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CONGRUENCE LATTICES OF ALGEBRAS OF FIXED SIMILARITY TYPE. I

RALPH S. FREESE, WILLIAM A. LAMPE AND WALTER FULLER TAYLOR

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We prove that if V is any infinite-dimensional vector space over any uncountable field F , then the congruence lattice (=subspace lattice) of V cannot be represented as a congruence lattice (of any algebra) without using at least $|F|$ operations. This refutes a long-standing conjecture—that one binary operation would always suffice.

Our result implies that the natural representation of $\text{Sub } V$ as a congruence lattice is the best one, i.e., uses the minimum number of operations. This result is easy to obtain—see § 1.

In the remainder of the paper we find further necessary conditions for the representability of an algebraic lattice L with $< \kappa$ operations. Necessary and sufficient conditions seem impossible at this stage of knowledge. Part II of this paper (by W. A. Lampe) gives some interesting sufficient conditions for representability of L with one binary operation, for instance: the unit element of L is compact.

The conjecture which we have refuted dates back at least to 1959 and the theorem of Grätzer and Schmidt ([6], [3], [14]) that every algebraic lattice can be represented as the congruence lattice $\text{Con } A$ of some unary algebra A ; the hope was to use say a single binary operation $+$ to code the unary operations $f(x)$ as $x + a$ (with $a \in A$ depending on f). (See e.g., [9], [8, p. 209].)

By some results of § 3 and Part II, every algebraic lattice can be embedded as a principal ideal in an algebraic lattice which is representable with one binary operation, and also in one requiring κ operations. This may partly explain why the conjecture resisted settlement for so long.

§ 1 contains the main result. §§ 2 and 3 contain refinements and variations on the ideas of § 1, and § 4 contains some open problems.

Some of these results were announced in [12], [13], and [18].

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1. The main result. All our results depend on the following lattice-theoretic property:

- (*) For every compact Ψ , there exist compact Θ, Φ such that $\Theta \vee \Phi \geq \Psi$ and $\Theta \wedge \Psi = \Phi \wedge \Psi = 0$.

It is easy to see that our subspace lattice obeys (*) (see the proof of Theorem 1); in fact all examples in this article will obey (*).

LEMMA 1. *If Con A satisfies (*), then A satisfies*

$$\exists x(t(x, p) = t(x, q)) \longrightarrow \forall x(t(x, p) = t(x, q))$$

(for x denoting (x_1, \dots, x_n) , $p, q \in A^m$ and t any $(n + m)$ -ary term, i.e., polynomial).

REMARKS. (*) is somewhat elusive—every algebraic lattice can be embedded as a principal ideal in an algebraic lattice obeying (*), and also in one where (*) fails. And so perhaps it is not surprising that Lemma 1 was overlooked for so long. It arose from a study of the lattices $L_{\lambda, \phi}$ of § 2.

Proof. We will assume that $t(a, p) = t(a, q)$ for some $a \in A^n$, and prove that $t(b, p) = t(b, q)$ for any $b \in A^n$. To see this, we will apply (*) to the compact congruence $\Psi = \Theta(a, b) \vee \Theta(p, q)$ (i.e., the congruence generated by $\{(a_1, b_1), \dots, (a_n, b_n), (p_1, q_1), \dots, (p_m, q_m)\}$). Taking Φ and θ as supplied by (*), we have

$$\Theta(a, b) \subseteq \Psi \subseteq \Theta \vee \Phi,$$

and so

$$a = s_0 \Theta s_1 \Phi s_2 \Theta s_3 \Phi \dots s_k = b.$$

Now since $s_0 \Theta s_1$, we have

$$t(s_1, p) \Theta t(s_0, p) = t(s_0, q) \Theta t(s_1, q),$$

and thus $t(s_1, p) \Theta t(s_1, q)$. Since $p \Psi q$, we also have $t(s_1, p) \Psi t(s_1, q)$. Finally, since $\Theta \cap \Psi = 0$, we have $t(s_1, p) = t(s_1, q)$. A similar argument (with Φ in place of Θ) yields $t(s_2, p) = t(s_2, q)$. Proceeding by induction, we obtain $t(b, p) = t(s_k, p) = t(s_k, q) = t(b, q)$.

