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BANACH SPACES WITH POLYNOMIAL NORMS

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A Banach space X is said to be in the class \mathcal{P}_{2n} if, for all elements x and y , $\|x + ty\|^{2n}$ is a polynomial in real t . These spaces generalize L_{2n} and are precisely those Banach spaces in which linear identities can occur. We shall discuss further properties of \mathcal{P}_{2n} spaces, often in terms of the permissible polynomials $p(t) = \|x + ty\|^{2n}$. For each n , the set of such polynomials forms a cone. All spaces in \mathcal{P}_2 are Hilbert spaces. If X is a two-dimensional real space in \mathcal{P}_4 , then it is embeddable in L_4 . This is not necessarily true for spaces with more dimensions or for \mathcal{P}_{2n} , $n \geq 3$. The question of embeddability is equivalent to the classical moment problem. All spaces in \mathcal{P}_{2n} are uniformly convex and uniformly smooth and thus reflexive. They obey generally weaker versions of the Hölder and Clarkson inequalities. Krivine's inequalities, shown to determine embeddability into L_p , $p \neq 2n$, fail in the even case.

1. Introduction. Throughout, we shall consider real Banach spaces, and, except where indicated, $L_{2n}(Y, \mu)$ with real-valued functions and real scalars. The phrase " X is embeddable in L_{2n} " is an abbreviation for " X is isometrically isomorphic to a subspace of $L_{2n}(Y, \mu)$ for some (Y, μ) ." Although \mathcal{P}_{2n} was introduced and motivated in [11], that paper and this one are largely independent.

2. Norm functions. Suppose $X = \langle x_1, \dots, x_m \rangle$ is the real vector space spanned by the x_i 's and ϕ is a real function of m real variables. Under what circumstances does $\|\sum u_i x_i\| = \phi(u_1, \dots, u_m)$ make $(X, \|\cdot\|)$ a Banach space? For $u = (u_1, \dots, u_m)$, let $\phi(u) = \phi(u_1, \dots, u_m)$. From the standard definition of the norm, it is evident that conditions (A), (B) and (C) are necessary and sufficient. (Here, t is an arbitrary real.)

(A) $\phi(u) \geq 0$ and $\phi(v) = 0$ implies $\phi(u) \equiv \phi(u + tv)$

(B) $\phi(tu) = |t|\phi(u)$

(C) $\phi(u) + \phi(v) \geq \phi(u + v)$.

Condition (C) is cumbersome to verify; the following lemma simplifies matters.

LEMMA 1. Conditions (A), (B) and (C) are equivalent to (A), (B) and (D).

(D) $\psi(t) = \phi(u + tv)$ is a convex function in t for all u and v .

Proof. Assume (A), (B) and (C) and fix u and v . Then for $0 \leq$

$\lambda \leq 1$, $\lambda\psi(t_0) + (1-\lambda)\psi(t_1) = \lambda\phi(\mathbf{u} + t_0\mathbf{v}) + (1-\lambda)\phi(\mathbf{u} + t_1\mathbf{v}) = \phi(\lambda\mathbf{u} + \lambda t_0\mathbf{v}) + \phi((1-\lambda)\mathbf{u} + (1-\lambda)t_1\mathbf{v}) \geq \phi(\mathbf{u} + (\lambda t_0 + (1-\lambda)t_1)\mathbf{v}) = \psi(\lambda t_0 + (1-\lambda)t_1)$. Conversely, assume (A), (B) and (D), then $\phi(\mathbf{u}) + \phi(\mathbf{v}) = \psi(0) + \psi(1) \geq 2\psi(1/2) = \phi(\mathbf{u} + \mathbf{v})$.

Observe that it is sufficient to check ϕ on all two-dimensional subspaces of X . For a discussion of a different condition on two-dimensional subspaces, see Dor [2]. We shall consider spaces X in \mathcal{P}_{2n} for which $p(t) = \|x + ty\|^{2n}$ is a polynomial in t of degree $2n$. When p is given in this way, we shall tacitly assume that $\|sx + ty\|^{2n} = s^{2n}p(t/s)$ for $s \neq 0$ and $\|y\|^{2n} = \lim_{t \rightarrow \infty} t^{-2n}p(t)$; that is, (B) is implicit.

THEOREM 1. *Suppose p is a nonnegative polynomial of degree $2n$. Let $X = \langle x, y \rangle$ and define $\|\cdot\|$ on X by $p(t) = \|x + ty\|^{2n}$. Then $(X, \|\cdot\|)$ is a Banach space if and only if $2np(t)p''(t) - (2n-1)(p'(t))^2 \geq 0$ for all t .*

Proof. With $\|sx + ty\|^{2n}$ defined as above, we need verify (A) and (D). Suppose $(X, \|\cdot\|)$ is a Banach space, then $\psi(t) = \|x + ty\| = p(t)^{1/2n} = f(t)$ is convex. If x and y are linearly dependent then $\|x + ty\| = |a + bt|$, and for $p(t) = (a + bt)^{2n}$, $2npp'' = (2n-1)(p')^2$. If x and y are linearly independent, then $f(t) > 0$ and f is convex if and only if $f''(t) = (2n)^{-2}(f(t))^{1-4n}(2np(t)p''(t) - (2n-1)(p'(t))^2) \geq 0$.

On the other hand, suppose $2np(t)p''(t) - (2n-1)(p'(t))^2 \geq 0$ and $\|\cdot\|$ is defined as above. If $\|sx + ty\| = 0$ for $(s, t) \neq (0, 0)$ then either $p(t_0) = 0$ or $\lim t^{-2n}p(t) = 0$. As the hypothesized condition is translation-invariant, assume $t_0 = 0$ in the first case. Since $p(t) \geq 0$ we have $p'(0) = 0$; let $p(t) = a_k t^k + o(t^k)$, $a_k \neq 0$, $k \geq 2$, for small t . Then $2np(t)p''(t) - (2n-1)(p'(t))^2 = -a_k^2 k(2n-k)t^{2k-2} + o(t^{2k-2})$ hence $k = 2n$, $p(t) = a_{2n} t^{2n}$ and $(X, \|\cdot\|)$ is a valid one-dimensional space. In the second case, let $p(t) = a_k t^k + o(t^k)$ for $k < 2n$, $a_k \neq 0$ and t large. Then $k = 0$ and $(X, \|\cdot\|)$ is again one-dimensional.

