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Let E be a closed subset of the compact Hausdorff X and let A be a closed separating subspace of $C_c(X)$. Let ρ be a dominator (strictly positive, l.s.c.) defined on $X \times T$, T the unit circle in \mathbb{C} . Conditions, formulated in terms of boundary measures, are discussed for approximate and exact solutions to the problem of finding ρ -dominated extensions in A of functions $g \in (A|_E)^-$ satisfying $\operatorname{re} tg(x) \leq \rho(x, t)$ on $E \times T$. Various interpolation theorems of Rudin-Carleson type for superharmonic dominators are incorporated into this framework.

We do not assume that A contains the constant functions. We denote $M(X) = C(X)^*$, the space of regular Borel measures on X .

We consider $N = M(E)$ as situated in $M(X)$ as the range of the projection $\pi_1 \mu = \mu|_E$ and denote the complementary projection $\pi_2 \mu = \mu|_{X \setminus E}$. Thus $(A|_E)^\perp$ is identified with the subspace $A^\perp \cap N$ in $M(X)$.

We call $\mu \in M(X)$ a *boundary measure* if $|\mu|$ is maximal with respect to the Choquet ordering as a measure of X (embedded by evaluation) in the w^* compact unit ball A_1^* . If $1 \in A$ then this is the same as $|\mu|$ being maximal on the state space S_A , as $X \subset S_A$, a w^* closed face of A_1^* .

For brevity we denote the boundary measures by $\partial_A M(X)$, or $\partial M(X)$, if A is understood, and in general, adopt the convention of writing $\partial_A S$ for $S \cap \partial_A M(X)$. Thus, $\partial_A A^\perp$ refers to the boundary measures annihilating A . The space A^* is the quotient space $M(X)/A^\perp$ and images under the quotient map are denoted $\hat{\mu}$ for $\mu \in M(X)$. A subset $S \subset M(X)$ is called *A -stable* if $\hat{S} = (\partial_A S)^\wedge$.

We call E an *interpolation set* if $A|_E$ is closed in $C(E)$. Gamelin [8] shows that E is an interpolation set if and only if there is a $k; 0 \leq k < \infty$, such that for each $m \in A^\perp$,

$$(1) \quad \|\pi_1 m + A^\perp \cap N\| \leq k \|\pi_2 m\|.$$

The best value of k is called the *extension constant*, $e(A, E)$.

In [10] Roth introduces a general framework for interpolation problems by means of a *dominator*, ρ , defined as a strictly positive l.s.c. extended real-valued function on $X \times T$ (T the unit circle in \mathbb{C}). We let

$$U = \{f \in C(X): \operatorname{re} tf(x)/\rho(x, t) \leq 1 \text{ for all } (x, t) \in X \times T\}$$

and write

$$\|f\|_\rho = \sup\{\rho(tf(x))/\rho(x, t): (x, t) \in X \times T\}$$

for the Minkowski functional of U . Thus $\|f\|_\rho \leq 1$ if and only if $\rho(tf(x)) \leq \rho(x, t)$, $(x, t) \in X \times T$. Then $\|\mu\|_\rho$, $\mu \in M(X)$, refers to the polar functional given by

$$\|\mu\|_\rho = \sup\{\rho(f, \mu): f \in U\}.$$

Since ρ is *l.s.c* and positive there is a constant c such that $\|f\|_\rho \leq c\|f\|$ (the uniform norm corresponding to $\rho \equiv 1$) and if ρ is bounded above the two are equivalent.

We say E is an *approximate ρ -interpolation set* for A if E is an interpolation set and for each $g \in (A|_E)^-$ and $\varepsilon > 0$ there is an $f \in A$ such that $f|_E = g$ and $\|f\|_\rho < \|g\|_\rho + \varepsilon$. We say E is an *exact ρ -interpolation set* if f can be chosen with $\|f\|_\rho = \|g\|_\rho$. It is shown in [5] that for bounded ρ , E is an approximate ρ -interpolation set for A if and only if for each $m \in A^\perp$,

$$(2) \quad \|\pi_1 m + A^\perp \cap N\|_\rho \leq \|-\pi_2 m\|_\rho.$$

If, in addition, the image \hat{U} of U^0 under the quotient map is *decomposable* by \hat{N} then E is an exact ρ -interpolation set. If there is an s , $0 \leq s < 1$, such that for each $m \in A^\perp$,

$$(3) \quad \|\pi_1 m + A^\perp \cap N\|_\rho \leq s \|-\pi_2 m\|_\rho$$

then the above holds and E is ρ -exact for A . Gamelin's results [8] can be phrased as follows: Let G be a compact set in $X \setminus E$ and let

$$\rho(G, k)(x, t) = \begin{cases} 1 & \text{for } (x, t) \in E \times T \\ k & \text{for } (x, t) \in G \times T \\ 1 \vee k & \text{otherwise.} \end{cases}$$

