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A well-known theorem of Hopkins and Levitzki states that any left artinian ring with identity element is left noetherian. The main theorem of this paper generalizes this to the situation of a hereditary torsion theory with associated idempotent kernel functor σ . It is shown that if a ring R with identity element has the descending chain condition on σ -closed left ideals, then R has the ascending chain condition on σ -closed left ideals.

The remainder of the paper generalizes some results of Faith and Walker concerning artinian and quasi-Frobenius rings. In the case that the localization functor \mathcal{L}_σ is exact the following are obtained: (1) a sufficient condition for the ring R to have the descending chain condition on σ -closed left ideals and (2) characterizations of the condition that every σ -torsion-free injective left R -module is codivisible (projective).

In this paper R always denotes ring with identity element, and unless denoted to the contrary, all modules are members of the category $R\text{-mod}$ of unital left R -modules.

A subfunctor σ of the identity functor on $R\text{-mod}$ is called a *left exact radical* (or *idempotent kernel functor*) if σ is left exact and $\sigma(M/\sigma(M))=0$ for every module M . Such a σ naturally determines a torsion class $\mathcal{T}_\sigma = \{M | \sigma(M) = M\}$ and a torsion-free class $\mathcal{F}_\sigma = \{M | \sigma(M) = 0\}$. The pair $(\mathcal{T}_\sigma, \mathcal{F}_\sigma)$ forms a hereditary torsion theory in the sense of [2], [10], [13], [14] and [15]. Then \mathcal{T}_σ is closed under submodules, homomorphic images, direct sums, and extensions of one member of \mathcal{T}_σ by another; and \mathcal{F}_σ is closed under submodules, direct products, injective hulls, and extensions of one member of \mathcal{F}_σ by another. Also associated with σ is the *localization functor* \mathcal{L}_σ as defined in [2], [4], [13] or [14]. The module $\mathcal{L}_\sigma(R)$ can be made into ring by defining multiplication in a natural way; this ring will be denoted by Q_σ . A torsion theory is called *perfect* in [2], [12] and [13] if $\mathcal{L}_\sigma(M) \cong Q_\sigma \otimes_R M$ for every module M . For additional details on the concepts discussed in this paragraph, the reader is referred to [2], [4], [9], [10], [13], [14], and their references.

A submodule N of M is called *σ -closed* if $M/N \in \mathcal{F}_\sigma$. The lattice of σ -closed submodules has been studied in [3], [5], [9], [12], [14], and [15]. Particular attention is usually given to chain conditions on σ -closed modules. We continue this investigation and focus our

attention on the descending chain condition for σ -closed submodules of ${}_R R$ (i.e., the σ -closed left ideal).

A well-known theorem of Hopkins [6] and Levitzki [8] states that any left artinian ring with identity element is left noetherian. Manocha [9] has generalized this result by showing that if σ is perfect and if R has the descending chain condition (dcc) on σ -closed left ideals, then R has the ascending chain condition (acc) on σ -closed left ideals. The main result of the first section (Theorem 1.4) removes the very restrictive hypothesis that σ is perfect from Manocha's result. Proofs of the result of Hopkins and Levitzki all seem to depend strongly on the nilpotence of the (Jacobson, or nil) radical; Manocha's proof uses the Hopkins-Levitzki result on Q_σ and depends only on a lattice isomorphism between the σ -closed left ideals of R and the left ideals of Q_σ (a consequence of σ being perfect). In our case where there is no restriction on σ , we can rely neither on nilpotence nor on a lattice isomorphism; our method of proof will depend on finding a substitute for actual nilpotence of the (Jacobson) radical and applying Goldman's results on modules of σ -finite length [5].

In the second section we generalize some results of Faith and Walker [1] to obtain a sufficient condition for R to have dcc on σ -closed left ideals when \mathcal{L}_σ is exact. In particular, we show in Theorem 2.3 that if \mathcal{L}_σ is exact and if each module in \mathcal{F}_σ is contained in a direct sum of finitely generated modules, then R has dcc on σ -closed left ideals.

In the third section we apply the results of the first two sections to answer the following question in the case where \mathcal{L}_σ is exact: For which σ is every injective module in \mathcal{F}_σ projective? Our answer to this question (given in Theorems 3.5 and 3.6) gives a generalization of an important theorem of Faith and Walker [1, Theorem 5.3] on quasi-Frobenius rings.

1. DCC implies ACC. In this section we show that if R has dcc on σ -closed left ideals, then R also has acc on σ -closed left ideals. In order to do this we first recall two definitions from [2] and [3]. A nonzero module M is σ -cocritical if $M \in \mathcal{F}_\sigma$ and every proper homomorphic image of M is in \mathcal{F}_σ . Nonzero submodules of σ -cocritical modules are σ -cocritical modules. If M is a nonzero module in \mathcal{F}_σ and if M has dcc on σ -closed submodules, then M contains a σ -cocritical submodule. A submodule N of a module M is called σ -critical if M/N is σ -cocritical. Thus a submodule N of a module M in \mathcal{F}_σ is σ -critical if and only if N is maximal among the proper σ -closed submodules of M . If there exist σ -cocritical modules, then there exist cyclic σ -cocritical modules; so we may define

$$V = \cap \{I \mid I \subseteq R, I \text{ } \sigma\text{-critical}\} .$$

Then V is σ -closed, and V is a proper two-sided ideal of R . If N is a σ -cocritical module, then $VN = 0$. We continue to use V as a standard notation in this section.