DEFINITION. $\#A$ denotes the number of nonnullary operations of A .

LEMMA 2. *If Con A obeys (*), then every block of every compact congruence of A has power $\leq \#A + \aleph_0$.*

Proof. Let S be a block of a compact congruence Θ ; thus $\Theta = \Theta(a, b)$ for some $a, b \in A^n$. Fixing $c \in S$ and considering arbitrary $d \in S$, we have $(c, d) \in \Theta(a, b)$, and hence, for some terms t_1, \dots, t_n and $p_1, \dots, p_n \in A^m$, we have

$$\begin{aligned} c &= t_1(a, b, p_1) \\ t_1(b, a, p_1) &= t_2(a, b, p_2) \\ &\vdots \\ t_{n-1}(b, a, p_{n-1}) &= t_n(a, b, p_n) = d \end{aligned}$$

(as follows from well known and easy descriptions of compact congruences—see e.g., [3, Theorem 10.4]).

To see our upper bound on $|S|$, it is clearly enough (by elementary cardinal arithmetic) to see that d depends only on the sequence (t_1, \dots, t_n) , and *not* on (p_1, \dots, p_n) . To see this we suppose we also have

$$\begin{aligned} c &= t_1(a, b, p'_1) \\ t_1(b, a, p'_1) &= t_2(a, b, p'_2) \\ &\vdots \\ t_{n-1}(b, a, p'_{n-1}) &= t_n(a, b, p'_n) = d' . \end{aligned}$$

The proof will be complete when we have shown $d = d'$. From $t_1(a, b, p_1) = c = t_1(a, b, p'_1)$ we obtain

$$t_2(a, b, p_2) = t_1(b, a, p_1) = t_1(b, a, p'_1) = t_2(a, b, p'_2) ,$$

(using Lemma 1 for the middle equality). Thus we have obtained $t_2(a, b, p_2) = t_2(a, b, p'_2)$; proceeding by induction yields $d = d'$.

The above lemma tells us that blocks cannot be too big. The next lemma says that sometimes there must be a large block.

LEMMA 3. *Suppose that $\Psi \in \text{Con } A$, $X \subseteq \text{Con } A$ is infinite, and for distinct $\Theta, \Phi \in X$, $\Theta \wedge \Phi = 0$ and $\Theta \vee \Phi = \Psi$. Then each nontrivial block of Ψ has power $\geq |X|$.*

Proof. Let Π be any partition of X into two-element sets, and take $\langle c, e \rangle \in \Psi$ with $c \neq e$. For each $B \in \Pi$, $\vee B = \Psi$, and so we have

$$c\theta_1 d\theta_2 \dots \theta_k e \quad (\text{each } \theta_i \in B) .$$

Of course we may assume that $d \neq c$. It will be enough to see that distinct B 's yield distinct d 's (for then the set of all d 's has power $\geq |\Pi| = |X|$, and certainly $c\Psi d$ holds for each d). But this is easy, since if we also had $c\theta'_1 d$ with $\theta'_1 \in B' \neq B$, then we would have $c = d$, since $\theta_1 \cap \theta'_1 = 0$.

THEOREM 1. *If V is an infinite-dimensional vector space over an uncountable field F , then $\text{Con } A \cong \text{Con } V$ implies that A has at least $|F|$ operations.*

Proof. We will see that $\text{Con } V$ (hence $\text{Con } A$) satisfies the hypotheses of Lemma 3, with \mathcal{V} actually compact. Fix one 2-dimensional subspace W and $|F|$ one-dimensional subspaces $U_\alpha \subseteq W$ ($\alpha < |F|$). With the usual identification of subspaces with congruences, we take $\mathcal{V} = W$ and $X = \{U_\alpha: \alpha < |F|\}$. Clearly $U_\alpha \vee U_\beta = W$ for $\alpha \neq \beta$, and so Lemma 3 yields a compact congruence with a block of power $|F|$.

