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# WALLMAN'S TYPE ORDER COMPACTIFICATION

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**For a completely regular ordered space $X$, the Stone-Čech order compactification $\beta_1(X)$ has been constructed by Nachbin. This compactification is a generalized concept of the ordinary Stone-Čech compactification $\beta(X)$ in the sense that if $X$ has the discrete order: $x \leqq y$ iff $x = y$, then $\beta_1 X = \beta X$. In this paper, for a convex ordered space $X$ with a semi-closed order, the Wallman order compactification $\omega_0(X)$ is constructed by the use of the concept of maximal bifilters. $\omega_0(X)$ is a $T_1$-compact ordered topological space in which $X$ is densely embedded in both the topological and order sense.**

Althought the order of $\omega_0(X)$ is not semi-continuous, in general, most of the corresponding properties of the ordinary Wallman compactification can be generalized. For example, it can be shown that for any compact ordered topological space $Y$ (with a closed order), a continuous increasing map from $X$ into $Y$ has a unique continuous increasing extension on $\omega_0(X)$, and if $\omega_0(X)$ has a closed order, then $X$ is a normally ordered space.

First, we fix some notations and terminologies: Let $(X, \leqq)$ be a partially ordered set. For a subset $A \subseteq X$, we write $d(A) = \{y \in X: y \leqq x \text{ for some } x \in A\}$ and $i(A) = \{y \in X: x \leqq y \text{ for some } x \in A\}$. In particular, if $A$ is a singleton set, say $\{x\}$, then we write $d(x)$ and $i(x)$ respectively. A subset $A$ of $X$ is decreasing (increasing, respectively) if $A = d(A)$ ($A = i(A)$, respectively). We say that a map $f$ from $X$ to a partially ordered space $Y$ is increasing if $x \leqq y$ in $X$ implies $f(x) \leqq f(y)$ in $X$. For a (partially) ordered topological space $(X, \mathscr{T})$ in the order $\leqq$, let

$$\mathscr{U} = \{U\varepsilon\colon U = i(U)\},$$
$$\mathscr{L} = \{U\varepsilon\colon U = d(U)\},$$

then $\mathscr{U}$ and $\mathscr{L}$ are evidently topologies for $X$, which are called the *upper*, *lower* topologies respectively ([6], [1]). We say that an ordered topological space $X$ is *convex* if $X$ has a subbase consisting of the sets in $\mathscr{U}$ and $\mathscr{L}$, or equivalently, if every open set in $X$ can be written as the intersection of an open decreasing set ([5]). Let $X$ be an ordered topological space. The partial order is said to be *upper* (*lower*) *semi-closed* if, for any $x \in X$, $i(x)(d(x)$, respectively) is closed. The partial order of $X$ is *semi-closed* if it is both upper and lower semi-closed. It is said to be *closed* if, its graph, the set of the points $(x, y)$ such that $x \leqq y$, is closed in the product space $X \times X$ ([4], [5] and [9]).

We recall that a filter $\mathscr{F}$ in a topological space $(X, \mathscr{T})$ is an open (*closed*) *filter* if $\mathscr{F}$ has a filter base consisting of open (*closed*) sets.

DEFINITION. Let $(X, \mathscr{T} \leqq)$ be an ordered topological space. Let $\mathscr{F}$ be a closed filter in $(X, \mathscr{U})$ and $\mathfrak{G}$ be a closed filter in $(X, \mathscr{L})$. A pair $(\mathscr{F}, \mathfrak{G})$ of closed filters $\mathscr{F}$ and $\mathfrak{G}$ is called to be a *bi-filter* on $X$ if $F \cap G \neq \emptyset$ for any $F \in \mathscr{F}$ and any $G \in \mathfrak{G}$.

For given two bi-filters $(\mathscr{F}_1, \mathfrak{G}_1)$ and $(\mathscr{F}_2, \mathfrak{G}_2)$, we define a relation $(\mathscr{F}_1, \mathfrak{G}_1) \subseteqq (\mathscr{F}_2, \mathfrak{G}_2)$ if and only if $\mathscr{F}_1 \subseteqq \mathscr{F}_2$ and $\mathfrak{G}_1 \subseteqq \mathfrak{G}_2$. We can easily remark that by Zorn's lemma, every bi-filter is contained in a maximal bi-filter. For an ordered topological space $X$, we write

$$\Gamma_{\mathscr{U}} X = \{A \subseteqq X\colon A \text{ is closed decreasing set}\},$$
$$\Gamma_{\mathscr{L}} X = \{A \subseteqq X\colon A \text{ is closed increasing set}\}.$$

The following two lemmas are analogous properties of maximal filters. Thus, the proofs are omitted.