Now suppose $p(t) > 0$. Let $\mathbf{u} = dx + by$, $\mathbf{v} = cx + ay$ be given; (D) will be satisfied provided $\psi(t)$ is convex, where

$$\psi^{2n}(t) = \|dx + by + t(cx + ay)\|^{2n} = |ct + d|^{2n} p((at + b)/(ct + d)).$$

(If $c = d = 0$, then ψ is a constant and so convex). Note that ψ^{2n} is again a positive polynomial of degree $2n$ so that ψ'' is continuous. It suffices, therefore, to check that $\psi''(t) \geq 0$ for $t \neq -d/c$. As above, $\psi''(t) \geq 0$ provided $2n\psi(t)\psi''(t) - (2n-1)(\psi'(t))^2 \geq 0$. A computation shows that this expression equals $(ad - bc)^2(ct + d)^{4n-4}(2np(u)p''(u) - (2n-1)(p'(u))^2)$, where $u = (at + b)/(ct + d)$. Thus, if $2npp'' - (2n-1)(p')^2 \geq 0$ then every ψ is convex and $(X, \|\cdot\|)$ is a Banach space.

It follows from Theorem 1 that the two-dimensional spaces in \mathcal{P}_{2n} are characterized by $p(t) = \|x + ty\|^{2n}$, and that a study of such polynomials is appropriate. Note also that generators may be chosen to make any computations easier; in general, (D) must be separately verified for each two-dimensional subspace.

3. The cone P_{2n} . Let P_{2n} consist of all polynomials p of degree $2n$ for which $p(t) \geq 0$ and $C_{2n}(p(t)) = 2np(t)p''(t) - (2n-1)(p'(t))^2 \geq 0$. If $p(t) = \Sigma \binom{2n}{k} a_k t^k$, then

$$C_{2n}(p(t)) = 4n^2(2n-1) \left(\left(\Sigma \binom{2n}{k} a_k t^k \right) \left(\Sigma \binom{2n-2}{k} a_{k+2} t^k \right) - \left(\Sigma \binom{2n-1}{k} a_{k+1} t^k \right)^2 \right).$$

We shall omit the subscript $2n$ when it is superfluous. As defined, $C_{2n}(p)$ is a polynomial with nominal degree $4n-2$; the coefficients for t^{4n-2} and t^{4n-3} actually vanish identically.

THEOREM 2. *The set P_{2n} is a closed cone.*

Proof. Suppose p is in P_{2n} . Then $C(p) \geq 0$ and for $\lambda \geq 0$, $\lambda p \geq 0$ and $C(\lambda p) = \lambda^2 C(p)$ so λp is in P_{2n} . If p_i is in P_{2n} , then $p_1 + p_2 \geq 0$ and $C(p_1 + p_2) = C(p_1) + C(p_2) + 2np_1'p_2' - (4n-2)p_1'p_2'$. Since $p_i p_i' \geq 0$ we have $(2np_i p_i')^{1/2} \geq (2n-1)^{1/2} |p_i|$ so that $2np_1'p_2' + 2np_1 p_2'' - (4n-2)p_1'p_2' = 2n((p_1'p_2')^{1/2} - (p_1 p_2'')^{1/2})^2 + 4n(p_1 p_1')^{1/2} (p_2 p_2')^{1/2} - (4n-2)|p_1 p_2| + (4n-2)(|p_1 p_2| - p_1 p_2) \geq 0$. Thus, P_{2n} forms a cone.

Associate $p(t) = \Sigma \binom{2n}{k} a_k t^k$ with the element (a_0, \dots, a_{2n}) in R^{2n+1} and pull back the usual topology. Convergence is then either pointwise or coefficientwise. If $\{p_m\}$ is a sequence of polynomials in P_{2n} and $p_m \rightarrow p$ then $C(p_m(t)) \rightarrow C(p(t))$. Hence P_{2n} is closed.

By the proof of Theorem 1, if $p(t)$ is in P_{2n} then so is

$$(ct + d)^{2n} p(at + b)/(ct + d).$$

For future reference, observe that, if p_1 and p_2 are in P_{2n} and $C((p_1 + p_2)(t_0)) = 0$ then $C(p_1(t_0)) = C(p_2(t_0)) = 0$, $p_1'(t_0)p_2(t_0) = p_1(t_0)p_2'(t_0)$ and $p_1'(t_0)p_2'(t_0) \geq 0$.

Since P_{2n} is a cone, it is natural to study its extreme elements. For $q(t) = (bt + c)^{2n}$, $C_{2n}(q) \equiv 0$. Suppose $q = p_1 + p_2$, with p_i in P_{2n} . If $b = 0$, then p_1 and p_2 must both be nonnegative constants. Suppose $b \neq 0$, then we may normalize $b = 1$ so $q(t) = (t + c)^{2n}$, hence $p_1(-c) = p_2(-c) = 0$. As in the proof of Theorem 1, it follows that $p_i(t) = r_i(t + c)^{2n}$ so each p_i is a multiple of q . We have proved that $(bt + c)^{2n}$ is

an extreme element in P_{2n} . Since P_{2n} is a cone, $\Sigma(b_k t + c_k)^{2n}$ is in P_{2n} . This is to be expected in light of Theorem 1 applied to the subspace of \mathcal{L}_{2n} generated by (b_1, b_2, \dots) and (c_1, c_2, \dots) .