Our first lemma is an analogue of the fact that, in a left artinian ring, the Jacobson radical is nilpotent.

LEMMA 1.1. *If R has dcc on σ -closed left ideals, then there exists a positive integer n such that $V^{n+q}/V^{n+q+1} \in \mathcal{F}_\sigma$ for all $q \geq 0$.*

Proof. Suppose not. Then there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that each $V^{n_i}/V^{n_i+1} \notin \mathcal{F}_\sigma$. Let $T_{n_i}/V^{n_i+1} = \sigma(V^{n_i}/V^{n_i+1})$. Choose a left ideal M_i of R containing T_{n_i} which is maximal with respect to the property that $M_i \cap V^{n_i} = T_{n_i}$. Via the natural map $R/T_{n_i} \rightarrow R/M_i$ we see that $V^{n_i}/T_{n_i} \in \mathcal{F}_\sigma$ is isomorphic to an essential submodule of R/M_i ; hence M_i is a σ -closed left ideal of R . For each positive integer j , let $N_j = \bigcap_{i=1}^j M_i$. Since intersections of σ -closed submodules are always σ -closed, then N_j is σ -closed. Now $N_j \supseteq T_{n_j} \supseteq V^{n_{j+1}} \supseteq V^{n_{j+1}}$. Furthermore $N_{j+1} \not\supseteq V^{n_{j+1}}$; for if $V^{n_i} \subseteq N_i \subseteq M_i$, then $V^{n_i}/V^{n_i+1} = T_{n_i}/V^{n_i+1} \in \mathcal{F}_\sigma$, which is contrary to the choice of the n_i 's. Therefore, for each positive integer j , $N_j \neq N_{j+1}$, and we have an infinite, strictly descending chain $\{N_j\}$ of σ -closed left ideals of R . This contradicts our hypothesis that R has dcc on σ -closed left ideals.

In [5] a module M is said to have σ -finite length if there exists a finite chain

$$(*) \quad 0 = M_n \subset M_{n-1} \subset M_{n-2} \subset \dots \subset M_0 = M$$

of submodules of M such that M_i/M_{i+1} is σ -cocritical for each $i = 0, 1, 2, \dots, n - 1$; we call the chain $(*)$ a σ -composition series of M . In [5] it is shown that (1) any two σ -composition series of a module of σ -finite length have the same number of terms and that (2) a module M has σ -finite length if and only if M has both acc and dcc on σ -closed submodules.

Our next lemma may be viewed as a specialization of [5, Proposition 2.10] and [3, Proposition 2.1(3)].

LEMMA 1.2. *Let M be a module for which 0 is an intersection of finitely many σ -critical submodules of M . Then there exist σ -cocritical submodules N_1, N_2, \dots, N_k of M such that $\sum_{i=1}^k N_i$ is an essential direct submodule of M and $M/(\sum_{i=1}^k N_i) \in \mathcal{F}_\sigma$.*

Proof. By hypothesis M is isomorphic to a submodule of a

direct sum of finitely many σ -cocritical modules. Since this direct sum clearly has σ -finite length, then by [5, Corollary 1.5 and Proposition 1.2] M has both acc and dcc on σ -closed submodules. We now use induction to choose the desired modules N_i .

Since M has dcc on σ -closed submodules, we can choose a σ -cocritical submodule N_1 of M . Let $0 \neq x \in N_1$. There exists a σ -critical submodule C_1 such that $x \notin C_1$ by hypothesis. Now $0 \neq N_1/(N_1 \cap C_1) \cong (C_1 + N_1)/C_1 \in \mathcal{S}_\sigma$; so, since N_1 is σ -cocritical $N_1 \cap C_1 = 0$. Suppose that N_1, N_2, \dots, N_t and C_1, C_2, \dots, C_t have been chosen such that N_i is σ -cocritical, C_i is σ -critical, $N_i \subseteq \bigcap_{j \leq i-1} C_j$, and $N_i \cap C_i = 0$ for each $i \leq t$. If $\bigcap_{j=1}^t C_j \neq 0$, then we can choose N_{t+1} to be a σ -cocritical submodule of $\bigcap_{j=1}^t C_j$. As in the discussion of case N_1 , we can find a σ -critical submodule C_{t+1} of M such that $N_{t+1} \cap C_{t+1} = 0$. Since M has dcc on σ -closed submodules, there exists an integer k such that $\bigcap_{j=1}^k C_j = 0$; so the inductive process stops after k steps. It follows from the construction that $\sum_{i=1}^k N_i$ is direct.