LEMMA 1. *Let $(\mathscr{F}, \mathfrak{G})$ be a maximal bi-filter, and $A \in \Gamma_{\mathscr{U}} X$. Then $A \in \mathscr{F}$ if and only if given $F \in \mathscr{F}$, $G \in \mathfrak{G}$, we have $A \cap F \cap G \neq \emptyset$. Moreover, a dual statement holds for $\mathfrak{G}$.*

LEMMA 2. *Let $(\mathscr{F}, \mathfrak{G})$ be a maximal bi-filter.*

(1) *Let $A_1$ and $A_2$ be in $\Gamma_{\mathscr{U}} X$ and $A_1 \cup A_2 \in \mathscr{F}$. Then either $A_1 \in \mathscr{F}$ or $A_2 \in \mathscr{F}$. Moreover, a dual statement holds for $\mathfrak{G}$.*

(2) *Let $A \in \Gamma_{\mathscr{U}} X$, $B \in \Gamma_{\mathscr{L}} X$ and $A \cup B = X$. Then either $A \in \mathscr{F}$, or $B \in \mathfrak{G}$.*

REMARK 1. Let $(X, \mathscr{T}, \leqq)$ be an ordered topological space with a semi-closed order. For each $x \in X$, we write

$$\mathscr{S}(d(x)) = \{A \text{ is a subset of } X\colon d(x) \subseteqq A\},$$
$$\mathscr{S}(i(x)) = \{A \text{ is a subset of } X\colon i(x) \subseteqq A\}.$$

Then every $\mathscr{S}(d(x))$ is a closed filter, but it need not be a maximal closed filter in $(X, \mathscr{U})$ under the inclusion relation. Moreover, a dual statement holds for $\mathscr{S}(i(x))$. $\mathscr{S}(d(x))$ is obviously a closed filter in $(X, \mathscr{U})$. In order to show that it need not be a maximal closed filter let us consider the following example:

Let $N = \{0, 1, 2\}$ be an ordered topological space with usual order and discrete topology. Then $\mathscr{S}(d(2))$ and $\mathscr{S}(d(1))$ are not maximal closed filters in $(N, \mathscr{U})$. However, if the order on $N$ is given as discrete, $\mathscr{S}(d(x))$ is a maximal closed filter for every $x \in N$.

LEMMA 3. *Let $(X, \mathscr{T}, \leqq)$ be an ordered topological space with a semi-closed order. Then for each $x \in X$, $(\mathscr{S}(d(x)), \mathscr{S}(i(x)))$ is a maximal bi-filter.*

*Proof.* Let $A \in \mathscr{S}(d(x))$ and $B \in \mathscr{S}(i(x))$. Then $d(x) \subseteqq A$ and $i(x) \subseteqq B$. Hence $A \cap B \neq \varnothing$. Therefore $(\mathscr{S}(d(x)), \mathscr{S}(i(x)))$ is a bi-filter. Suppose that there exists a bi-filter $(\mathscr{F}, \mathfrak{G})$ such that $(\mathscr{S}(d(x)), \mathscr{S}(i(x))) \subsetneqq (\mathscr{F}, \mathfrak{G})$. It follows that $\mathscr{S}(d(x)) \subsetneqq \mathscr{F}$ or $\mathscr{S}(i(x)) \subsetneqq \mathfrak{G}$.

Suppose that $\mathscr{S}(d(x)) \subsetneqq \mathscr{F}$. Then there exists an $F \in \mathscr{F}$ such that $F \notin \mathscr{S}(d(x))$. Hence $d(x) \nsubseteq F$. Since $\mathscr{F}$ is a closed filter in $(X, \mathscr{U})$, there exists a decreasing closed set $A$ such that $A \in \mathscr{F}$ and $A \subseteqq \mathscr{F}$. Hence $d(x) \nsubseteq A$ and $x \notin A$. Therefore $i(x) \subseteqq X - A$ or $X - A \in \mathscr{S}(i(x))$. It follows that $X - A \in \mathfrak{G}$. Hence $A \cap (X - A) = \varnothing$. It is a contradiction. Similarly in the case that $\mathscr{S}(i(x)) \subsetneqq \mathfrak{G}$, we have a contradiction. Therefore $(\mathscr{S}(d(x)), \mathscr{S}(i(x)))$ is a maximal bi-filter.