To see that $\text{Con } V (= \text{Sub } V)$ obeys (*), let U be any finite-dimensional subspace of V with basis $\{v_1, \dots, v_n\}$. Extend this to a linearly independent subset $\{v_1, \dots, v_{2n}\}$ of V . Now take U' to be spanned by $\{v_1 + v_{n+1}, \dots, v_n + v_{2n}\}$ and U'' to be spanned by $\{v_{n+1}, \dots, v_{2n}\}$. Then certainly $U \subseteq U' + U''$ and $U \cap U' = U \cap U'' = \{0\}$.

And so Lemma 2 tells us that $|F| \leq \#A + \aleph_0 = \#A$.

2. Refinements. In this section we present further means of finding large congruence blocks and thus seeing (*via* Lemma 2) that certain algebraic lattices obeying (*) require many operations in their congruence representations. Lemma 4 below may be proved in a manner similar to Lemma 3. Lemma 5 is far more general than Lemma 4, but also more complicated. Lemma 6 below produces large congruence blocks in an entirely different manner.

LEMMA 4. *Suppose that $\mathcal{V} \in \text{Con } A$, $X \subseteq \text{Con } A$ and a partition Π of X obey these conditions:*

(1) *for each $B \in \Pi$, $\mathcal{V} \leq \vee B$;*

(2) *for distinct $B, B' \in \Pi$, $\Theta \in B$ and*

$\Phi \in B'$, $\Theta \wedge \Phi = 0$.

Then every nontrivial block of $\vee X$ has power $\geq |\Pi|$.

In what follows, κ is an infinite cardinal. We say that a poset P is κ -directed if and only if each $S \subseteq P$ with $|S| < \kappa$ has an upper bound in P .

THEOREM 2. *If $\text{Con } A$ satisfies (*) and the compact elements of $\text{Con } A$ are κ^+ -directed, then $\kappa \leq \#A + \aleph_0$.*

Proof. By Lemma 2, it will be enough to produce a compact congruence with one block of power $\geq \kappa$. For this, we use (*) and κ^+ -directedness. For $\alpha \leq \kappa$ we recursively define compact congruences $\mathcal{V}_\alpha, \Theta_\alpha, \Phi_\alpha$ as follows:

$$\begin{aligned}\Psi_\alpha &\geq \Psi_\beta, \Theta_\beta, \Phi_\beta & (\alpha > \beta) \\ \Theta_\alpha \vee \Phi_\alpha &\geq \Psi_\alpha \\ \Theta_\alpha \wedge \Psi_\alpha &= \Phi_\alpha \wedge \Psi_\alpha = 0\end{aligned}$$

(the first condition is possible by κ^+ -directedness, the last two by (*)). It is clear that if we take $X = \{\theta_\alpha, \Phi_\alpha : \alpha < \kappa\}$ and $\Pi = \{\{\theta_\alpha, \Phi_\alpha\} : \alpha < \kappa\}$, then the hypotheses of Lemma 4 are fulfilled, and moreover $\vee X \subseteq \Psi_\kappa$. Thus Ψ_κ is a compact congruence with a block of power $\geq \kappa$.

To illustrate Theorem 2, we suppose V is a vector space of dimension κ^+ over some field. We let S be the lattice of all subspaces of V of dimension $\leq \kappa$, and let L be the lattice of all ideals of S . Theorem 2 says that $\#A \geq \kappa$ whenever $L \cong \text{Con } A$.