If $2n = 2$, $C_2(a_2 t^2 + 2a_1 t + a_0) = 4(a_0 a_2 - a_1^2)$ so that $p \geq 0$ implies $C_2(p) \geq 0$. Hence the extreme elements of P_2 are precisely $(bt + c)^2$. Surprisingly enough, the same is true for $2n = 4$.

THEOREM 3. *The extreme functions of P_4 are $(bt + c)^4$; indeed, if p is in P_4 then $p(t) = (b_0 t + c_0)^4 + (b_1 t + c_1)^4 + c_2^4$ for some b_i and c_i .*

Proof. Write $p(t) = \Sigma \binom{4}{k} a_k t^k$, then $(48)^{-1} C_4(p(t)) = (a_2 a_4 - a_3^2) t^4 + (2a_1 a_4 - 2a_2 a_3) t^3 + (a_0 a_4 + 2a_1 a_3 - 3a_2^2) t^2 + (2a_0 a_3 - 2a_1 a_2) t + a_0 a_2 - a_1^2$. If $p(t_0) = 0$, then, as before, $p(t) = a_4(t - t_0)^4$. If $C(p(t_0)) = 0$, then with $q(t) = p(t - t_0)$, $C(q(0)) = 0$. As the conclusion is invariant under translation, assume $t_0 = 0$. In this case, since $C(p) \geq 0$, $a_0 a_2 = a_1^2$ and $a_0 a_3 = a_1 a_2$. As $a_0 = p(0) \neq 0$, let $a_1 = r a_0$, then $a_2 = r^2 a_0$ and $a_3 = r^3 a_0$. If $a_4 = r^4 a_0 + s$ then $C(p(t)) = s a_0 t^2 (rt + 1)^2$, so $s \geq 0$ and $p(t) = a_0 (rt + 1)^4 + s t^4$. (In general $p(t) = a_0 (r(t - t_0) + 1)^4 + s(t - t_0)^4$.) If the degree of $C(p(t))$ is less than four, then by a similar argument, $p(t) = a_4(t + r)^4 + s$, $s \geq 0$. Finally, suppose that $C(p(t))$ is a positive quartic and let $p_\lambda(t) = p(t) - \lambda$, then $C(p_\lambda(t)) = C(p(t)) - 4\lambda p''(t)$. Since p'' is quadratic, and $pp'' > 0$, $(4p''(t))^{-1} C(p(t))$ is continuous, goes to infinity quadratically in t , and achieves a minimum $\lambda_0 > 0$ at $t = t_0$. Thus $p(t) - \lambda_0$ is in P_4 , $C(p_\lambda(t_0)) = 0$; hence $p(t) = \lambda_0 + a_0(r(t - t_0) + 1)^4 + s(t - t_0)^4$, which may be rewritten as in the conclusion.

By considering $(ct + d)^4 p((at + b)/(ct + d))$ instead of p , we may replace c_i^4 by $s^4(ct + d)^4$ for any pre-selected c and d . It would be nice if this pattern continued for $2n \geq 6$; unfortunately, this is not the case.

THEOREM 4. *If $n \geq 3$ then there exists a polynomial p in P_{2n} which cannot be written $p(t) = \Sigma(b_k t + c_k)^{2n}$.*

Proof. Fix n and let $p(t) = t^{2n} + t^2 + 1$. A computation shows that $C_{2n}(p(t)) = (8n^3 - 20n^2 + 12n)t^{2n} + (8n^3 - 4n^2)t^{2n-2} + (4 - 4n)t^2 + 4n$. Since $n \geq 3$, each term but $(4 - 4n)t^2$ is positive. For $|t| \leq 1$, $(4 - 4n)t^2 + 4n \geq 0$; for $|t| \geq 1$, $(8n^3 - 4n^2)t^{2n-2} + (4 - 4n)t^2 > (8n^3 - 4n^2 - 4n)t^2 > 0$. Thus $C_{2n}(p(t)) \geq 0$ and p is in P_{2n} .

Suppose $t^{2n} + t^2 + 1 = \Sigma(b_k t + c_k)^{2n}$; from the coefficient of t^4 and t^2 , $0 = \Sigma b_k^4 c_k^{2n-4}$ and $1 = \binom{2n}{2} \Sigma b_k^2 c_k^{2n-2}$. Since $n \geq 3$, the first implies that $b_k c_k = 0$ for each k , and this contradicts the second.

The coefficient 1 for t^2 is not the best possible. The following proposition provides a sharp estimate.

PROPOSITION 1. *If $t^{2n} + \alpha t^{2k} + 1$ is in P_{2n} , then*

$$0 \leq \alpha \leq 2n(2n-1)c(k, n),$$

where $(c(k, n))^n = (2k)^{-k}(2n-2k)^{k-n}(2k-1)^{n-2k}(2n-2k-1)^{2k-n}$.

Outline of proof. Suppose $p_\alpha(t) = t^{2n} + \alpha t^{2k} + 1$ has the largest α , then $C_{2n}(p_\alpha(t)) \geq 0$ and $C_{2n}(p_\alpha(t_0)) = 0$ for some t_0 . Hence the derivative vanishes at t_0 as well. This gives two quadratic equations in α which may be solved simultaneously. After eliminating an extraneous solution, the bound is derived.

We see then that there are extreme functions in P_{2n} , $n \geq 3$, which are not of the form $(bt + c)^{2n}$.

PROPOSITION 2. *The extreme rays of P_6 are generated by*

$$(ct+d)^{2n}f_\lambda((at+b)/(ct+d)),$$

where $f_\lambda(t) = t^6 + 6\lambda t^5 + 15\lambda^2 t^4 + 20\lambda^3 t^3 + 15\lambda^2 t^2 + 6\lambda t + 1$, and $|\lambda| \leq 1/2$ or $|\lambda| = 1$.

Outline of proof. As in Theorem 3, we consider special cases and then subtract various $(ct+d)^6$'s. Then f_λ are those polynomials for which $C_6(f_\lambda(0)) = 0$ and $C_6(f)$ is at most quartic.