It remains to show that $M/(\sum_{i=1}^k N_i) \in \mathcal{S}_\sigma$. To do this it is sufficient to show by induction that $M/((\sum_{i=1}^t N_i) + (\bigcap_{i=1}^t C_i)) \in \mathcal{S}_\sigma$ for each $t = 1, 2, \dots, k$. Since M/C_1 is σ -cocritical and $(N_1 + C_1)/C_1 \neq 0$, then $M/(C_1 + N_1) \in \mathcal{S}_\sigma$; so the first case is established. We now assume that the result is true for all integers $< t$. Since $0 \neq (N_t + C_t)/C_t \subseteq ((\bigcap_{i=1}^{t-1} C_i) + C_t)/C_t$ and M/C_t is σ -cocritical, then

$$(**) \quad \left(\left(\bigcap_{i=1}^{t-1} C_i \right) + C_t \right) / (N_t + C_t) \in \mathcal{S}_\sigma.$$

Since $N_t \subseteq \bigcap_{i=1}^{t-1} C_i$, for each $x \in \bigcap_{i=1}^{t-1} C_i$ we obtain $(N_t + \bigcap_{i=1}^t C_i : x) = ((N_t + C_t) \cap (\bigcap_{i=1}^{t-1} C_i) : x) = ((N_t + C_t) : x)$. Thus by (**) we obtain $(\bigcap_{i=1}^{t-1} C_i) / (N_t + \bigcap_{i=1}^t C_i) \in \mathcal{S}_\sigma$. Since \mathcal{S}_σ is closed under homomorphic images, we have $((\sum_{i=1}^{t-1} N_i) + (\bigcap_{i=1}^{t-1} C_i)) / ((\sum_{i=1}^t N_i) + (\bigcap_{i=1}^t C_i)) \in \mathcal{S}_\sigma$. Thus from the induction hypothesis and the exact sequence

$$0 \longrightarrow \frac{\sum_{i=1}^{t-1} N_i + \bigcap_{i=1}^{t-1} C_i}{\sum_{i=1}^t N_i + \bigcap_{i=1}^t C_i} \longrightarrow \frac{M}{\sum_{i=1}^t N_i + \bigcap_{i=1}^t C_i} \longrightarrow \frac{M}{\sum_{i=1}^{t-1} N_i + \bigcap_{i=1}^{t-1} C_i} \longrightarrow 0$$

we obtain $M/((\sum_{i=1}^t N_i) + (\bigcap_{i=1}^t C_i)) \in \mathcal{S}_\sigma$ as desired.

As an immediate consequence of Lemma 1.2, we have the following analogue for σ of the structure theorem for semisimple rings with dcc.

COROLLARY 1.3. *If R has dcc on σ -closed left ideals, then there exist σ -cocritical submodules $A_1/V, A_2/V, \dots, A_k/V$ such that $\sum_{i=1}^k (A_i/V)$ is a direct essential submodule of R/V and $(R/V) \oplus_{i=1}^k (A_i/V) \in \mathcal{S}_\sigma$.*

We can now obtain the main result of this section.

THEOREM 1.4. *Let R have dcc on σ -closed left ideals. If a module B has dcc on σ -closed submodules, then B also has acc on σ -closed submodules. In particular, R has acc on σ -closed left ideals.*

Proof. Let B be a module with dcc on σ -closed submodules. Let $I_0 = \sigma(B)$. For $j \geq 1$, define I_j by I_j/I_{j-1} is a minimal, nonzero, σ -closed submodule of B/I_{j-1} ; such an I_j exists whenever $0 \neq B/I_{j-1}$ (as $B/I_{j-1} \in \mathcal{F}_\sigma$ and has dcc on σ -closed submodules). Moreover, I_j/I_{j-1} is σ -cocritical by the minimality. It is sufficient to show that $I_s = B$ for some index s ; for then B/I_0 has acc on σ -closed submodules by [5, Proposition 1.2], and hence B has acc on σ -closed submodules (as the lattice of σ -closed submodules of B/I_0 is clearly isomorphic to the lattice of σ -closed submodules of B).

Assume for contradiction that $I_j \neq B$ for each $j \in \mathbb{Z}^+$, which \mathbb{Z}^+ denotes the set of positive integers. Set $m_0 = 0$, and let $m_{t+1} = \max \Gamma_t$, where $\Gamma_t = \{j \in \mathbb{Z}^+ \mid Vx \subseteq I_{m_t} \text{ for some } x \in I_j - I_{j-1}\}$. Note that $m_t + 1 \in \Gamma_t$ as $V(I_{m_t+1}/I_{m_t}) = 0$. Inductively, assume that m_t exists; we show via the next three paragraphs that m_{t+1} exists.