LEMMA 5. *Suppose that $\Psi, Z \in \text{Con } A$ and a family P of finite subsets of $\text{Con } A$ obey these conditions:*

- (1) $|\cup P|$ is regular and infinite;
- (2) Ψ is compact and $\Psi \not\leq Z$;
- (3) $\forall B \in P, \Psi \leq (\vee B) \vee Z$;
- (4) each $B \in P$ is minimal for condition (3);
- (5) $\forall T \subseteq \cup P$, if $|T| = |\cup P|$, then $\bigwedge T \leq Z$.

Then $\vee(\cup P) \vee Z$ has a block of power $\geq |\cup P|$.

REMARKS. Of course the finiteness of the sets $B \in P$ follows from (2) and (4). This lemma is more versatile than Lemma 4 since, firstly, P need not be a partition, secondly, the disjointness condition (5) has been weakened considerably, and thirdly, 0 has been replaced by an arbitrary $Z \not\leq \Psi$. Obviously the new assumption (1) makes no difference for our applications in conjunction with Lemma 2, because each singular cardinal is the supremum of all smaller regular cardinals.

Proof. By (2) we have $a, b \in A^n$ with

$$\Psi = \Theta(a, b) = \Theta(a_1, b_1) \vee \cdots \vee \Theta(a_n, b_n).$$

Since each $(a_i, b_i) \in \Psi$, (3) says that for each $B \in P$

$$a_i = x_0 \Theta_1 x_1 \Theta_2 x_2 \cdots x_k = b_i$$

for some $x_1, x_2, \dots \in A$ and some $\Theta_1, \Theta_2, \dots \in B \cup \{Z\}$. For each i we form $Q_i \subseteq A$ as follows: take one such sequence for each $B \in P$, and then let Q_i be the set of all x_1, x_2, \dots occurring in all these sequences. Obviously each Q_i is contained in a single $\vee(\cup P) \vee Z$ block, and so it will be enough to see that some $|Q_i| \geq |\cup P|$.

Clearly it is enough to show that $Q = Q_1 \cup \cdots \cup Q_n$ has power $\geq |X|$,

where $X = \bigcup P$. To see this we define

$$\tau: (Q \times Q - Z) \longrightarrow \text{power set of } X$$

via

$$(p, q) \longmapsto \{\theta \in X: p\theta q\}.$$

We will clearly be done if we can show that $|\text{Range } \tau| \geq |X|$. Condition (5) implies that each $|\tau(p, q)| < |X|$, and so by the regularity of $|X|$ it will be enough to see that $|\bigcup \text{Range } \tau| = |X|$. We will in fact show that $\bigcup \text{Range } \tau = X$.

For each $\theta \in X = \bigcup P$, we have $B \in P$ and

$$a_i = x_0 \theta_1 x_1 \theta_2 x_2 \cdots x_k = b_i \quad (1 \leq i \leq n)$$

(as above) with $\theta \in B$ and each $\theta_j \in B \cup \{Z\}$. Now among these relations must occur $x_j \theta x_{j+1}$ with $(x_j, x_{j+1}) \notin Z$, for otherwise we could eliminate θ in favor of Z , contradicting the minimality of B in (4). But clearly $\theta \in \tau(x_j, x_{j+1})$, and hence $X \subseteq \text{Range } \tau$.

LEMMA 6. *Suppose that C is the set of compact elements of $\text{Con } A$, $X \subseteq \text{Con } A$ and X is infinite. If $\bigvee X = 1$ and $|X| < |C|$, then some congruence in X has a block of power $\geq \kappa$, for each regular κ satisfying $|X| < \kappa \leq |C|$. (In particular, for $\kappa = |X|^+$.)*

Proof. Fixing such a regular κ , we let Y be the set of finite joins of congruences from X . Thus Y is directed and $|Y| < \kappa$. Fixing any $a \in A$, we have, since A is the unique block of 1,

$$A = [a]_1 = \bigcup_{\theta \in Y} [a]_\theta.$$

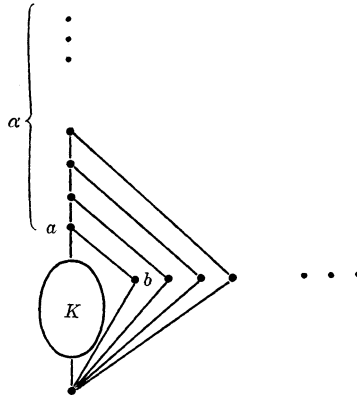
One easily checks that $|C| \leq |A|$, and so the regularity of κ yields $|[a]_\theta| \geq \kappa$ for some $\theta \in Y$.