As Proposition 2 is not directly relevant to the rest of this paper and its proof is tedious, we omit the details. The general question of finding the extreme rays of P_{2n} for $n \geq 4$ remains open.

Let Q_{2n} denote the closure of the cone of polynomials of the form $\sum_{j=1}^R (b_j t + c_j)^{2n}$; $Q_{2n} \subseteq P_{2n}$ with equality if and only if $2n = 2$ or 4. As any $2n+2$ distinct $2n$ th powers are linearly dependent, we may assume that $R \leq 2n+1$. Suppose $q(t) = \Sigma \binom{2n}{k} a_k t^k$ is in Q_{2n} . Then $q = \lim q_m$, where $q_m(t) = \sum_{j=1}^{2n+1} (b_j^{(m)} t + c_j^{(m)})^{2n}$. Since $\Sigma (b_j^{(m)})^{2n} \rightarrow a_{2n}$ and $\Sigma (c_j^{(m)})^{2n} \rightarrow a_0$, we may take $|b_j^{(m)}| < M$, $|c_j^{(m)}| < M$. Thus there exists a convergent subsequence with limit b_j and c_j so that one may write $q(t) = \sum_{j=1}^{2n+1} (b_j t + c_j)^{2n}$ for all q in Q_{2n} . Similar considerations apply for the generalization of Q_{2n} to several variables.

4. Subspaces of L_{2n} . In [11] we showed that $L_{2n}(Y, \mu)$ is in \mathcal{P}_{2n} , that is, $\|f + tg\|^{2n} = \int |f + tg|^{2n} d\mu$ is a polynomial in t for all f and g . The converse, as we shall see, is false. Suppose that

$X = \langle x, y \rangle$ is a two-dimensional space in \mathcal{P}_{2n} , then $p(t) = \|x + ty\|^{2n}$ is in P_{2n} . Suppose that X is embeddable in $L_{2n}(Y, \mu)$, then $p(t) = \Sigma \binom{2n}{k} a_k t^k = \int (f + tg)^{2n} d\mu = \|f + tg\|^{2n}$. By Hölder's inequality, since $\int f^{2n} d\mu < \infty$ and $\int g^{2n} d\mu < \infty$, $\int f^{2n-k} g^k d\mu < \infty$ so that the integral can be broken up and $a_k = \int f^{2n-k} g^k d\mu$. Let $Y_0 = \{s \in Y: f(s) = 0\}$, $Z = Y - Y_0$; let $d\nu = f^{2n} d\mu$ and $h = gf^{-1}$ on Z . Then we have $a_k = \int h^k d\nu$, $0 \leq k \leq 2n - 1$, and $a_{2n} = \int_Z h^{2n} d\nu + \int_{Y_0} g^{2n} d\mu$. If $\Phi(r) = \nu(h^{-1}\{(-\infty, r]\})$, then $a_k = \int_{-\infty}^{\infty} s^k d\Phi$ for $0 \leq k \leq 2n - 1$ and $a_{2n} \geq \int_{-\infty}^{\infty} s^{2n} d\Phi$.

Conversely, suppose there exists a nonnegative measure Φ and a_k 's so that $a_k = \int_{-\infty}^{\infty} t^k d\Phi$ and $a_{2n} \geq \int_{-\infty}^{\infty} t^{2n} d\Phi$. Define (Y, μ) as follows: $Y = R \cup \{p_0\}$, $\mu = \Phi$ on R and $\mu\{p_0\} = a_{2n} - \int_{-\infty}^{\infty} s^{2n} d\Phi$. Let $(f(s), g(s)) = (1, s)$ on R and $(0, 1)$ on $\{p_0\}$. Then $\|f + tg\|^{2n} = \int_{-\infty}^{\infty} (1 + st)^{2n} d\Phi + (a_{2n} - \int_{-\infty}^{\infty} s^{2n} d\Phi) t^{2n} = \Sigma \binom{2n}{k} t^k \int_{-\infty}^{\infty} s^k d\Phi + (a_{2n} - \int_{-\infty}^{\infty} s^{2n} d\Phi) t^{2n} = \Sigma \binom{2n}{k} a_k t^k = p(t)$.

Fortunately, this transforms the embedding problem into the classical moment problem, which has been studied extensively. The complete solution is known, see for example Akhiezer [1] p. 71, and we may combine this solution with the previous discussion to obtain the following theorem.

THEOREM 5. *Let X be a two-dimensional Banach space in \mathcal{P}_{2n} with generators x and y and let $p(t) = \|x + ty\|^{2n} = \Sigma \binom{2n}{k} a_k t^k$. Define the $(n+1) \times (n+1)$ matrix $B = (b_{ij})$ by $b_{ij} = a_{i+j}$ for $0 \leq i, j \leq n$. Then X is embeddable in L_{2n} if and only if the matrix B is positive semidefinite. Further, X is embeddable in L_{2n} if and only if p is in Q_{2n} .*

Proof. The positive semidefiniteness of B is equivalent to the solution of the described moment problem. If p is in Q_{2n} then X is embeddable in \mathcal{L}_{2n}^{2n+1} in the obvious fashion. If X is embeddable in L_{2n} , then by approximating $d\Phi$ by a sequence of point masses, we see that p is in Q_{2n} .

COROLLARY 6. *If X is two-dimensional space in \mathcal{P}_4 , then X is embeddable in L_4 . There are two-dimensional spaces in \mathcal{P}_{2n} , $n \geq 3$, which are not embeddable in L_{2n} .*

Proof. Combine Theorems 3, 4 and 5.

The case for higher dimensions is less clearcut. Professor J. H. B. Kemperman [6] has pointed out, using techniques from [4] and [5], that the analogous moment problem in more than one variable has a solution which requires knowledge of all polynomials $f(u_1, \dots, u_p)$ of total degree $2n$ which are nonnegative for all real u_i .