In what follows, we assume that $(X, \mathscr{T}, \leqq)$ is a convex ordered topological space with a semi-closed order. Let $\omega_0(X)$ be the collection of all maximal bi-filters $(\mathscr{F}, \mathfrak{G})$ on $X$. For given closed decreased set $A$, and closed increasing set $B$ in $X$, define

$$A^d = \{(\mathscr{F}, \mathfrak{G}) \in \omega_0(X) \colon A \in \mathscr{F}\} ,$$
$$B^i = \{(\mathscr{F}, \mathfrak{G}) \in \omega_0(X) \colon B \in \mathfrak{G}\} .$$

Then it is easy to see that $\{A^d \colon A \in \Gamma_{\mathscr{U}}X\}$ forms a closed base for a topology, say $\mathscr{W}_{\mathscr{U}}$, on $\omega_0(X)$. Similarly, the family $\{B^i \colon B \in \Gamma_{\mathscr{L}}X\}$ forms a closed base for a topology, say $\mathscr{W}_{\mathscr{L}}$, on $\omega_0(X)$. Let $\mathscr{W}$ be the smallest topology containing $\mathscr{W}_{\mathscr{U}}$ and $\mathscr{W}_{\mathscr{L}}$. Then every basic open set $(\omega_0(X), \mathscr{W})$ can be written in the form $\omega_0(X) - (A^d \cup B^i)$ for some $A \in \Gamma_{\mathscr{U}}X$ and some $B \in \Gamma_{\mathscr{L}}X$. We also note that $(A_1 \cap A_2)^d = A_1^d \cap A_1^d$ for $A_1$, $A_2$ in $\Gamma_{\mathscr{U}}X$ and $(B_1 \cap B_2)^d = B_1^d \cap B_2^d$ for $B_1$, $B_2$ in $\Gamma_{\mathscr{L}}X$. We define an order relation $\leqq$ on $\omega_0(X)$ as follows:

$(\mathscr{F}_1, \mathfrak{G}_1) \leqq (\mathscr{F}_2, \mathfrak{G}_2)$ if and only if $\mathscr{F}_1 \supseteqq \mathscr{F}_2$ and $\mathfrak{G}_1 \subseteqq \mathfrak{G}_2$. Then obviously $\leqq$ is a partial order on $\omega_0(X)$. Hence $(\omega_0(X), \mathscr{W}, \leqq)$ is an ordered topological space.

REMARK 2. Let $(\omega_0(X), \mathscr{W}, \leqq)$ be the ordered topological space obtained in the above. Let $A \in \Gamma_{\mathscr{U}}X$ and $B \in \Gamma_{\mathscr{L}}X$. Then $A^d$ is a closed decreasing set and $B^i$ is a closed increasing set in $\omega_0(X)$. Moreover, $\omega_0(X)$ is a convex ordered topological space.

LEMMA 4. *Let $(X, \mathscr{T}, \leqq)$ be a convex ordered topological space with a semi-closed order. Then the map $\Phi: X \to \omega_0(X)$ defined by $\Phi(x) = (\mathscr{S}(d(x)), \mathscr{S}(i(x)))$ for any $x \in X$ is a dense embedding into $(\omega_0(X), \mathscr{W}, \leqq)$.*

*Proof.* First, we show that $\Phi$ is an order isomorphism into $\omega_0(X)$. To show that $\Phi$ is one to one, let $x \neq y$ in $X$. Then $x \not\leqq y$ or $y \not\leqq x$. If $x \not\leqq y$ then $y \notin i(x)$ or $i(y) \not\subseteq i(x)$. It follows that $i(x) \notin \mathscr{S}(i(y))$ or $\mathscr{S}(i(x)) \not\subseteq \mathscr{S}(i(y))$. Hence $(\mathscr{S}(d(x)), \mathscr{S}(i(x)) \neq (\mathscr{S}(d(y)), \mathscr{S}(i(y))))$. Similarly, if $y \not\leqq x$ then $\Phi(x) \neq \Phi(y)$. Clearly, $\Phi$ is increasing. It is also immediate that if $\Phi(x) \leqq \Phi(y)$, then $x \leqq y$. Hence $\Phi$ is an order isomorphism into $\omega_0(X)$. Secondly, we show that $\Phi$ is a dense homeomorphism from $X$ into $\Phi(X)$. We observe the following: For a given closed decreasing set $A$,

$$\begin{aligned} A^d \cap \Phi(X) &= \{(\mathscr{S}(d(x)), \mathscr{S}(i(x))): A \in \mathscr{S}(d(x))\} \\ &= \{\Phi(x): x \in A\} = \Phi(A) . \end{aligned}$$