Then E is an approximate $\rho(G, k)$ -interpolation set for all such G if and only if (1) holds and if, in addition, $e(A, E) < 1$ then E is an exact ρ -interpolation set for any continuous T -invariant ρ such that $\rho > e(A, E)$ on $X \times T$. This was obtained in abstract form using polar techniques by Ando [3].

In [6] Briem shows that if E is a subset of the Choquet boundary, $\partial_A X$, then E is an interpolation set if and only if (1) holds only for $m \in \partial_A A^\perp$. Further, if X is metrizable then (1) holds for $\partial_A A^\perp$ if and only if E is an approximate $\rho(G, k)$ -interpolation set for each compact $G \subset \partial_A X \setminus E$. The A -stability of the unit ball $M_1(X)$ (Hustad's theorem [9]) and of $N = M(E)$ (since $E \subset \partial_A X$) are

essential here. If (1) holds for $\tilde{e}(A, E) < 1$ (again, \tilde{e} is the smallest k such that (1) holds for all $m \in \partial A^\perp$) then E is $\rho(G, k)$ exact for any $G \subset \partial_A X \setminus E$ and $k > \tilde{e}$.

If (1) holds for all $m \in \partial_A A^\perp$ with $k = 0$ this can be expressed as

$$(4) \quad m \in \partial_A A^\perp \text{ implies } \pi_1 m \in A.$$

The set E is called an M -set if $M(E)$ is A -stable and (4) holds. Roth [10] shows that if E is an M -set and ρ is a bounded A -superharmonic (if $1 \in A$ this means $\rho(x, t) \geq \int \rho(\cdot, t) d\mu$ for any $\mu \in M_1^+(X)$ and $\hat{\mu} = x \in X \subset A_1^*$) dominator then E is an exact ρ -interpolation set for A . This generalizes the Alfsen-Hirsberg theorem [2] which deals with T -invariant ρ and $E \subset \partial_A X$.

In this note we consolidate these results by showing that for E an interpolation set with $M(E)$ A -stable and ρ A -superharmonic then E is an approximate ρ -interpolation set if and only if (2) holds for $m \in \partial_A A^\perp$. If in addition \hat{U} is decomposable by \hat{N} in A^* then the interpolation is exact. This is the case if ρ is bounded and (3) holds for $m \in \partial_A A^\perp$. (If ρ is bounded and (2) or (3) holds then E is already an interpolation set.) We give a measure theoretic condition for the decomposability of \hat{U} and show by means of simple examples of $A(K)$ spaces that exactness of interpolation can be deduced in this way even though equality holds in (2) which, of course, precludes the use of (3).

1. Hustad-Roth stability theorems. Let A be a closed separating subspace of $C(X)$. Define $\Phi: C(X) \rightarrow C(X \times T)$ by $\Phi f(x, t) = tf(x)$. By separating we shall mean that the range of $\Phi|_A$ separates the points of $X \times T$. This assumption can be avoided, as is shown in Fuhr-Phelps [7], but at the expense of additional technicalities. If $\nu \in M(X \times T)$ then the Hustad map is given by

$$\mu = \Phi^* \nu \in M(X); \mu(f) = \int_{X \times T} tf(x) d\nu(x, t).$$

If $\Phi = \Phi|_A$ has range $B \subset C(X \times T)$ and ν is a maximal probability measure on $X \times T \subset B_1^*$ representing $\tilde{L} \in B_1^*$ then Hustad's theorem says $\mu = \Phi^* \nu$ belongs to $\partial_A M(X)_1$ with $\hat{\mu} = L = \phi^* \tilde{L}$. We combine this with the following observations concerning T -invariant A -superharmonic dominators to obtain a general stability theorem due to Roth [11].