Specifically, one transforms the polynomial $p(t_1, \dots, t_p) = \|x_0 + t_1 x_1 + \dots + t_p x_p\|^{2n}$ for a space $X = \langle x_0, \dots, x_p \rangle$ into a family of equations $a(m_1, \dots, m_p) = \int \dots \int_{t_1^{m_1} \dots t_p^{m_p}} d\mu$; $m_1 + \dots + m_p < 2n$, with inequality if $\sum m_i = 2n$. Suppose $f(u_1, \dots, u_p) \geq 0$ for all real u_i and $f(u_1, \dots, u_p) = \sum b(m_1, \dots, m_p) u_1^{m_1} \dots u_p^{m_p}$, where the sum is taken over all m_i , $\sum m_i \leq 2n$. Then certainly $\int \dots \int f(u_1, \dots, u_p) d\mu = \sum a(m_1, \dots, m_p) b(m_1, \dots, m_p) \geq 0$. It turns out this condition holding for all such f is sufficient for the existence of a measure with the desired property.

Since X is real, it is unreasonable to embed X in an L_{2n} space with complex scalars; one might, however, embed X in an $L_{2n}(Y, \mu)$ space with real scalars but complex-valued functions. This situation is taken care of by the following theorem.

THEOREM 7. *There is an isometry from the space of all complex-functions in $L_{2n}(Y, \mu)$, taken with real scalars, into real $L_{2n}(Z, \nu)$, where (Z, ν) consists of $2n + 1$ copies of (Y, μ) .*

Proof. It is well known that \mathcal{L}_2^2 is embeddable in any infinite-dimensional Banach space. Let x and y be orthogonal generators of \mathcal{L}_2^2 and let \bar{x} and \bar{y} be their isometric images in \mathcal{L}_{2n} . Then $(t^2 + u^2)^n = \|tx + uy\|^{2n} = \|\bar{t}\bar{x} + \bar{u}\bar{y}\|^{2n} = \sum (b_k t + c_k u)^{2n}$; by the remarks at the end of §3, we may say that $(t^2 + u^2)^n = \sum_{k=1}^{2n+1} (b_k t + c_k u)^{2n}$. Define the mapping ϕ from $L_{2n}(Y, \mu)$ with complex-valued functions to $L_{2n}(Z, \nu)$ as follows: if $f = g + ih$ is the decomposition into real and imaginary parts, then $\phi(f) = b_k g + c_k h$ on the k th copy of (Y, μ) . For real λ_i , $\phi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \phi(f_1) + \lambda_2 \phi(f_2)$; $\|\phi(f)\|^{2n} = \sum_{k=1}^{2n+1} \int_Y (b_k g + c_k h)^{2n} d\mu = \int_Y (g^2 + h^2)^n d\mu = \int_Y |f|^{2n} d\mu = \|f\|^{2n}$ so ϕ is an isometry.

We may actually choose b_k and c_k by: $b_k + ic_k = a(n) \exp(2\pi k i (n+1)^{-1})$, where $a(n) = 2 \binom{2n}{n} (2n+1)^{-1/2n}$. Hilbert has proved that b_k and c_k may be chosen to be rational; see Ellison [3] p. 11 for an extended discussion. In any case, it suffices to consider embeddings into real L_{2n} .

5. **A counterexample.** The remaining case for embedding is the three-dimensional one for \mathcal{P}_4 . We shall construct a three-dimensional space in \mathcal{P}_4 which is not embeddable in L_4 . Consequently, there are spaces with arbitrarily large dimensions which are not embeddable in L_4 . This example is drastically simplified from the one appearing in the author's thesis.

Suppose $X = \langle x, y, z \rangle$ and a polynomial $p(u, v)$ with total degree 4 is given. Let $\|\cdot\|$ be defined on X by $\|x + uy + vz\|^4 = p(u, v)$; $\|tx + uy + vz\|^4$ for $t \neq 1$ is defined in the usual way. In view of Lemma 1, we need check (A), (B) and (D) on every two-dimensional subspace of X . Conditions (A) and (B) will be automatic. A two-dimensional subspace of X is either $\langle y, z \rangle$ or $\langle x + ay + cz, by + dz \rangle$ for some a, b, c, d . Thus, for $f(u, v) = (p(u, v))^{1/4}$, it suffices to show that $\psi(t) = f(a + bt, c + dt)$ is convex for all a, b, c, d . (We consider $\langle y, z \rangle$ separately.) Adopt the usual convention that $f_1(u, v) = (\partial/\partial u)f(u, v)$, $f_{22}(u, v) = (\partial^2/\partial v^2)f(u, v)$, etc. Then $\psi''(t) = (b^2f_{11} + 2bdf_{12} + d^2f_{22})(a + bt, c + dt)$. Hence it suffices to show that $f_{11} \geq 0$, $f_{22} \geq 0$ and $f_{11}f_{22} \geq f_{12}^2$ at all points in the plane. If we can verify this for $f = p^{1/4}$ then $(X, \|\cdot\|)$ will be a Banach space.