Suppose not. Then for an infinite set Ω of indices $j > m_t + 1$, we may choose $x_j \in I_j - I_{j-1}$ such that $Vx_j \subseteq I_{m_t}$. By Corollary 1.3 R/V contains an essential submodule of the form $\bigoplus_{i=1}^k (A_i/V)$, where each A_i/V is σ -cocritical and $(R/V)/\bigoplus_{i=1}^k (A_i/V) \in \mathcal{F}_\sigma$. If for each $i = 1, 2, \dots, k$ we have $A_i x_j \subseteq I_{m_t}$, then $(\sum_{i=1}^k A_i)x_j \subseteq I_{m_t}$; so $0 \neq (Rx_j + I_{m_t})/I_{m_t} \in \mathcal{F}_\sigma$. But $B/I_{m_t} \in \mathcal{F}_\sigma$ by construction, which yields a contradiction. Thus for at least one of the A_i/V , $A_i x_j \not\subseteq I_{m_t}$.

Next assume that, for any such A_i/V with $A_i x_j \not\subseteq I_{m_t}$, we have $(A_i x_j + I_{m_t}) \cap I_{j-1} \cong I_{m_t}$. Since $(A_i x_j + I_{m_t})/I_{m_t} \subseteq B/I_{m_t} \in \mathcal{F}_\sigma$ and A_i/V is σ -cocritical, we see that the natural epimorphism $A_i/V \rightarrow (A_i x_j + I_{m_t})/I_{m_t}$ is an isomorphism. Thus $(A_i x_j + I_{m_t})/I_{m_t}$ is σ -cocritical, and we have

$$\begin{aligned} & (A_i x_j + I_{m_t}) / ((A_i x_j + I_{m_t}) \cap I_{j-1}) \\ & \cong ((A_i x_j + I_{m_t})/I_{m_t}) / (((A_i x_j + I_{m_t}) \cap I_{j-1})/I_{m_t}) \in \mathcal{F}_\sigma \end{aligned}$$

by assumption. But we also have $(A_i x_j + I_{m_t}) / ((A_i x_j + I_{m_t}) \cap I_{j-1}) \cong (A_i x_j + I_{j-1}) / I_{j-1} \subseteq B/I_{j-1} \in \mathcal{F}_\sigma$. We conclude that $A_i x_j \subseteq I_{j-1}$ for any A_i/V with $A_i x_j \not\subseteq I_{m_t}$. Now for each of the remaining A_i/V , we have $A_i x_j \subseteq I_{m_t} \subseteq I_{j-1}$. Hence $(\sum_{i=1}^k A_i)x_j \subseteq I_{j-1}$, which leads to a contradiction as $R/\sum_{i=1}^k A_i \in \mathcal{F}_\sigma$ and $B/I_{j-1} \in \mathcal{F}_\sigma$.

We have now established that, for each $j \in \Omega$, there exists a left ideal A_j of R and an $x_j \in I_j - I_{j-1}$ such that $A_j x_j \not\subseteq I_{m_t}$,

$(A_jx_j + I_{m_t})/I_{m_t}$ is σ -cocritical, and $(A_jx_j + I_{m_t}) \cap I_{j-1} = I_{m_t}$. One easily checks that $\sum_{j \in \Omega} [(A_jx_j + I_{m_t})/I_{m_t}] \subseteq B/I_{m_t}$ is direct. Let $\Omega = \{j_1, j_2, \dots\}$. Set $M_1 = B$, and for $u > 1$ choose M_u maximal with respect to $M_u \supseteq \sum_{i=u}^{\infty} (A_{j_i}x_{j_i} + I_{m_t})$, $M_u \subseteq M_{u-1}$, and $M_u \cap (\sum_{i=1}^{u-1} A_{j_i}x_{j_i} + I_{m_t}) = I_{m_t}$. Then then set $\{M_u\}_{u=1}^{\infty}$ forms a strictly descending chain of σ -closed submodules of B , which contradicts our assumption that B has dcc on σ -closed submodules. Hence m_{t+1} exists.

Since $I_{m_{t+1}}/I_{m_t}$ is σ -cocritical, $V(I_{m_{t+1}}/I_{m_t}) = 0$; so for each $t > 0$, $m_{t+1} \geq m_t + 1 > m_t$. Hence the sequence $\{m_t\}_{t=1}^{\infty}$ is strictly increasing and infinite. By Lemma 1.1 there exists a positive integer n such that $V^{n+q}/V^{n+q+1} \in \mathcal{F}_{\sigma}$ for all $q \geq 0$. Let $x \in I_{m_{n+1}} - I_{m_n}$. Then $Vx \not\subseteq I_{m_{n-1}}$; thus we have $v_1x \notin I_{m_{n-1}}$ for some $v_1 \in V$. But $Vv_1x \not\subseteq I_{m_{n-2}}$. So we inductively obtain $v_2, v_3, \dots, v_{n-1} \in V$ such that $Vv_{n-i} \cdots v_2v_1x \not\subseteq I_{m_{n-i}}$ for each $i = 1, 2, \dots, n-1$. In particular, $Vv_{n-1}v_{n-2} \cdots v_2v_1x \not\subseteq I_{m_0} = I_0$; hence $V^n x \not\subseteq I_0$. However, since $x \in I_{m_{n+1}}$, we have that $V^{m_{n+1}}x \subseteq I_0$ as I_w/I_{w-1} is σ -cocritical for all $w \geq 1$. It follows that there exists an integer $d \geq n$ such that $V^d x \not\subseteq I_0$, but $V^{d+1}x \subseteq I_0$.

