorem 3.2 and the discussion following Definition 3.3. For the other, it suffices to prove the "converse" of Theorem 3.2 for every Lie p-algebra L which contains no bad p-subalgebras. Thus, let B be a maximal triangulable p-subalgebra of such a Lie p-algebra L and write B as $B = T + U$ where T is a torus and $U = \mathrm{Nil}\, B$. Then form $N = N(U)$, noting that $N \supset B$. We claim that U is a maximal nil p-subalgebra of L and that $B = N = N(U)$, thereby establishing the "converse" of Theorem 3.2 for L. Note first that T is a maximal torus of N. For if S is a torus of N containing T, then $S \oplus U$ is a triangulable p-subalgebra and contains the maximal triangulable p-subalgebra $T + U$, so that $S \oplus U = T \oplus U$ and $S = T$. Thus, the centralizer $C(T)$ of T in N is a Cartan subalgebra of N (see [7, Theorem 2.14]) and we have the root space decomposition $N = C(T) \oplus \sum_{\alpha \neq 0} N_\alpha$ of N with respect to T. For $x \in C(T)$, we have $x = x_s + x_n$ with $x_s \in T$. Since x_n centralizes T and is a nilpotent element of N normalizing U, the p-subalgebra $\langle x_n, U \rangle$ generated by x_n and U is nil and $T + \langle x_n, U \rangle$ is consequently a triangulable p-subalgebra containing the maximal triangulable p-subalgebra B. Thus, $T + \langle x_n, U \rangle = B = T + U$. This shows that $x_n \in U$, and therefore that $x_n \in C(T) \cap U$. It follows that $C(T) = T \oplus V$ where V is the nil p-subalgebra $V = C(T) \cap U$. Now take any element $x \in N_\alpha$ with $\alpha \neq 0$. Since $(\mathrm{ad}\, x^p) N_\beta \subset N_{\beta + p\alpha}$, we have $(\mathrm{ad}\, x^p) N_\beta \subset N_\beta$ for all β, where $0 = [\mathrm{ad}\, t, \mathrm{ad}\, x^p] = \mathrm{ad}\, [t, x^p]$ so that $0 = (\mathrm{ad}\, t)^2 x^p$ and therefore $0 = (\mathrm{ad}\, t) x^p$ for all $t \in T$. Thus, $x^p \in C(T) = T \oplus V$. It follows that x^p is in $(T \oplus V) \cap \{y \in L \mid [y, x] = 0\} = T_0 \oplus W$ where $T_0 = \{y \in T \mid [y, x] = 0\}$ and W is the nil p-subalgebra $W = \{y \in T \mid [y, x] = 0\}$. Note in this connection that for any element y of $T \oplus V$, y equals $y = y_s + y_n$ with $y_s \in T$, $y_n \in V$ and $[y, x] = 0$ if and only if $[y_s, x] = 0$ and $[y_n, x] = 0$. What we have shown is that $x^p \in T_0 \oplus W$. Choose t in $T_\pi = \{s \in T \mid s^p = s\}$ such that $[t, x] = x$. This is possible since $\{0\} \neq \alpha(T) = \alpha(kT_\pi)$ and since $\alpha(s) \in \{0, 1, \cdots, p - 1\}$ for $s \in T_\pi$. For one can choose $s \in T_\pi$ such that $\alpha(s) \neq 0$ and let $t = \alpha(s)^{-1} s$, so that $\alpha(t) = 1$ and $[t, x] = x$. We then have $T = T_0 \oplus kt$ where T_0 is defined as the centralizer of x in T. We have shown that $x^p \in T_0 \oplus W$, so it follows that $x^{p^e} \in T_0 \oplus W_0$ for all $e \geqq 1$, $W_0 =$ Center W. Since the p-subalgebra $kt \oplus T_0 \oplus W_0 \oplus kx$ of L cannot be a bad p-subalgebra of L, by our hypothesis, we must have $x^{p^e} = 0$ for some e. But then x is nilpotent and normalizes U, so that $T \oplus \langle x, U \rangle$ is triangulable and $x \in U$ as before. It follows that U contains every x in every N_α with $\alpha \neq 0$, so that N is just $N = T \oplus U$. That is, $B = N$. It remains to show that U is a maximal nil p-subalgebra of L. Suppose that it were not and let $U \subsetneqq M$ where M is a maximal nil p-subalgebra of L. By Engel's theorem, ad U represented on the quotient space M/U has a nonzero eigenvector $m + U$ with eigenvalue 0, so that $m \notin U$ and $(\operatorname{ad} U)m + U = 0 + U$ or $[U, m] \subset U$. That is, we have $m \in N(U)$. But m is nilpotent and we have seen that $N(U) = T \oplus U$. Thus, $m \in U$, which contradicts our choice of m such that $m \notin U$. Thus, U is a maximal nil p-subalgebra of L and the maximal triangulable p-subalgebra B has the asserted form $B = N(U)$.