Similarly, for a given closed increasing set $B$, $B^i \cap \Phi(X) = \Phi(B)$. Since $X$ is a convex ordered topological space, $\Phi$ is evidently a homeomorphism from $X$ onto $\Phi(X)$.

To show that $\Phi(X)$ is a dense subset of $\omega_0(X)$, let $\omega_0(X) - (A^d \cup B^i)$ be a nonempty basic open set, where $A \in \Gamma_{\mathscr{U}} X$ and $B \in \Gamma_{\mathscr{L}} X$. Then there exists a maximal bi-filter $(\mathscr{F}, \mathfrak{G})$ such that $(\mathscr{F}, \mathfrak{G}) \in \omega_0(X) - (A^d \cup B^i)$. It follows that $(\mathscr{F}, \mathfrak{G}) \notin A^d$ and $(\mathscr{F}, \mathfrak{G}) \notin B^i$. Hence $A \notin \mathscr{F}$ and $B \notin \mathfrak{G}$. By Lemma 2, $A \cup B \neq X$. Therefore $(X - A) \cap (X - B) \neq \emptyset$. Let $y \in (X - A) \cap (X - B)$. Then it is easy to show that $\Phi(y) \in \omega_0(X) - (A^d \cup B^i)$. Hence $\Phi(X) \cap (\omega_0(X)) - (A^d \cup B^i) \neq \emptyset$. Hence $\Phi(X)$ is a dense subset of $\omega_0(X)$. This completes the proof.

LEMMA 5. *$(\omega_0(X), \mathscr{W}, \leqq)$ is a $T_1$-compact ordered space.*

*Proof.* First, we show that $\omega_0(X)$ is a $T_1$-space. Suppose that $(\mathscr{F}_1, \mathfrak{G}_1) = (\mathscr{F}_2, \mathfrak{G}_2)$ in $\omega_0(X)$. Without loss of generality we may assume that $\mathscr{F}_1 \not\subseteq \mathscr{F}_2$. Then there exists an $F_1 \in \mathscr{F}_1$ such that $F_1 \notin \mathscr{F}_2$. Since $\mathscr{F}_1$ is a closed filter in $(X, \mathscr{U})$, there exists a closed decreasing set $A_1$ such that $A_1 \in \mathscr{F}_1$ and $A_1 \subseteq F_1$. Hence $A_1 \notin \mathscr{F}_2$. It follows that $(\mathscr{F}_1, \mathfrak{G}_1) \in A_1^d$ and $(\mathscr{F}_2, \mathfrak{G}_2) \notin A_1^d$. Therefore $\omega_0(X) - A_1^d$ is an open neighborhood of $(\mathscr{F}_2, \mathfrak{G}_2)$ in $\omega_0(X)$ such that $(\mathscr{F}_1, \mathfrak{G}_1) \notin \omega_0(X) - A_1^d$. Since $\mathscr{F}_1 \not\subseteq \mathscr{F}_2$, we may consider the following two cases:

*Case* 1. $\mathscr{F}_2 \not\subseteq \mathscr{F}_1$: By the same method as before, there exists an open neighborhood of $(\mathscr{F}_1, \mathfrak{G}_1)$, which does not contain $(\mathscr{F}_2, \mathfrak{G}_2)$.

*Case* 2. $\mathscr{F}_2 \subseteq \mathscr{F}_1$; then $\mathfrak{G}_2 \nsubseteq \mathfrak{G}_1$. Hence there exists a closed incleasing set $B_2$ such that $B_2 \in \mathfrak{G}_2$ and $B_2 \notin \mathfrak{G}_1$ It follows that $(\mathscr{F}_2, \mathfrak{G}_2) \in B_2^i$ and $(\mathscr{F}_1, \mathfrak{G}_1) \notin B_2^i$. Therefore, $\omega_0(X) - B_2^i$ is an open neighborhood of $(\mathscr{F}_1, \mathfrak{G}_1)$ in $\omega_0(X)$, which does not contain $(\mathscr{F}_2, \mathfrak{G}_2)$. Hence $\omega_0(X)$ is a $T_1$-space.