Thus let ρ be a strictly positive l.s.c. extended real-valued function on X such that for each $x \in X$ and $\mu \in M_1^+(X)$ with $\hat{\mu} = x \in A^*$, we have $\rho(x) \geq \int_X \rho d\mu$, that is, ρ is A -superharmonic. If $U = \{f \in C(X); \text{ref}/\rho \leq 1\}$ then U^0 is a w^* compact convex subset of the

positive cone $M^+(X)$, and we let \hat{U} be the quotient image in A^* . Take \bar{R}^+ to be the one-point-compactification of R^+ and

$$\begin{aligned} X_0 &= \{(x, s) \in X \times \bar{R}^+ : \rho(x) \leq s \leq +\infty\}, \\ Y_0 &= \{(x, \rho(x)) \in X_0 : \rho(x) < \infty\}, \\ Y_\infty &= \{(x, \rho(x)) \in X_0 : \rho(x) = +\infty\}. \end{aligned}$$

Since ρ is l.s.c., $Y_0 \cup Y_\infty$ and Y_∞ are both G_δ subsets of X_0 so that Y_0 is a Borel set. Define

$$\psi: C(X) \longrightarrow C(X_0); \psi f(x, s) = f(x)/s,$$

and let $\theta = \psi|_A$ with (not necessarily closed) range $B \subset C(X_0)$. Since ρ is strictly positive ψ is bounded and θ^* is one-to-one from B^* into A^* . Let

$$\phi_0: X_0 \longrightarrow B_1^*$$

be the evaluation map and let $\hat{V} = w^* - \overline{co}\phi_0(X_0)$.

PROPOSITION 1.1. *Let ρ be a T -invariant A -superharmonic dominator on X as above.*

(1) ϕ_0 is one-to-one on $X_0 \setminus (X \times \{\infty\})$, $X \times \{\infty\} = \phi_0^{-1}(0)$, and $\theta^* \hat{V} = \hat{U}$.

(2) If ν is a maximal probability measure on \hat{V} then $\nu[\phi_0(Y_0) \cup \{0\}] = 1$ and ν may be identified with the measure on Y_0 given by $\nu \circ \phi_0$.

(3) If ν is as in (2) and $\mu = \psi^* \nu$ then for any bounded Borel function h on X

$$\int_X h d\mu = \int_{Y_0} (h(x)/\rho(x)) d\nu(x, \rho(x)).$$

In particular, $\mu \in U^0$.

(4) Let $\mu_0 \in M_1^+(X)$ with $\hat{\mu}_0 = x_0 \in X \subset A_1^*$ and define $\tilde{\mu}_0 \in M(X_0)$ by

$$\tilde{\mu}_0(F) = (1/\rho(x_0)) \int_X F(x, \rho(x)) \rho(x) d\mu_0(x).$$

Then for any bounded Borel function h on X

$$\int_{X_0} (h(x)/s) d\tilde{\mu}_0(x, s) = (1/\rho(x_0)) \int_X h d\mu_0.$$

In particular $\tilde{\mu}_0 \geq 0$, $\tilde{\mu}_0(X_0) = \tilde{\mu}_0(Y_0) \leq 1$, and $\tilde{\mu}_0$ represents $(x_0, \rho(x_0)) \in \hat{V}$.

(5) If ν is maximal on \hat{V} then $\mu = \psi^* \nu$ is maximal on $K = \overline{co}X \subset A^*$.

Proof. (1) The separation theorem shows $\hat{U} = w^*\overline{co}\{x/s: (x, s) \in X_0\}$. Now

$$\theta^* \circ \phi_0(x, s) = x/s \in A^*$$

so the rest of (1) follows from the fact that A separates points in X . For (2) let $p = 1 - \chi_{\{0\}}$ on \hat{V} and note that the lower envelope $\check{\rho}$ is the Minkowski functional of \hat{V} . Since ν is maximal,

$$1 = \nu[\{x: p(x) = \check{\rho}(x)\}] = \nu[\{x: \check{\rho}(x) = 1 \text{ or } 0\}].$$

Now $\lambda \geq 1$ implies $\phi_0(x, \lambda s) = (1/\lambda)\phi_0(x, s)$, so that

$$\nu[\phi_0(Y_0) \cup \{0\}] = 1.$$

If $f \in C(X)$ then $\psi^*\nu(f) = \int_{X_0} (f(x)/s) d\nu(x, s) = \int_{Y_0} (f(x)/\rho(x)) d\nu(x, \rho(x))$ and so (3) holds.