Now $(Rx + I_0/I_0)$ is the homomorphic image of R/V^{d+1} via $r + V^{d+1} \xrightarrow{\alpha} rx + I_0$. We note that $0 \neq \alpha(V^d/V^{d+1}) \subseteq B/I_0 \in \mathcal{F}_{\sigma}$. However, since $d \geq n$, $\alpha(V^d/V^{d+1}) \in \mathcal{F}_{\sigma}$. This contradicts the fact that $\mathcal{F}_{\sigma} \cap \mathcal{F}_{\sigma} = 0$. Hence $I_s = B$ for some s as desired.

2. Finitely generated injective modules in F_{σ} . In this section we study the relationship of finiteness conditions on injective hulls of cyclic modules and the dcc on σ -closed left ideals, where \mathcal{L}_{σ} is exact. We obtain generalizations of several results of Faith and Walker [1].

A module M is called σ -finitely generated if M has a finitely generated submodule N such that $M/N \in \mathcal{F}_{\sigma}$. Any finitely generated module is σ -finitely generated.

We use $E(M)$ to denote the injective hull of a module M , and we let ϕ_M be the natural homomorphism from M into $\mathcal{L}_{\sigma}(M)$ (see [2], [4], [13] or [14]). If σ is perfect, then the correspondence $K \rightarrow \mathcal{L}_{\sigma}(K)$ gives a lattice isomorphism from the lattice of σ -closed submodules K of M to the lattice of Q_{σ} -submodules of submodules of $\mathcal{L}_{\sigma}(M)$; the inverse isomorphism is given by $X \rightarrow \phi_M^{-1}(X)$ for each Q_{σ} -submodule X of $\mathcal{L}_{\sigma}(M)$ — see [2], [4], [13] or [14]. If \mathcal{L}_{σ} is exact and R has acc on σ -closed left ideals, then Q_{σ} must be a left noetherian ring by this lattice isomorphism (with $R = M$), and thus Q_{σ} will contain a maximal two-sided nilpotent ideal N .

THEOREM 2.1. *Let \mathcal{L}_{σ} be exact, and let R have acc on σ -closed*

left ideals. Let N be the maximal nilpotent ideal of Q_σ . If $E(R/\phi_R^{-1}(N))$ is σ -finitely generated, then R has dcc on σ -closed left ideals.

Proof. Let $N' = \phi_R^{-1}(N)$. Then N' is a nilpotent, two-sided ideal of R ; since σ is perfect, N' is also σ -closed and $N = \mathcal{L}_\sigma(N') = Q_\sigma \otimes_R N$. Let J be the injective hull of Q_σ/N as a Q_σ -module. Since σ is perfect, we have

$$R/N' \xrightarrow{\phi_{R/N'}} Q_\sigma \otimes_R (R/N') \cong Q_\sigma/Q_\sigma \otimes_R N' = Q_\sigma/N \subseteq J \subseteq E(R/N').$$

Since $E(R/N')$ is σ -finitely generated by hypothesis, then $E(R/N')$ has acc on σ -closed submodules by [9, Proposition 3.20]. Since σ is perfect, then $E(R/N')/J \in \mathcal{F}_\sigma$ by [2, Proposition 17.1], hence J also has acc on σ -closed R -submodules. Since every Q_σ -submodule of J is σ -closed as an R -submodule of J (as σ is perfect) then J has acc on Q_σ -submodules. Consequently J is finitely generated as a Q -module. By [1, Theorem 2.2] Q_σ is a left artinian ring. Thus R has dcc on σ -closed left ideals via the lattice isomorphism between the lattice of σ -closed left ideals of R and the lattice of left ideals of Q_σ .

COROLLARY 2.2. *Let \mathcal{L}_σ be exact, and let R have acc on σ -closed left ideals. If injective hulls of cyclic modules in \mathcal{F}_σ are finitely generated, then R has dcc on σ -closed left ideals.*

It is now easy to obtain the main result of this section.

THEOREM 2.3. *Let \mathcal{L}_σ be exact. If each module in \mathcal{F}_σ is contained in a direct sum of finitely generated modules, then R has dcc on σ -closed left ideals.*

Proof. By [15, Theorem 1.2] R has acc on σ -closed left ideals. Let E be the injective hull of a cyclic module in \mathcal{F}_σ . By hypothesis, E is contained in a direct sum of finitely generated modules; so E is finitely generated by [1, Proposition 2.4]. The result now follows from Corollary 2.3.