Now we show that $\omega_0(X)$ is a compact space. Let $\{A_\alpha^d, B_\beta^i: \alpha \in \Gamma, \beta \in \Delta\}$ be a family of subbasic closed sets having a finite intersection property. Since $A_\alpha^d \cap B_\beta^i \neq \emptyset$ implies $A_\alpha \cap B_\beta \neq \emptyset$, $\{A_\alpha, B_\beta: \alpha \in \Gamma, \beta \in \Delta\}$ has a finite intersection property. Let $\mathscr{A}$ be the filter generated by $\{A_\alpha: \alpha \in \Gamma\}$ and $\mathscr{B}$ be the filter generated by $\{B_\beta: \beta \in \Delta\}$. Then $(\mathscr{A}, \mathscr{B})$ is obviously a bi-filter, and hence there exists a maximal bi-filter $(\mathscr{F}, \mathfrak{G})$ containing $(\mathscr{A}, \mathscr{B})$. It follows that $A_\alpha \in \mathscr{F}$ and $B_\beta \in \mathfrak{G}$ for all $\alpha \in \Gamma$ and all $\beta \in \Delta$. Therefore $(\mathscr{F}, \mathfrak{G}) \in A_\alpha^d$ and $(\mathscr{F}, \mathfrak{G}) \in B_\beta^i$. That is, $(\mathscr{F}, \mathfrak{G}) \in A_\alpha^d \cap B_\beta^i$ for all $\alpha$ and all $\beta$. It follows that $(\mathscr{F}, \mathfrak{G}) \in \bigcap_{\alpha,\beta} (A_\alpha^d \cap B_\beta^i)$. Hence $(\omega_0(X), \mathscr{W})$ is compact.

By Lemmas 4 and 5, we have the following theorem:

THEOREM 1. *Let $(X, \mathscr{T}, \leqq)$ be a convex ordered topological space with a semi-closed order. Then $(\omega_0(X), \mathscr{W}, \leqq)$ is a $T_1$-compact ordered space in which $X$ is densely embedded.*

REMARK 3. In the proof of Lemma 5, we see that $(\omega_0(X), \mathscr{W}, \leqq)$ is an ordered topological space which has either a lower semi-closed order or an upper semi-closed order. We note that a compact ordered space with a lower semi-closed order need not have a semi-closed order. For example, let $Z^+$ be the set of all natural numbers with the usual ordering and the cofinite topology. Then obviously $Z^+$ is compact and its order is lower semi-closed. But its order is not a semi-closed order because it is not upper semi-closed. In particular, this shows that a $T_1$-compact ordered space need not have a semi-closed order. We also note that if the given order on $X$ in Theorem 1 is discrete, then it reduces to the Wallman compactification of $(X, \mathscr{T})$ in the general topology.

Let $(X, \mathscr{T}, \leqq)$ be an ordered topological space with a semi-closed order and $(Y, \mathscr{T}', \leqq')$ a compact ordered space with a closed order, and let $f: X \to Y$ be a continuous increasing map. Define $\mathscr{F}^*$ to be the filter generated by a family $\{A$ is a closed decreasing set in $Y: f^{-1}(A) \in \mathscr{F}\}$, and $\mathfrak{G}^*$ to be the filter generated by a family $\{B$ is a closed increasing set in $Y: f^{-1}(B) \in \mathfrak{G}\}$.

LEMMA 6. *Under the above assumption, $(\mathscr{F}^*, \mathfrak{G}^*)$ is a bi-filter on $Y$ and there exists a unique point $y$ in $Y$ such that $y \in \cap\{F \cap G: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\}$.*