(4): If $F \in C(X_0)$ and $0 \leq F \leq 1$ then

$$0 \leq \tilde{\mu}_0(F) \leq (1/\rho(x_0)) \int_X \rho d\mu_0 \leq 1.$$

Thus $\tilde{\mu}_0 \geq 0$, $\tilde{\mu}_0(X_0) \leq 1$ and $\mu_0[\{x: \rho(x) = +\infty\}] = 0$. For $F = \psi h$,

$$\begin{aligned} \tilde{\mu}_0(F) &= \int_{X_0} (h(x)/s) d\tilde{\mu}_0(x, s) \\ &= (1/\rho(x_0)) \int_X h d\mu_0. \end{aligned}$$

(5): Let f be a continuous convex function of K and denote the upper envelope of f by $\hat{f}(K)$, where [1, I. 3.6]

$$\hat{f}(K)(x_0) = \sup\{\mu(f): \mu \in M_1^+(X) \text{ and } \hat{\mu} = x_0 \in A^*\}.$$

If $g = \psi(f|_X)$ then $g \in C(X_0)$ with $g \equiv 0$ on $X \times \{\infty\}$. If $\tilde{\mu}_0 = x_0$ and $\tilde{\mu}_0$ is as in (4) then $\tilde{\mu}_0$ represents $(x_0, \rho(x_0)) \in \hat{V}$ and the upper envelope, $\hat{g}(\hat{V})$, satisfies

$$\hat{g}(\hat{V})(x_0, \rho(x_0)) \geq \sup\{\tilde{\mu}_0(g): \hat{\mu}_0 = x_0\} = (1/\rho(x_0))\hat{f}(K)(x_0)$$

by part (4). Thus, using part (3), and [1, I. 4.5],

$$\int_X [\hat{f}(K) - f] d\mu = \int_{Y_0} [\hat{f}(K) - f]/\rho d\nu \leq \int_{Y_0} [\hat{g}(\hat{V}) - g] d\nu = 0$$

since ν is maximal. Hence, μ is maximal on K .

We now consider the case where ρ is defined on $X \times T$. We say such a ρ is *A-superharmonic* if for each $(x, t) \in X \times T$ and $\mu \in M(X \times T)_1^+$ with

$$\int_{X \times T} sf(y) d\mu(y, s) = tf(x) \text{ for all } f \in A$$

we have $\rho(x, t) \geq \int_{X \times T} \rho d\mu$.

THEOREM 1.2 (Hustad-Roth). *If ρ is an A -superharmonic dominator then U^0 is A -stable.*

Proof. Let $\Phi: C(X) \rightarrow C(X \times T)$; $\Phi f(x, t) = tf(x)$ and let

$$U^1 = \{F \in C(X \times T): reF(x, t)/\rho(x, t) \leq 1\}$$

and $\phi = \Phi|_A$ with range B .

Let $\Psi: C(X \times T) \rightarrow C(X_0)$; $\Psi F(x, t, s) = F(x, t)/s$, where X_0 is the closed epigraph of ρ in $(X \times T) \times \bar{R}^+$. Now $\Phi U \subset U^1$ and $\phi(A \cap U) = B \cap U^1$. Given $L \in \hat{U}$, let $\tilde{L} \in (U^1)^\wedge \subset B^*$ and $L' \in \hat{V}$ (as in Proposition 1.1) with $\theta^* L' = \tilde{L}$ and $\phi^* \tilde{L} = L$. We have $B_1^* = w^*\overline{co}(X \times T)$ and the hypothesis says ρ on $X \times T$ is B -superharmonic. Hence the results of Proposition 1.1 apply. Thus if ν' is maximal on \hat{V} representing L' then 1.1 (3) and (5) show $\nu = \Psi^* \nu'$ is maximal on B_1^* representing $\tilde{L} \in (U^1)^\wedge$. Then $\mu = \phi^* \nu \in U^0$ and $\hat{\mu} = L \in \hat{U}$. Furthermore, Hustad's theorem shows μ is a boundary measure.

If $1 \in A$ then the condition for A -superharmonicity is somewhat simpler.

PROPOSITION 1.3. *If $1 \in A$ then ρ is A -superharmonic if and only if for each $\mu \in M_1^+(X)$ with $\hat{\mu} = x$,*

$$\rho(x, t) \geq \int_X \rho(\cdot, t) d\mu.$$

Proof. If ρ is A -superharmonic and $\mu \in M_1^+(X)$ with $\hat{\mu} = x$ we can embed X as $X \times \{t\} \subset X \times T$ so that the measure μ satisfies

$$\int_{X \times T} sf(y) d\mu = tf(x)$$

and hence

$$\rho(x, t) \geq \int_{X \times \{t\}} \rho(x, t) d\mu = \int_X \rho(\cdot, t) d\mu.$$