3. A generalization of quasi-Frobenius rings. A ring is called quasi-Frobenius (QF) if it is both left and right artinian and left self-injective. A well-known theorem of Faith and Walker [1] states that a ring is QF if and only if every injective module is projective. It is also known [11, page 37] that R is QF if and only if R is left artinian (or noetherian) and R is a cogenerator of R -mod. In this section we generalize these results.

We call a module W an \mathcal{F}_σ -cogenerator if every member of \mathcal{F}_σ can be embedded in a product of copies of W . Following [10], [12], and their references, we say that a module C is σ -codivisible if and only if $\text{Ext}_R^1(C, F) = 0$ for every $F \in \mathcal{F}_\sigma$. By [12, Theorem 8] a module C is σ -codivisible if and only if $C/\sigma(R)C$ is a projective $R/\sigma(R)$ -module.

PROPOSITION 3.1. *If every injective module in \mathcal{F}_σ is σ -codivisible (projective), then R has acc on σ -closed left ideals and $R/\sigma(R)$ (R) is an \mathcal{F}_σ -cogenerator.*

Proof. Let $M \in \mathcal{F}_\sigma$ be injective. By assumption M is projective as an $R/\sigma(R)$ -module (R -module). Thus M is a direct summand of a direct sum of countably generated modules. By Kaplansky's theorem [7] M is a direct sum of countably generated modules. Hence R has acc on σ -closed left ideals by [15, Theorem 1.2].

Now let $N \in \mathcal{F}_\sigma$. Then $E(N)$ is σ -codivisible (projective) by hypothesis, which implies that N is contained in a direct sum of copies of $R/\sigma(R)$ (R). So $R/\sigma(R)$ (R) is an \mathcal{F}_σ -cogenerator.

PROPOSITION 3.2. *If R has dcc on σ -closed left ideals and $R/\sigma(R)$ is an \mathcal{F}_σ -cogenerator, then every injective module in \mathcal{F}_σ is codivisible.*

Proof. By Theorem 1.4 R has acc on σ -closed left ideals. Let $M \in \mathcal{F}_\sigma$ be injective. By [15, Theorem 1.2] M is a direct sum of indecomposable modules. Thus we may assume that M is indecomposable (as a direct sum of σ -codivisible modules is σ -codivisible). Since $M \in \mathcal{F}_\sigma$ and R has dcc on σ -closed left ideals, M contains a σ -cocritical submodule N . By assumption $M = E(N)$ is embedded in a direct product U of copies of $R/\sigma(R)$. Choose a projection map $p: U \rightarrow R/\sigma(R)$ such that $p(N) \neq 0$. We see that the restriction of p to N is one-to-one as N is σ -cocritical and $R/\sigma(R) \in \mathcal{F}_\sigma$. Since N is essential in M , p must also be one-to-one on M . Consequently, M is isomorphic to a direct summand $p(M)$ of $R/\sigma(R)$; this implies M is σ -codivisible since $R/\sigma(R)$ is.

Let $W \in \mathcal{F}_\sigma$ be an injective module that cogenerates the torsion-free class \mathcal{F}_σ . Using a proof similar to the one just given, one easily shows that if R has dcc on σ -closed left ideals and W is σ -codivisible (projective), then every injective module in \mathcal{F}_σ is σ -codivisible (projective).

COROLLARY 3.3. *R has dcc on σ -closed left ideals if and only if every injective module in \mathcal{F}_σ is a direct sum of injective hulls of σ -cocritical modules.*

Proof. The proof of the “only if” part is contained in the proof of Proposition 3.2.

Suppose that $I_1 \supseteq I_2 \supseteq \dots$ is a descending chain of σ -closed left ideals of R , and let $I = \bigcap_{n=1}^\infty I_n$. Since $R/I \in \mathcal{F}_\sigma$ is cyclic, it follows from the hypothesis that $E(R/I)$ contains a finite, essential, direct sum M of σ -cocritical submodules. By [5, Corollary 1.5] $M \cap (R/I)$ has σ -finite length.

We claim that

$$(***) \quad M \cap (R/I) \cap (I_1/I) \supseteq M \cap (R/I) \cap (I_2/I) \supseteq \dots$$

is a descending chain of σ -closed submodules of $M \cap (R/I)$. To see this, let f_j be the natural composition

$$(R/I)/M \cap (I_j/I) \longrightarrow (R/I)/(I_j/I) \longrightarrow R/I_j .$$

Let g_j be the restriction of f_j to $(M \cap (R/I))/(M \cap (I_j/I))$. Then $\ker g_j = \ker f_j \cap [M/(M \cap (I_j/I))] = [(I_j/I)/(M \cap (I_j/I))] \cap [M/(M \cap (I_j/I))] = 0$; so g_j is a monomorphism into $R/I_j \in \mathcal{F}_\sigma$. Hence $(M \cap (R/I))/(M \cap (I_j/I)) \in \mathcal{F}_\sigma$ for each j .