*Proof* It is straightforward that $(\mathscr{F}^*, \mathfrak{G}^*)$ is a bi-filter in $Y$. Since $Y$ is compact, $\{F \cap G: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\}$ has a limit point $y$, that is,

$$\begin{aligned} y &\in \cap \{\overline{F \cap G}: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\} \\ &\subseteqq \cap \{A \cap B: A \in \mathscr{B}_{\mathscr{F}^*}, B \in \mathscr{B}_{\mathfrak{G}^*}\} \\ &\subseteqq \cap \{F \cap G: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\} \end{aligned}$$

where $\mathscr{B}_{\mathscr{F}^*}$ is a filter base for $\mathscr{F}^*$ consisting only of decreasing closed sets, and $\mathscr{B}_{\mathfrak{G}^*}$ is a filter base for $\mathfrak{G}^*$ consisting only of increased closed sets. Hence there exists a $y$ in $Y$ such that $y \in \cap \{F \cap G: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\}$. In order to show the uniqueness of $y$, suppose that there exist $x \neq y$ in $Y$ such that $x$ and $y$ are elements of $\cap \{F \cap G: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\}$. Then we may assume that $x \nleqq y$. Hence $i(x) \cap d(y) = \varnothing$. Since $Y$ is a compact ordered space with a closed order, there exists an open increasing neighborhood $U$ of $x$ and an open decreasing neighborhood $V$ of $y$ such that $U \cap V = \varnothing$. Hence $(Y - U) \cup (Y - V) = Y$, and hence $f^{-1}(Y - U) \cup f^{-1}(Y - V) = X$. Since $f$ is a continuous increasing map, $f^{-1}(Y - U) \in \mathscr{F}$ or $f^{-1}(Y - V) \in \mathfrak{G}$ by Lemma 2. By the definition of $\mathscr{F}^*$ and $\mathfrak{G}^*$, $(Y - U) \in \mathscr{F}^*$ or $(Y - V) \in \mathfrak{G}^*$. If $(Y - U) \in \mathscr{F}^*$, then $x \in Y - U$, and hence $x \notin U$, which contradicts the fact that $x \in U$. Similarly, in the case that $(Y - V) \in \mathfrak{G}^*$, we have a contradiction. Hence $x = y$.

THEOREM 2. *Let $(X, \mathscr{T}, \leqq)$ be a convex ordered topological space with a semi-closed order, and $(Y, \mathscr{T}', \leqq')$ a compact ordered space with a closed order. For a continuous increasing map $f: X \to Y$, there exists a unique continuous increasing map $\bar{f}$ from $\omega_0(X)$ into $Y$ such that $\bar{f} \circ \Phi = f$, where $\Phi$ is the embedding: $X \to \omega_0(X)$.*

*Proof.* For given $(\mathscr{F}, \mathfrak{G}) \in \omega_0(X)$, let $\mathscr{F}^*$ and $\mathfrak{G}^*$ be the filters given as before. By Lemma 6, there exists a unique point $y \in \cap \{F \cap G: F \in \mathscr{F}^*, G \in \mathfrak{G}^*\}$. We show that the map $\bar{f}: \omega_0(X) \to Y$ defined $\bar{f}(\mathscr{F}, \mathfrak{G}) = y$ is the required map. Indeed, (1): $\bar{f} \circ \Phi = f$; let $x$ be any point of $X$. It is easy to see that $[\mathscr{S}(d(x))]^* = \mathscr{S}(d(f(x)))$ and $[\mathscr{S}(i(y))]^* = \mathscr{S}(i(f(x)))$. Hence $([\mathscr{S}(d(x))]^*, [\mathscr{S}(i(x))]^*) = (\mathscr{S}(d(f(x))), \mathscr{S}(i(f(x))))$. It follows that $(\bar{f} \circ \Phi)(x) = \bar{f}((\mathscr{S}(d(x)), \mathscr{S}(i(x)))) = f(x)$. (2): $\bar{f}$ is a continuous map: Since $\omega_0(X)$ and $Y$ are convex ordered spaces, it is sufficient to show that $\bar{f}$ is continuous from $(\omega_0(X), \mathscr{W}_{\mathscr{U}})$ into $(Y, \mathscr{L})$. For a fixed point $(\mathscr{F}, \mathfrak{G}) \in \omega_0(X)$, let $U$ be an open decreasing neighborhood of $\bar{f}((\mathscr{F}, \mathfrak{G}))$ in $Y$. Then $Y - U$ is a closed increasing set, which does not contain $\bar{f}((\mathscr{F}, \mathfrak{G}))$.