By definition of Y , we have $\theta = \theta_1 \vee \cdots \vee \theta_n$ with each $\theta_j \in X$. Let

$$Q_0 = \{a\} \quad \text{and} \quad Q_{i+1} = \{x \notin Q_i: (\exists y \in Q_i)(\exists j)x\theta_j y\}.$$

Since $[a]_\theta = \bigcup_{i \in \omega} Q_i$ and κ is regular, there exists i with $|Q_{i+1}| \geq \kappa$; taking the least such i , we also have $|Q_i| < \kappa$. For $b \in Q_i$ and each j , let $R_{bj} = \{x \in Q_{i+1}: b\theta_j x\}$. By the regularity of κ , we may choose b and j with $|R_{bj}| \geq \kappa$, and clearly $|[b]_{\theta_j}| \geq \kappa$.

To illustrate how Lemmata 5 and 6 may be applied, we will present several algebraic lattices, all of the same general form, namely: $L_{\alpha, K}$ is the ideal lattice of



for a limit ordinal α and a join-semilattice K which we will specify further below. Obviously the lattices $L_{\alpha, K}$ satisfy (*), and so we may apply Lemma 2 to them.

We first note that if $\alpha = \beta + \gamma$ with $\gamma < |\beta| + |K|$, then Lemmate 2 and 6 immediately yield $\#A \geq |\beta| + |K|$ for $\text{Con } A \cong L_{\alpha, K}$. An example where neither Lemma 4 nor Lemma 5 is useful is obtained by taking $\alpha = \omega$ and $K = \omega_1$.

By way of historical comment, we should remark that Lampe's pioneering example [12] was precisely $L_{\alpha, K}$ with $\alpha = \omega_1 + \omega$, $K = \emptyset$. (He actually used Lemmate 2 and 4, which apply equally well to this example.)

The lattices $L_{\alpha, \emptyset}$ arose quite naturally during Lampe's attempts to represent each algebraic lattice as the congruence lattice of a groupoid. It is possible to view $L_{\omega, \emptyset}$ as a partial lattice in such a way that a lattice L obeys (*) if and only if for every compact a there is a zero-preserving embedding σ of $L_{\omega, \emptyset}$ into L with a in the range of σ . Each of Lampe's attempts foundered on a lattice which had some $L_{\alpha, \emptyset}$ as a *partial* sublattice.

Next we give three examples using various features of Lemma 5, where neither Lemma 4 nor Lemma 6 can be applied. These examples have $\alpha = \omega_1$ and $|K| = \omega_1$. In each case we will see that $\#A \geq \omega_1$ for $L_{\alpha, K} \cong \text{Con } A$ (and of course $\#A = \omega_1$ is possible, by Grätzer-Schmidt).

For our first example we take $K = \omega_1^*$ (ω_1 upside down), and we apply Lemma 5 with $Z = 0$, $\Psi = a$ and $P = \{\{b, \alpha\}: \alpha \in K\}$. In this example P is not a partition, and only a weak disjointness assumption (5) holds.