Conversely, if $\mu \in M_1^+(X \times T)$ and represents tx then, since $1 \in A$, we have $\overline{tco}X = tS_A(S_A)$ the state space of A is a face of A_1^* . Hence $\text{supp } \mu \subset X \times \{t\}$ and the measure $\mu_1(f) = \int_{X \times T} f(x) d\mu$ represents x so that

$$\rho(x, t) \geq \int_X \rho(\cdot, t) d\mu_1 = \int_{X \times T} \rho d\mu.$$

2. Dominated interpolation. If E is a compact subset of X we let

$$M = \{f \in C(X) : f|_E = 0\}$$

and denote $M \cap A$ by E^\perp . It is well known that E is an interpolation set for A if and only if $A + M$ is closed in $C(X)$ and this in turn is equivalent to \hat{N} being w^* (or norm) closed in A^* . The following characterization of approximate ρ -interpolation sets follows from results in [5; 4.2]. We denote $N = M(E) \subset M(X)$.

THEOREM 2.1. *Let ρ be a (strictly positive l.s.c) dominator on X such that either ρ is bounded or E is an interpolation set. The following are equivalent:*

- (i) E is an approximate ρ -interpolation set for A ,
- (ii) $A + M$ is closed in $C(X)$ and

$$(A + M) \cap (U + M) = (A \cap U + M)^-,$$

- (iii) $\hat{U} \cap \hat{N} = (U^0 \cap N)^\wedge$,
- (iv) $\|\mu + A^\perp \cap N\|_\rho = \|\mu + A^\perp\|_\rho$ for all $\mu \in N$,
- (v) $\|\pi_1 m + A^\perp \cap N\|_\rho \leq \|-\pi_2 m\|_\rho$ for all $m \in A^\perp$.

For $x \in A^*$ we write $\|x\|_\rho$ for the Minkowski functional of \hat{U} so that if $\hat{\mu} = x$

$$\|x\|_\rho = \|\mu + A^\perp\|_\rho.$$

The set U^0 is *split*, that is, $\|\mu\|_\rho = \|\pi_1 \mu\|_\rho + \|\pi_2 \mu\|_\rho$ [10, 5].

PROPOSITION 2.2. *Let N and U^0 be A -stable sets in $M(X)$. Then for $\mu \in \partial_A M(X)$,*

- (1) $\|\mu + A^\perp\|_\rho = \|\mu + \partial A^\perp\|_\rho = \|\hat{\mu}\|_\rho$,
- (2) $\|\mu + N + A^\perp\|_\rho = \|\pi_2 \mu + \pi_2 \partial A^\perp\|_\rho$ ($\pi_2 \mu = \mu|_{X \setminus E}$),
- (3) If $\|\mu\|_\rho = \|\hat{\mu}\|_\rho$ then

$$\|\pi_i \mu\|_\rho = \|(\pi_i \mu)^\wedge\|_\rho \quad (i = 1, 2).$$

Proof. If $\mu \in \partial M(X)$ and $\|\hat{\mu}\|_\rho \leq r$ then $\mu = r\nu + m$ with $\nu \in U^0$ and $m \in A^\perp$. The stability of U^0 shows we can assume $\nu \in \partial U^0$, so that $m \in \partial A^\perp$. Then (1) follows. If $\mu = r\nu + \eta + \zeta$ with $\nu \in \partial U^0$, $\eta \in \partial N$, $\zeta \in A^\perp$, then $\zeta \in \partial A^\perp$ and $\pi_2 \mu = r\pi_2 \nu + \pi_2 \zeta \in r\pi_2 U^0 + \pi_2 \partial A^\perp$. Conversely, if $\pi_2 \mu = r\nu + \pi_2 \zeta$, $\nu \in \partial U^0$, $\zeta \in \partial A^\perp$ then

$$\mu = r\nu + (\pi_1\mu - \pi_1\zeta) + \zeta \in rU^0 + \partial N + \partial A^\perp.$$

For (3), we have

$$\begin{aligned} \|\pi_1\mu\|_\rho &\geq \|(\pi_1\mu)^\wedge\|_\rho = \|\pi_1\mu + A^\perp\|_\rho = \|\mu - \pi_2\mu + A^\perp\|_\rho \\ &\geq \|\mu\|_\rho - \|\pi_2\mu + A^\perp\|_\rho \geq \|\mu\|_\rho - \|\pi_2\mu\|_\rho = \|\pi_1\mu\|_\rho. \end{aligned}$$

Since we do not assume $1 \in A$, we take the *Choquet boundary*, $\partial_A X$, to be $X \cap \text{ext} A_1^*$. There are two main instances where the A -stability of N can be deduced.