Thus $d(\bar{f}((\mathscr{F}, \mathfrak{G}))) \cap (Y - U) = \varnothing$. Let $W$ be an open decreasing set and $V$ an open increasing set such that $d(\bar{f}((\mathscr{F}, \mathfrak{G}))) \subseteqq W$, $Y - U \subseteqq V$ and $W \cap V = \varnothing$. Then $(Y - W) \cup (Y - V) = Y$. Therefore $f^{-1}(Y - W) \cup f^{-1}(Y - V) = X$. Furthermore, $[f^{-1}(Y - W)]^i \cup [f^{-1}(Y - V)]^d = \omega_0(X)$. Since $\bar{f}((\mathscr{F}, \mathfrak{G})) \notin Y - W$, $(\mathscr{F}, \mathfrak{G}) \notin [f^{-1}(Y - W)]^i$. Hence $\omega_0(X) - [f^{-1}(Y - W)]^i$ is an open decreasing neighborhood of $(\mathscr{F}, \mathfrak{G})$ in $(\omega_0(X), \mathscr{W}_{\mathscr{U}})$. And clearly, $\bar{f}(\omega_0(X) - [f^{-1}(Y - W)]^i) \subseteqq U$. Therefore $\bar{f}$ is continuous from $(\omega_0(X), \mathscr{W}_{\mathscr{U}})$ into $(Y, \mathscr{L})$. Dually, $\bar{f}$ is continuous from $(\omega_0(X), \mathscr{W}_{\mathscr{U}})$ into $(Y, \mathscr{U})$. Finally, (3): $\bar{f}$ is an increasing map: Suppose that $(\mathscr{F}_1, \mathfrak{G}_1) \leqq (\mathscr{F}_2, \mathfrak{G}_2)$ and $\bar{f}((\mathscr{F}_1, \mathfrak{G}_1)) \nleq \bar{f}((\mathscr{F}_2, \mathfrak{G}_2))$. Since $Y$ is a compact ordered space with a closed order, there exists an open increasing neighborhood $U$ of $\bar{f}((\mathscr{F}_1, \mathfrak{G}_1))$ and an open decreasing neighborhood $V$ of $\bar{f}((\mathscr{F}_2, \mathfrak{G}_2))$ such that $U \cap V = \varnothing$. Thus $\bar{f}((\mathscr{F}_1, \mathfrak{G}_1)) \notin V$. Since $\bar{f}$ is continuous from $(\omega_0(X), \mathscr{W}_{\mathscr{U}})$ into $(Y, \mathscr{L})$, there exists a closed increasing set $A$ in $X$ such that $\omega_0(X) - A^i$ is an open decreasing set containing $(\mathscr{F}_2, \mathfrak{G}_2)$ and $\bar{f}(\omega_0(X) - A^i) \subseteqq V$. Since $(\mathscr{F}_1, \mathfrak{G}_1) \leqq (\mathscr{F}_2, \mathfrak{G}_2)$, $(\mathscr{F}_1, \mathfrak{G}_1) \in \omega_0(X) - A^i$. It follows that $\bar{f}((\mathscr{F}_1, \mathfrak{G}_1)) \in V$, which contradicts the fact that $\bar{f}((\mathscr{F}_1, \mathfrak{G}_1)) \notin V$. Therefore $\bar{f}((\mathscr{F}_1, \mathfrak{G}_1)) \leqq \bar{f}((\mathscr{F}_2, \mathfrak{G}_2))$. In particular, the uniqueness of $\bar{f}$ is straightforward (see [7], page 97, Theorems 14, 19).

THEOREM 3. *Let $(X, \mathscr{T}, \leqq)$ be a compact convex ordered space with a semi-closed order. Then $(X, \mathscr{T}, \leqq)$ is isomorphic with $(\omega_0(X), \mathscr{W}, \leqq)$.*

*Proof.* Let $(\mathscr{F}, \mathfrak{G})$ be a maximal bi-filter on $X$. Then $\{F \cap G: F \in \mathscr{F}, G \in \mathfrak{G}\}$ has a limit point, say $x$, in $X$. It follows that $\{x\} \subseteqq \cap \{A \cap B: A \in \mathscr{B}_{\mathscr{F}}, B \in \mathscr{B}_{\mathfrak{G}}\}$, where $\mathscr{B}_{\mathscr{F}}$ and $\mathscr{B}_{\mathfrak{G}}$ are closed bases of $\mathscr{F}$ in $(X, \mathscr{U})$ and $\mathfrak{G}$ in $(X, \mathscr{L})$ respectively. Since $X$ has a semi-closed order, we have $(\mathscr{F}, \mathfrak{G}) \subseteqq (\mathscr{S}(d(x)), \mathscr{S}(i(x)))$. By the maximality of $(\mathscr{F}, \mathfrak{G})$, $(\mathscr{F}, \mathfrak{G}) = (\mathscr{S}(d(x)), \mathscr{S}(i(x)))$. Hence $\Phi(X) = \omega_0(X)$, that is, $(X, \mathscr{T}, \leqq)$ is iseomorphic with $(\omega_0(X), \mathscr{W}, \leqq)$.