By [5, Proposition 1.2] the chain (***) must terminate. Since $\bigcap_{n=1}^\infty (I_n/I) = 0$, then there exists a positive integer k such that $M \cap (R/I) \cap (I_k/I) = 0$. Since M is essential in $E(R/I)$, then $(R/I) \cap (I_k/I) = 0$, and hence $I_k = I$. Therefore, the chain $I_1 \supseteq I_2 \supseteq \dots$ terminates.

As usual, we call the torsion class \mathcal{F}_σ a TTF class if \mathcal{F}_σ is closed under direct products. If \mathcal{F}_σ is a TTF class, then there exists a (necessarily unique and idempotent) ideal T in the filter $F(\mathcal{F}_\sigma) = \{I \mid R/I \in \mathcal{F}_\sigma\}$. If N is a σ -cocritical module in \mathcal{F}_σ , then TN is a simple module. Indeed, $TN \neq 0$ since $N \in \mathcal{F}_\sigma$; and if K is a nonzero submodule of TN , we must have $TN/K = T(TN/K) = 0$ as TN is σ -cocritical. Thus in the TTF case we have the following result.

COROLLARY 3.4. *Let \mathcal{F}_σ be a TTF class. Then R has dcc on σ -closed left ideals if and only if every injective module in \mathcal{F}_σ is a direct sum of injective envelopes of simple modules.*

In case \mathcal{L}_σ is exact, we can strengthen Propositions 3.1 and 3.2 considerably.

THEOREM 3.5. *If \mathcal{L}_σ is exact, then the following statements are equivalent:*

- (1) *R has dcc on σ -closed left ideals, and $R/\sigma(R)$ is an \mathcal{F}_σ -cogenerator.*
- (2) *R has acc on σ -closed left ideals, and $R/\sigma(R)$ is an \mathcal{F}_σ -cogenerator.*

(3) Every injective module in \mathcal{F}_σ is σ -codivisible.

(4) R has dcc on σ -closed left ideals, and any injective \mathcal{F}_σ -cogenerator in \mathcal{F}_σ is σ -codivisible.

(5) R has acc on σ -closed left ideals, and any injective \mathcal{F}_σ -cogenerator in \mathcal{F}_σ is σ -codivisible.

Furthermore, any of these five equivalent statements imply that Q_σ is a QF ring.

REMARK. In analogy with QF rings, one might expect to find that $R/\sigma(R)$ is σ -injective (that is, $R/\sigma(R) = Q_\sigma$) and hence that $R/\sigma(R)$ is QF when the hypotheses of Theorem 3.5 are satisfied. However, it is trivial to give examples where this is not the case. In particular, let R be then 2×2 upper triangular matrix ring over a field F , and let \mathcal{F}_σ be the class of all modules annihilated by the top row of R ; then R and σ satisfy the hypotheses of Theorem 3.5, $R = R/\sigma(R)$ is not QF, and Q_σ is the full 2×2 matrix ring over F .

Proof of 3.5. That (1) implies (3) is Proposition 3.2. That (3) implies (2) follows by Proposition 3.1.

If (2) holds, then σ is perfect, and hence Q_σ is left noetherian via (2). We claim that Q_σ is a cogenerator in the category Q_σ -mod of unital left Q_σ -modules. Since any left Q_σ -module M is in \mathcal{F}_σ when viewed as an R -module, then M is embedded in a direct product of copies of $R/\sigma(R)$. Thus there is an $R/\sigma(R)$ -monomorphism $\alpha: M \rightarrow N$, where N is a direct product of copies of Q_σ . Let $q \in Q_\sigma$, let $m \in M$, and consider $(qm)\alpha - q((m)\alpha)$. Since $Q_\sigma/(R/\sigma(R)) \in \mathcal{F}_\sigma$, there is a left ideal $K \in F(\mathcal{F}_\sigma) = \{I \mid R/I \in \mathcal{F}_\sigma\}$ such that $Kq \subseteq R/\sigma(R)$. Now for any $k \in K$, we have $k((qm)\alpha - q((m)\alpha)) = k(qm)\alpha - kq((m)\alpha) = (kqm)\alpha - (kqm)\alpha = 0$. Hence α is a Q_σ -monomorphism; that is, Q_σ is a cogenerator for Q_σ -mod. Consequently, Q_σ is a QF ring [11, page 373]; so Q_σ is left artinian. Since σ is perfect, it follows that R has dcc on σ -closed left ideals, and (1) follows.

(3) \Rightarrow (4). In view of (3), any injective \mathcal{F}_σ -cogenerator is certainly codivisible. Moreover, R has dcc on σ -closed left ideals since we have shown that (3) implies (1).

That (4) implies (5) follows from Theorem 1.4.

(5) \Rightarrow (2). Let $W \in \mathcal{F}_\sigma$ be an injective \mathcal{F}_σ -cogenerator. Since W is σ -codivisible by (5), then W is a direct summand of a direct sum of copies of $R/\sigma(R)$; hence $R/\sigma(R)$ must also be an \mathcal{F}_σ -cogenerator.