THEOREM 8. *For $X = \langle x, y, z \rangle$, let $\|tx + uy + vz\|^4 = t^4 + 6t^2(u^2 + v^2) + (u^2 + v^2)^2$. Then $(X, \|\cdot\|)$ is a Banach space which is not embeddable in L_4 .*

Proof. Note that $\|tx + uy + vz\| > 0$ unless $t = u = v = 0$ so that (A) is satisfied. On $\langle y, z \rangle$, $\|uy + vz\| = (u^2 + v^2)^{1/2}$ so $\langle y, z \rangle$ is isometric to ℓ_2^2 and (D) is satisfied. In general, let $f = p^{1/4}$, then $16f_{11} = p^{-7/4}(4pp_{11} - 3p_1^2)$, $16f_{22} = p^{-7/4}(4pp_{22} - 3p_2^2)$ and $16f_{12} = p^{-7/4}(4pp_{12} - 3p_1p_2)$. We must show that $4pp_{ii} - 3p_i^2 \geq 0$ and that

$$\begin{aligned} (4pp_{11} - 3p_1^2)(4pp_{22} - 3p_2^2) - (4pp_{12} - 3p_1p_2)^2 \\ = 4p(4p(p_{11}p_{22} - p_{12}^2) - 3p_1^2p_{22} + 6p_1p_2p_{12} - 3p_2^2p_{11}) \\ = 4pD(p) \geq 0. \end{aligned}$$

For $p(u, v) = \|x + uy + vz\|^4 = 1 + 6(u^2 + v^2) + (u^2 + v^2)^2$ let $w = u^2 + v^2$, then $p = 1 + 6w + w^2$, $p_1 = 4u(3 + w)$, $p_2 = 4v(3 + w)$, $p_{11} = 4(3 + w + 2u^2)$, $p_{12} = 8uv$, $p_{22} = 4(3 + w + 2v^2)$. Hence

$$4pp_{11} - 3p_1^2 = 16(3(1 - u^2)^2 + v^2(19 + 12u^2 + u^4) + v^4(9 + 2u^2) + v^6) \geq 0$$

and similarly $4pp_{22} - 3p_2^2 \geq 0$. Further, $p_{11}p_{22} - p_{12}^2 = 48(w + 3)(w + 1)$ and $p_1^2p_{22} - 2p_1p_2p_{12} + p_2^2p_{11} = 64w(w + 3)^3$, hence

$$\begin{aligned} D(p) &= 192(w + 3)(w + 1)(w^2 + 6w + 1) - 192w(w + 3)^3 \\ &= 192(w + 3)(w - 1)^2 \geq 0. \end{aligned}$$

Thus $(x, \|\cdot\|)$ is a Banach space.

If X were embeddable in L_4 , then for some f, g and h , $t^4 + 6t^2(u^2 + v^2) + (u^2 + v^2)^2 = \int_Y (tf + ug + vh)d\mu$, so $\int f^4 = \int g^4 = \int h^4 = \int f^2 g^2 = \int f^2 h^2 = 1$, $\int g^2 h^2 = 1/3$. The first five equations imply that $f^2 = g^2$ and $f^2 = h^2 \mu - \text{a.e.}$; this is contradicted by the sixth. Alternatively, in the spirit of the moment problem, $0 \leq \int_Y (f^2 - g^2 - h^2)^2 d\mu = -1/3$. Either proof shows that X is not embeddable in L_4 .

One can make a lengthy plausibility argument that the set of polynomials $p(t, u, v) = \|tx + uy + vz\|^4$ has 15 degrees of freedom for spaces in \mathcal{S}_4 and 14 for spaces in L_4 . The last degree of freedom manifests itself here as the coefficient of $u^2 v^2$.

6. Other properties of \mathcal{S}_{2n} . Since $Q_{2n} \subseteq P_{2n}$, with strict inclusion for $n \geq 3$, it is not obvious that spaces in \mathcal{S}_{2n} are necessarily as "nice" as spaces in L_{2n} . For example, $L_{2n}(Y, \mu)$ is uniformly convex and uniformly smooth (see Lindenstrauss and Tzafriri [10] p. 127 for definition) and hence reflexive. Hölder's inequality says that, if $\int f^{2n} = \int g^{2n} = 1$, then $\left| \int f^k g^{2n-k} \right| \leq 1$ for $0 \leq k \leq 2n$. Thus if $q(t) = 1 + \sum_{k=1}^{2n-1} \binom{2n}{k} a_k t^k + t^{2n}$ is in Q_{2n} , then $|a_k| \leq 1$; indeed, $1 \geq a_k \geq r(k)$, where $r(2j) = 0$, $r(2j+1) = -1$. Clarkson's inequality states that $\|f + g\|^{2n} + \|f - g\|^{2n} \geq 2(\|f\|^{2n} + \|g\|^{2n})$; if $q(t) = \sum_{k=0}^{2n} \binom{2n}{k} a_k t^k$ is in Q_{2n} , then $q(1) + q(-1) \geq 2(q(0) + a_{2n})$. As a whole, these properties extend to \mathcal{S}_{2n} , although numerical constants are generally weaker.

Koehler [7] defined a G_{2n} space to be a Banach space on which a $2n$ -fold inner product $\langle x_1, \dots, x_{2n} \rangle$ is defined, satisfying certain regularity conditions. In [11] it was shown that G_{2n} spaces and \mathcal{S}_{2n} spaces coincide. Koehler [8] proved that G_{2n} spaces are uniformly convex. That is, \mathcal{S}_{2n} spaces are uniformly convex and thus reflexive. To prove uniform smoothness and the other regularity conditions we need the analogue to Hölder's inequality.

THEOREM 9. *If $p(t) = 1 + \sum_{k=1}^{2n-1} \binom{2n}{k} a_k t^k + t^{2n}$ is in P_{2n} , then there are constants so that $m(k, 2n) \leq a_k \leq M(k, 2n)$.*

Proof. Since $p^{1/2n}(t)$ is convex, by the triangle inequality on the space induced by p , $(1 - |t|)^{2n} \leq p(t) \leq (1 + |t|)^{2n}$, so for $t \geq 0$, $(t-1)^{2n} \leq p(t) \leq (t+1)^{2n}$. The set of $2n-1$ equations $\sum_{k=1}^{2n-1} \binom{2n}{k} a_k j^k = p(j) - 1 - j^{2n}$, $1 \leq j \leq 2n-1$, has a Vandermonde determinant, hence $\binom{2n}{k} a_k$ may be expressed in terms of $p(j) - 1 - j^{2n}$. Since $p(j)$ is bounded one

obtains bounds on α_k which are, in general, wildly generous.