For our next example we take $K = K_0 \cup \{p, c\}$, with K_0 = the lattice of all finite or co-countable subsets of ω_1 , and p and c two new points. The natural order of K_0 is extended to K by letting $x \leq c$ for all $x \in K$, and letting $x \leq p$ if and only if x is a finite subset of ω_1 . Let $\Psi = c$, $P = \{\{p, x\}: x \text{ a coatom of } K_0\}$ and Z = the join in

$L_{\omega_1, K}$ of all the atoms in K_0 . Then one may easily check that Ψ, P, Z satisfy the hypotheses of Lemma 5 (whenever $L_{\omega_1, K}$ is identified with $\text{Con } A$). We could not have Z any smaller and still satisfy (5), for if $Y < Z$, then $Y \not\leq$ some atom $e \in K_0$, but clearly $e = \bigwedge T$ for a suitable set T of coatoms.

For our third example we let $X = \{x_\alpha: \alpha < \omega_1\}$ and let $FL(X)$ be the free lattice generated by X , and then we take K to be the lattice of dual ideals of $FL(X)$, ordered by reverse set inclusion. Let $\Psi \in K$ be the dual ideal of $FL(X)$ generated by the set $\{x_0 \vee x_\alpha: 0 < \alpha < \omega_1\}$. We will think of $FL(X)$ as a sublattice of K (i.e., K is a special kind of completion of $FL(X)$). Straightforward calculation shows that the hypotheses of Lemma 5 hold for $\Psi, P = \{x_0, x_\alpha: 0 < \alpha < \omega_1\}$ and $Z =$ the ideal of K generated by $\{\bigwedge T: T \subseteq X, |T| = \omega_1\}$ (whenever $L_{\omega_1, K}$ is identified with $\text{Con } A$). Note that in this example Ψ and Z are not comparable.

Of course, further examples could be obtained by embedding the ideals of K into other lattices obeying (*), e.g., a subspace lattice. Incidentally, we are unable to decide whether $L_{\omega_1, \emptyset}$ or L_{ω_1, ω_1} can be represented with \aleph_0 operations.

3. Further speculations. We first remark that obviously (*) can be weakened without harm to

(**) For every compact Ψ there exist compact $\Theta_1, \dots, \Theta_n$ such that $\Theta_1 \vee \dots \vee \Theta_n \geq \Psi$ and $\Theta_1 \wedge \Psi = \dots = \Theta_n \wedge \Psi = 0$.

Notice also that if some principal dual ideal D of L requires κ operations for a representation, then the same is true of L , since $\text{Con } A \cong L$ implies $\text{Con } A/\theta \cong D$ for some θ . (And so one easily sees that *every* algebraic lattice L is embeddable in one which requires κ operations.) Thus in searching for a characterization of " L is representable with $\leq \kappa$ operations" we need to examine all principal dual ideals of L . Define $C(L)$ to be the set of compact elements of L and then define

$\sigma(L) = \Sigma\{|\bigcup P|: (\exists \Psi, Z \in L) P, \Psi \text{ and } Z \text{ are as in Lemma 5, and there exists a compact element } \geq \bigvee (\bigcup P) \bigvee Z\}.$

$\tau(L) = \begin{cases} |C(L)| & \text{if } \exists X \subseteq C(L) \text{ with } |X| < |C(L)| \text{ and } \bigvee X = 1; \\ 0 & \text{otherwise.} \end{cases}$

$\rho(L) = \Sigma\{\sigma(D) + \tau(D): D \text{ a principal dual ideal of } L \text{ obeying } (**)\}.$

Clearly all our previous results combine to show that $\#A + \aleph_0 \geq \rho(\text{Con } A)$. This last inequality summarizes all our available information on this topic. Thus, for example, we see that $\rho(L) \leq \aleph_0$ is a necessary condition for congruence representability with \aleph_0 operations; but we would hardly conjecture that it is a sufficient condition.

4. Problems.

(1) Is the inequality $\#A + \aleph_0 \geq \rho(\text{Con } A)$ sharp, in the sense that every algebraic lattice is isomorphic to some $\text{Con } A$ with $\#A + \aleph_0 = \rho(\text{Con } A) + \aleph_0$? (See § 3 just above.)

(2) If no principal dual ideal of L obeys (**) of § 3, is L representable with $\leq \aleph_0$ operations? (See § 3.)