4. Maximal triangulable p-ideals of L. The *radical* Rad L of L is the unique maximal solvable ideal of L, and it is easily seen to be a p-ideal of L containing the nil radical Nil L. It is convenient to also define the *toral radical* of L to be the maximal toral ideal Tor L of L.

PROPOSITION 4.1. *Suppose that a torus T of L is contained in the center of an ideal I of L. Then T is contained in the center of L. In particular,* Tor L *is the maximal torus of the center of L.*

Proof. Since $[T, L] \subset I$, we have $[T, [T, L]] = \{0\}$. But ad T is diagonalizable, so that $[T, L] = \{0\}$.

We say that L is *semisimple* if Rad $L = \{0\}$.

PROPOSITION 4.2. *L is semisimple if and only if* Nil $L = \{0\}$ *and* Tor $L = \{0\}$.

Proof. One direction is trivial. For the other, suppose that $R =$ Rad L is not $\{0\}$ and choose n such that $R^{(n)} = \{0\}$ and $R^{(n-1)} \neq \{0\}$. Then the p-ideal $A = \langle R^{(n-1)} \rangle$ is abelian. If the maximal torus T of A is not $\{0\}$, then Tor $L \neq \{0\}$ by Proposition 4.1. Otherwise A is a nil p-ideal of L and Nil L is not $\{0\}$.

We let $L_1 = \operatorname{Nil} L$, $L_2/L_1 = \operatorname{Tor}(L/L_1)$, $L_3/L_2 = \operatorname{Nil}(L/L_2)$, etc. We then have a sequence $L_1 \subset L_2 \subset L_3 \subset \cdots \subset L_n = L_{n+1} = \cdots$ of p-

ideals of L contained in Rad L. Since Nil $L/L_n=\{0\}$ and Tor $L/L_n=\{0\}$, L/L_n is semisimple by Proposition 4.2, so that the series stabilizes at $L_n = \text{Rad}\, L$. This series may be called the *ascending nil-toral series* for L. Its counterpart in characteristic 0 always stabilizes $\text{Rad}\, L = L_2 = L_3 = \cdots$.

THEOREM 4.3. *Let I be a maximal triangulable p-ideal of L. Then $L_2 \subset I \subseteqq L_3$.*

Proof. Write $I = T \oplus U$ where T is a torus and $U = \text{Nil}\, I$. Since $U \oplus \text{Nil}\, L$ is a nil p-ideal, $T + U + \text{Nil}\, L$ is a triangulable p-ideal containing I, so that $I = T + U + \text{Nil}\, L$ and $\text{Nil}\, L \subset U$. Since $\text{Tor}\, L/L_1 = L_2[\text{Nil}\, L$ is a central torus in $L/\text{Nil}\, L$ is a central torus in L/Nil, we have $L_2 = T_2 \oplus \text{Nil}$ where T_2 is a torus and $[T_2, L] \subset \text{Nil}\, L \subset U$. Thus, U is a p-ideal of $I + L_2 = I + T_2$. Since I/U is a toral ideal of $(I + T_2)/U$, $(I + T_2)/U$ is the sum of two commuting tori I/U and $(U + T_2)/U$, by Proposition 4.1. Thus $(I + T_2)/U$ is a torus, so that $I + L_2 = I + T_2$ is a triangulable p-ideal containing the maximal triangulable p-ideal I. Thus, $L_2 \subset I$. We claim that $I \subset L_3$. Since I/U is a torus, $[I, I]$ is a nil ideal of L and $[I, I] \subset \text{Nil}\, L = L_1$. That is, $(I + L_1)/L_1$ is an abelian ideal of L/L_1, so that the maximal torus $(T + L_1)/L_1$ of $(I + L_1)/L_1$ is central in L/L_1, by Proposition 4.1, and is therefore contained in $\text{Tor}\, L/L_1 = L_2/L_1$. Thus, $T \subset L_2$ and it follows that $I/L_2 = (U + L_2)/L_2$ is contained in $\text{Nil}\, L/L_2 = L_3/L_2$. But then we have $I \subset L_3$, as asserted.

As an example, let k have characteristic $p = 2$ and let L be the Lie p-algebra $L = ke_- + kh + ke_+$ where $[e_-, e_+] = h$, h is central, $(e_-)^p = (e_+)^p = 0$ and $h^p = h$. *Then L is nilpotint, and $I = ke_- + kh$ and $J = kh + ke_+$ are two distinct maximal triangulable p-ideals of L such that $L_2 \subsetneqq I \subsetneqq L_3$, $L_2 \subsetneqq J \subsetneqq L_3$.* This Lie p-algebra has the p-representation

$$e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ [e_-, e_+] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = h$$

since $p = 2$. The reader should note, in this example, that $L_1 = 0$, $L_2 = kh$, $L_3 = L$.

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