THEOREM 3.6. Assume that (i) R has dcc on σ -closed left ideals, (ii) R is an \mathcal{F}_σ -cogenerator, and (iii) if M contains an essential σ -cocritical submodule N which is isomorphic to a submodule of a direct product of copies of $\sigma(R)$, then M is isomorphic to a submodule

of a projective module. Then every injective module in \mathcal{F}_σ is projective. The converse is true if \mathcal{L}_σ is exact.

Proof. Let $M \in \mathcal{F}_\sigma$ be an injective module. As in the proof of Proposition 3.2, we may assume that $M = E(N)$, where N is σ -cocritical. By (ii) M is embedded in a direct product U of copies of R . If there is no projection map $p: U \rightarrow R$ such that $p(N) \not\subseteq \sigma(R)$, then N is embedded in a direct product of copies of $\sigma(R)$. Thus by (iii) M is isomorphic to a submodule of a projective module; this implies that M is projective, as M is given to be injective. Now assume that there is a projection map $p: U \rightarrow R$ such that $p(N) \not\subseteq \sigma(R)$. Then the restriction of p to N is one-to-one, as N is σ -cocritical. Since N is essential in M , we also have that p is one-to-one on M . Consequently M is projective as it is isomorphic to a direct summand of R .

For the converse assume that \mathcal{L}_σ is exact and that every injective module in \mathcal{F}_σ is projective. By Proposition 3.1, R is an \mathcal{F}_σ -cogenerator. By assumption every module in \mathcal{F}_σ is contained in a projective module, namely its injective hull. Thus (iii) holds trivially, and (i) holds by Theorem 2.4.

REMARKS. We note that conditions (i), (ii), and (iii) are independent; that is, there exist σ such that \mathcal{L}_σ is exact and any two of (i), (ii) or (iii) hold while the remaining condition fails. Moreover, each of the following conditions is sufficient to imply condition (iii) of Proposition 3.6.

- (1) For each σ -cocritical module N , $\text{Hom}_R(N, \sigma(R)) = 0$.
- (2) \mathcal{F}_σ is a TTF class.
- (3) $Z(R) \cap \sigma(R) = 0$, where $Z(R)$ denotes the singular submodule of R .
- (4) $\sigma(R)$ contains no nilpotent ideals of R .

As a question related to the ideas in this paper, one might ask whether every injective module in \mathcal{F}_σ being σ -codivisible is equivalent to every σ -codivisible module in \mathcal{F}_σ being injective. We easily resolve this question in our closing result.

PROPOSITION 3.7. *The following statements are equivalent.*

- (1) Every σ -codivisible module in \mathcal{F}_σ is injective.
- (2) $R/\sigma(R)$ is a QF ring.
- (3) Every injective module in \mathcal{F}_σ is σ -codivisible, and \mathcal{F}_σ is closed under homomorphic images.

Proof. (1) \Rightarrow (2). Let X be a projective $R/\sigma(R)$ -module. As an R -module, $X \in \mathcal{F}_\sigma$, and X is σ -codivisible by [12, Theorem 8]. By

(1) X is injective as an R -module, and hence X is also injective as an $R/\sigma(R)$ -module. That every projective $R/\sigma(R)$ -module is injective is well-known to imply that $R/\sigma(R)$ is QF.

(2) \Rightarrow (1). Let $X \in \mathcal{F}_\sigma$ be σ -codivisible. Then X is projective as an $R/\sigma(R)$ -module by [12, Theorem 8]. Hence X is an injective $R/\sigma(R)$ -module since $R/\sigma(R)$ is QF by assumption. Since $X \in \mathcal{F}_\sigma$, then X is also injective as an R -module by [9, Proposition 4.8].

(2) \Rightarrow (3). If $M \in \mathcal{F}_\sigma$ is injective, then M is also injective as an $R/\sigma(R)$ -module. Hence M is a projective $R/\sigma(R)$ -module by (2). This implies M is σ -codivisible by [12, Theorem 8].

If Y is an R -homomorphic image of $M \in \mathcal{F}_\sigma$, then Y is also an $R/\sigma(R)$ -module as $\sigma(R)M = 0$. However, $R/\sigma(R)$ is a cogenerator for $R/\sigma(R)$ -mod by (2), which implies that $Y \subseteq \text{II}R/\sigma(R)$. Hence $Y \in \mathcal{F}_\sigma$.

(3) \Rightarrow (2). Let M be injective as an $R/\sigma(R)$ -module. Since \mathcal{F}_σ is closed under homomorphic images, every $R/\sigma(R)$ -module when viewed as an R -module is in \mathcal{F}_σ . Thus by [9, Proposition 4.8] M is injective as an R -module. By assumption M is σ -codivisible; and therefore, as an $R/\sigma(R)$ -module M is projective [12, Theorem 8]. Thus $R/\sigma(R)$ is QF as every injective $R/\sigma(R)$ -module is projective.

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