We recall that an ordered topological space $(X, \mathscr{T}, \leqq)$ is *normally ordered* if, for every two disjoint subsets $A$, $B$ of $X$, where $A$ is a decreasing closed set and $B$ is an increasing closed set, there exist two disjoint open sets $U$ and $V$ such that $U$ contains $A$ and is decreasing, and $V$ contains $B$ and is increasing [5].

THEOREM 4. *Let $(X, \mathscr{T}, \leqq)$ be a convex ordered topological space with a semi-closed order. If $\omega_0(X)$ has a closed order, then $X$ is a normally ordered space.*

*Proof.* Clearly, $\omega_0(X)$ is a normally ordered space. Let $A$ and $B$ be two disjoint subsets of $X$, where $A$ is a decreasing closed set and $B$ is an increasing closed set. Thus $A^d \cap B^i = \varnothing$. Since $\omega_0(X)$ is normally ordered, there exists an open decreasing set $W$ and an open increasing set $W'$ in $\omega_0(X)$ such that $A^d \subseteqq W$, $B^i \subseteqq W'$ and $W \cap W' = \varnothing$. Further, $W$ and $W'$ could be written in the form: $W = \bigcup_j (\omega_0(X) - B_j^i)$ and $W' = \bigcup_j (\omega_0(X) - A_j^d)$, where $B_j$ in $\Gamma_{\mathscr{U}} X$ and $A_j$ in $\Gamma_{\mathscr{L}} X$. Since $A^d$ and $B^i$ are compact, $A^d \subseteqq \bigcup_{j=1}^n (\omega_0(X) - B_j^i) = \omega_0(X) - \bigcap_{j=1}^n B_j^i = \omega_0(X) - (\bigcap_{j=1}^n B_j)^i$. Similarly, $B^i \subseteqq \omega_0(X) - (\bigcap_{j=1}^m A_j)^d$. Let $U = X - (\bigcap_{j=1}^n B_j)$ and $V = X - (\bigcap_{j=1}^m A_j)$. Then $U$ is an open decreasing set and $V$ is an open increasing set. Let $x \in A$. Then $d(x) \subseteqq A$, and hence $(\mathscr{S}(d(x)), \mathscr{S}(i(x))) \in A^d$. Since $A^d \subseteqq \omega_0(X) - (\bigcap_{j=1}^m B_j)^i$, $(\mathscr{S}(d(x)), \mathscr{S}(i(x))) \notin (\bigcap_{j=1}^n B_j)^i$. It follows that $\bigcap_{j=1}^n B_j \notin \mathscr{S}(i(x))$. Hence $i(x) \nsubseteqq \bigcap_{j=1}^n B_j$. Therefore $x \in X - \bigcap_{j=1}^n B_j$. Hence $A \subseteqq U$. Similarly, $B \subseteqq V$. Since $[\omega_0(X) - (\bigcap_{j=1}^n B_j)^i] \cap [\omega_0(X) - (\bigcap_{j=1}^m A_j)^d] = \varnothing$, we have $U \cap V = \varnothing$. Hence $X$ is a normally ordered space.

REMARK 4. If the given order on $X$ is discrete, then the previous results reduce the corresponding results in the general topology. However, we do not know whether the converse of Theorem 4 is true. We finally note that, in [2], a compact ordered space $\beta_0 X$ with a closed order for a completely regular ordered space $X$ is constructed. It immediately follows that given the following diagram:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\beta_0"'] & \omega_0(X) \arrow[dl, dashed, "\bar{\beta}_0"] \\ \beta_0 X & \end{tikzcd}$$

there exists a continuous increasing map $\bar{\beta}_0$ from $\omega_0(X)$ onto $\beta_0(X)$ such that $\bar{\beta}_0 \circ \Phi = \beta_0$. Furthermore, if $\omega_0(X)$ has a closed order, $\beta_0 X$ and $\omega_0(X)$ are iseomorphic under $\bar{\beta}_0$ such that the above diagram commutes.

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