(3) Is every distributive algebraic lattice L representable with \aleph_0 operations? ... one operation? ... as Con (some lattice)? This last property holds for L finite [1, p. 83] [4, p. 95]. (See also [2], [5], [15]–[17] for further information.) A *groupoid* representation exists if L is the ideal lattice of a distributive lattice (see Part II of this paper).

Notice that if 1 is true, then so is 2, and if 2 is true, then so is the first part of 3. We would be surprised if 1 is true.

(4) If the set of compact elements of L is countable, then is L representable with $< \aleph_0$ operations?

(5) If below each compact element of L there are $\leq \aleph_0$ compact elements, then is L representable with \aleph_0 operations? ... $< \aleph_0$ operations? ... one operation? (W. Hanf showed [7] (see also [8, p. 106]) that the corresponding assertions hold for representations as *subalgebra lattices*.)

(6) If A has countable similarity type, must there exist B of finite similarity type with $\text{Con } B \cong \text{Con } A$?

(7) If A has finite similarity type, must there exist a groupoid B with $\text{Con } B \cong \text{Con } A$? (Lampe conjectures yes.) Note that Hanf [7] proved the corresponding assertion for subalgebra lattices, and McKenzie has proved a slightly weaker assertion for Con : there exists B of type (2,1) with $\text{Con } B \cong \text{Con } A$ (see Lampe's survey paper in the Proceedings of the 1977 Esztergom Colloquium).

(8) Characterize the class of lattices isomorphic to congruence lattices of groupoids. ... of algebras with $< \kappa$ operations.

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Marilyn Breen, <i>The dimension of the kernel of a planar set</i>	15
Kenneth Alfred Byrd, <i>Right self-injective rings whose essential right ideals are two-sided</i>	23
Patrick Cousot and Radhia Cousot, <i>Constructive versions of Tarski's fixed point theorems</i>	43
Ralph S. Freese, William A. Lampe and Walter Fuller Taylor, <i>Congruence lattices of algebras of fixed similarity type. I</i>	59
Cameron Gordon and Richard A. Litherland, <i>On a theorem of Murasugi</i>	69
Mauricio A. Gutiérrez, <i>Concordance and homotopy. I. Fundamental group</i>	75
Richard I. Hartley, <i>Metabelian representations of knot groups</i>	93
Ted Hurley, <i>Intersections of terms of polycentral series of free groups and free Lie algebras</i>	105
Roy Andrew Johnson, <i>Some relationships between measures</i>	117
Oldrich Kowalski, <i>On unitary automorphisms of solvable Lie algebras</i>	133
Kee Yuen Lam, <i>K O-equivalences and existence of nonsingular bilinear maps</i>	145
Ernest Paul Lane, <i>PM-normality and the insertion of a continuous function</i>	155
Robert A. Messer and Alden H. Wright, <i>Embedding open 3-manifolds in compact 3-manifolds</i>	163
Gerald Ira Myerson, <i>A combinatorial problem in finite fields. I</i>	179
James Nelson, Jr. and Mohan S. Putcha, <i>Word equations in a band of paths</i>	189
Baburao Govindrao Pachpatte and S. M. Singare, <i>Discrete generalized Gronwall inequalities in three independent variables</i>	197
William Lindall Paschke and Norberto Salinas, <i>C*-algebras associated with free products of groups</i>	211
Bruce Reznick, <i>Banach spaces with polynomial norms</i>	223
David Rusin, <i>What is the probability that two elements of a finite group commute?</i>	237
M. Shafii-Mousavi and Zbigniew Zielezny, <i>On hypoelliptic differential operators of constant strength</i>	249
Joseph Gail Stampfli, <i>On selfadjoint derivation ranges</i>	257
Robert Charles Thompson, <i>The case of equality in the matrix-valued triangle inequality</i>	279
Marie Angela Vitulli, <i>The obstruction of the formal moduli space in the negatively graded case</i>	281