Alternatively, a sequence of polynomials with unbounded a_k 's has a subsequence from which can be deduced the existence of \bar{p} in P_{2n} , $\bar{p}(t) = \sum_{k=1}^{2n-1} \binom{2n}{k} \bar{a}_k t^k$, not all \bar{a}_k 's equal to zero. This yields a contradiction.

It follows that the set of all points (a_1, \dots, a_{2n-1}) , A , in R^{2n-1} so that $1 + \sum_{k=1}^{2n-1} \binom{2n}{k} a_k t^k + t^{2n}$ is in P_{2n} forms a closed (Theorem 2) and bounded (Theorem 9) set. Thus functionals, such as $p(1)$, achieve maxima and minima on A .

The actual values of $m(k, 2n)$ and $M(k, 2n)$ can be found in a few instances. Since $p(t)$ in P_{2n} implies $p(-t)$ and $t^{2n}p(1/t)$ are in P_{2n} , $m(2j+1, 2n) = -M(2j+1, 2n)$, $m(2n-k, 2n) = m(k, 2n)$ and $M(2n-k, 2n) = M(k, 2n)$. As L_{2n} spaces are in \mathcal{S}_{2n} , $M(k, 2n) \geq 1$ and $m(k, 2n) \leq r(k)$. These coefficients are a two-dimensional property; consequently $m(k, 2n)$ and $M(k, 2n)$ are already determined for $2n = 2$ or 4.

In any case, $a_1 = \lim_{t \rightarrow \infty} t^{-1}(\|x + ty\| - \|x\|)$, so $|a_1| \leq 1$ and $M(1, 2n) = -m(1, 2n) = 1$. Further, $C(p(0)) = (2n)^2(2n-1)(a_0 a_2 - a_1^2)$ so $a_2 \geq 0$ and $m(2, 2n) = 0$. The condition in Theorem 9 is, for general p in P_{2n} , $a_k \leq M(k, 2n) a_0^{1-\alpha} a_{2n}^\alpha$, where $\alpha = k/2n$. From the convexity of x^α , extreme values are attained on extreme elements in P_{2n} . In this way, considering Proposition 2, one can show that $M(3, 6) = -m(3, 6) = 1$ and $M(2, 6) = 5^{-5/3}(1565 + 496\sqrt{10})^{1/3} \cong 1.000905$. The general problem remains open.

THEOREM 10. *If X is in \mathcal{S}_{2n} then X is uniformly convex, uniformly smooth and so is reflexive.*

Proof. The uniform convexity follows from Koehler, or by noting that $\|x\| = \|y\| = 1$, $\|x + y\| = 2$ implies $\|x + ty\| = 1 + t$ for $t \geq 0$ so $p(t) = (1+t)^{2n}$ and $\|x - y\| = 0$. Since the set of coefficients A_ε for which $\|x\| = \|y\| = 1$, $\|x - y\| \geq \varepsilon$ is compact, $\|x + y\|$ achieves a maximum, which is strictly less than 2.

For uniform smoothness, let $\|x\| = \|y\| = 1$. For $t \leq \tau$, by Taylor's theorem, $\|x + ty\| + \|x - ty\| = 2 + (2n-1)(a_2 - a_1^2)t^2 + o(t^2)$. Thus $1/2(\|x + ty\| + \|x - ty\|) - 1 \leq c\tau^2 + o(\tau^2)$ so X is uniformly smooth.

If X is any Banach space, suppose $t = \|y\| \geq \|x\| = 1$ and $u = \|x + y\| \geq \|x - y\| = v$. Then $u + v \geq 2t$ so $u^p + v^p \geq u^p + (2t - u)^p \geq 2t^p \geq t^p + 1$. That is, $\|x + y\|^p + \|x - y\|^p \geq \|x\|^p + \|y\|^p$ with equality if and only if $\|x\| = \|y\| = \|x + y\| = \|x - y\| = 1$. In this case,

by the triangle inequality, $\|x + ry\| \equiv 1$ for $|r| \leq 1$ so X cannot be in \mathcal{S}_{2n} . Thus, by the compactness of A , $\|x + y\|^{2n} + \|x - y\|^{2n} \geq c(n)(\|x\|^{2n} + \|y\|^{2n})$ for x and y in X in \mathcal{S}_{2n} . Taking $x = 0$, $c(n) \leq 2$.

THEOREM 11. *If X is in \mathcal{S}_{2n} for $n \leq 3$ then $\|x + y\|^{2n} + \|x - y\|^{2n} \geq 2(\|x\|^{2n} + \|y\|^{2n})$, but this is not necessary true for $n \geq 4$.*

Proof. For $n \leq 2$, X is embeddable in L_{2n} . For $n = 3$, let $\|x + ty\|^6 = \sum_{k=0}^6 \binom{6}{k} a_k t^k$ then $\|x + y\|^6 + \|x - y\|^6 - 2\|x\|^6 - 2\|y\|^6 = 30(a_2 + a_4) \geq 0$ since $m(2, 6) = m(4, 6) = 0$.

Fix $n \geq 4$ and set $p_\varepsilon(t) = 1 + \varepsilon(t^2 - 3t^4 + t^6) + t^{2n}$ and $\|x + ty\|^{2n} = p_\varepsilon(t)$, then $\|x + y\|^{2n} + \|x - y\|^{2n} - 2(\|x\|^{2n} + \|y\|^{2n}) = -2\varepsilon > 0$ for $\varepsilon > 0$. A computation shows that $C_{2n}(p_\varepsilon(t)) = 4n^2(2n - 1)t^{2n-2} + \varepsilon(g(t) + \varepsilon h(t))$, where $g(t) = 4n^2(2n - 1)t^{2n-2}(t^2 - 3t^4 + t^6) + 2n(1 + t^{2n})(2 - 36t^2 + 30t^4) - 4n(2n - 1)t^{2n-1}(2t - 12t^3 + 6t^5)$ and $h(t) = 2n(t^6 - 3t^4 + t^2)(30t^4 - 36t^2 + 2) - (2n - 1)(6t^5 - 12t^3 + 2t)^2$.

As $n \geq 4$, the highest order term of $g + \varepsilon h$ is

$$2n(4n^2 - 26n + 42)t^{2n+4},$$

there exist ε_0 and R so that for $0 \leq \varepsilon \leq \varepsilon_0$ and $|t| > R$, $(g + \varepsilon h)(t) \geq 0$ and thus $C_{2n}(p_\varepsilon(t)) > 0$. As $(g + \varepsilon h)(0) = 4n$, for $0 \leq \varepsilon \leq \varepsilon_0$ and $|t| < \delta$ or $|t| > R$, $C_{2n}(p_\varepsilon(t)) > 0$. On the remaining (compact) set, t^{2n-2} is positive and $|g| + \varepsilon_0|h|$ is bounded, so for some further reduced range of ε , $C_{2n}(p_\varepsilon) > 0$ and p_ε is in P_{2n} .

For $n = 4$ take $\varepsilon = .04$, then $p_\varepsilon(t) = t^8 + .04t^6 - .12t^4 + .04t^2 + 1$. A direct computation shows that $C_8(p_\varepsilon(t)) = 64(t^{12} + 1) + 11.5392(t^{10} + t^2) + 9.68(t^8 + t^4) + 447.9104t^6$. If we factor out $.64t^6$ and let $u = t^2 + t^{-2}$, then we obtain $u^3 - 18.03u^2 + 12u + 735.92 = q(u)$. (The range for $t^2 + t^{-2}$ is $u \geq 2$.) Clearly $q(2) > 0$, and q achieves its minimum when $u = u_0 = 6.01 + \sqrt{32.1201} \cong 11.67$. Since $q(u_0) \cong 9.79 > 0$, $C(4) \leq 1.96$. This bound is not sharp. This example also shows that $m(4, 8) < 0$.

The question of describing spaces dual to spaces in \mathcal{S}_{2n} also remains open. Indeed it is false, in general, that the dual space to a subspace of $L_p(Y, \mu)$ is necessarily embeddable in L_q , $p^{-1} + q^{-1} = 2$. For example, if $p = 2n/(2n - 1)$, $x = (1, 1, 0)$, $y = (1, 0, 1)$ and X is the subspace of ℓ_p^3 generated by x and y , then X^* is not even in \mathcal{S}_{2n} , let alone L_{2n} . We omit the proof.

7. Krivine inequalities. Krivine [9] has described necessary and sufficient conditions for a space to be embeddable in L_p provided p is not an even integer. Krivine's proof does not apply when $p = 2n$ because it involves the Taylor series remainder of $\cos x$. Theorem 12 discusses this case and provides an underlying reason for this

failure when viewed in conjunction with Corollary 6.

THEOREM (Krivine). *If $2r - 2 < p < 2r \leq 4k$ then a necessary and sufficient condition for X to be embeddable in L_p is that (1) holds for all elements x_i and all choices of real scalars r_i with $\sum r_i = 0$. The sum is taken as the i_j 's range independently from 1 to m and as the ε_j 's range over all choices of sign ± 1 . The sum has $m^{2k}2^{k-1}$ terms.*

$$(1) \quad (-1)^r \sum_{i_1=1}^m \cdots \sum_{i_{2k}=1}^m r_{i_1} \cdots r_{i_{2k}} \sum_{\varepsilon_j} \|x_{i_1} + \varepsilon_2 x_{i_2} + \cdots + \varepsilon_{2k} x_{i_{2k}}\|^p \geq 0.$$

THEOREM 12. *If $4k > 2n$ and X is in \mathcal{P}_{2n} , then the sum in (1), taken with $p = 2n$, is identically zero.*

Proof. By Theorem 11 in [11], it suffices to verify any linear identity on one space in \mathcal{P}_{2n} , say C . Since in (1) all elements are combined with real coefficients, by Theorem 7, we may embed C isometrically in R . It therefore suffices to check that (2) holds in R .

$$(2) \quad \sum_{i_1=1}^m \cdots \sum_{i_{2k}=1}^m r_{i_1} \cdots r_{i_{2k}} \sum_{\pm} (t_{i_1} \pm t_{i_2} \pm \cdots \pm t_{i_{2k}})^{2n} = 0.$$

Because of the signs in the inner sum, we may rewrite this in the form $\sum_j d_j t_{i_1}^{\pi_j(1)} \cdots t_{i_{2k}}^{\pi_j(2k)}$, where j indexes all partitions of $2n$ into $2k$ even integers and d_j is the positive multinomial coefficient. If we now exchange the order of summation, then (2) becomes (3).

$$(3) \quad \sum_j d_j \prod_{s=1}^{2k} \left(\sum_{i_s=1}^m r_{i_s} t_{i_s}^{\pi_j(s)} \right) = 0.$$

Fix j ; since $4k > 2n$, at least one of the $\pi_j(s)$'s is zero. Thus, one term in the product is $\sum r_j = 0$, each term in the sum vanishes and (3) is verified.

For $2n \geq 4$, there are spaces in \mathcal{P}_{2n} which are not embeddable in L_4 , so that Krivine's inequalities do not extend. For $4k = 2n$ and $X = L_{2n}(Y, \mu)$, it is not hard to show that the left hand side of (1) becomes $\left(\int \sum r_i x_i^2 d\mu \right)^{2k}$ which is nonnegative. If, on the other hand, X is the space in Theorem 8, $x_1 = x$, $x_2 = y$, $x_3 = z$, $r_1 = -2$, $r_2 = r_3 = 1$, then $\sum_i^3 \sum_i^3 r_i r_j \sum \|x_i \pm x_j\|^4 = -16$. It is possible that a careful study of Krivine's inequality for such borderline cases could lead to an embedding theorem for L_p , $p = 2n$.

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