PROPOSITION 2.3. *Let E be a closed subset of X such that either*

(a) $E \subset \partial_A X$ or

(b) $E = F \cap X$, F a w^* closed face of A_1^* .

Then N is A -stable.

Proof. In the case (a) each probability measure on E is maximal and so the result follows since $\overline{co}E$ spans \hat{N} . In case (b) each maximal probability measure μ with $\hat{\mu} \in \overline{co}E$ has its support on $(\text{ext } F)^- \subset E$.

THEOREM 2.4. *Let E be a closed subset of X such that either*

(a) $E \subset \partial_A X$, or

(b) $E = F \cap X$, F a closed face of A_1^* .

Let ρ be an A -superharmonic dominator such that either ρ is bounded or E is an interpolation set. Then the following are equivalent:

(i) E is an approximate ρ -interpolation set,

(ii) $\|\mu + A^\perp \cap N\|_\rho = \|\mu + \partial A^\perp\|_\rho$ for all $\mu \in \partial N$,

(iii) $\|\pi_1 m + A^\perp \cap N\|_\rho \leq \|-\pi_2 m\|_\rho$ for all $m \in \partial A^\perp$.

Proof. The hypotheses imply that U^0 and N are A -stable and so 2.2. (1) shows for $\mu \in \partial M$,

$$\|\mu + A^\perp\|_\rho = \|\mu + \partial A^\perp\|_\rho.$$

Thus (i) \Rightarrow (ii) \Leftrightarrow (iii) follows from 2.1. If (ii) holds and $x \in \hat{U} \cap \hat{N}$ then choose $\mu \in \partial N$ with $\hat{\mu} = x$ and $\mu \in U^0 + A^\perp$. Then

$$\|\mu + A^\perp \cap N\|_\rho = \|\mu + \partial A^\perp\|_\rho = \|\mu + A^\perp\|_\rho \leq 1$$

so that $\mu = \nu + m$; $\nu \in U^0$, $m \in A^\perp \cap N$. Hence $\nu \in N$ and $\hat{\mu} = x = \hat{\nu} \in (U^0 \cap N)^\wedge$. Thus 2.1 (iii) holds and hence (i) is shown.

The exactness of ρ -interpolation is characterized by the sum

$A \cap U + E^\perp$ (E^\perp the ideal of functions in $C(X)$ vanishing on E) being closed in A , a condition which is implied by the decomposability of \hat{U} by \hat{N} in A^* [5; Theorem 3.2]. If E is an interpolation set (so that \hat{N} if w^* closed in A^*) then \hat{U} is said to be *decomposable* by \hat{N} if there is an $\alpha \geq 1$ such that each $x \in \hat{U}$ is a convex combination of elements y, z with $y \in \hat{U} \cap \hat{N}, z \in \hat{U}$ and $\|z\| \leq \alpha \|z + \hat{N}\|$ (dual uniform norm).

The condition for decomposability, and hence exact interpolation, can be formulated in terms of representing measures in $M(X)$. We illustrate this for boundary measures in the case where ρ is superharmonic.

THEOREM 2.5. *Let E be a closed subset of X and A a closed separating subspace such that either*

(a) $E \subset \partial_A X$, or

(b) $E = F \cap X, F$ a closed face of A_1^* ,

and let ρ be an A -superharmonic dominator such that either ρ is bounded or E is an interpolation set.

If for each $x \in \hat{U}$ there is a $\mu \in \partial_A U^0$ with $\hat{\mu} = x$ and

$$\|\pi_2 \mu + \partial A^\perp\| \leq \alpha \|\pi_1 \mu + \pi_2 \partial A^\perp\|$$

(α a constant independent of μ) then E is an exact ρ -interpolation set.)

Proof. Given $x \in \hat{U}$ choose a boundary measure μ satisfying $\hat{\mu} = x$, $\|\hat{\mu}\|_\rho = \|\mu\|_\rho$ and $\|\pi_2 \mu + \partial A^\perp\| \leq \alpha \|\pi_1 \mu + \pi_2 \partial A^\perp\|$. Now $\|\mu\|_\rho = \|\pi_1 \mu\|_\rho + \|\pi_2 \mu\|_\rho$ shows that μ is a convex combination of $\mu_1 \in U^0 \cap N$ and $\mu_2 \in U^0$, scalar multiples of $\pi_1 \mu, \pi_2 \mu$ respectively. Thus, $\|\mu_2 + \partial A^\perp\| \leq \alpha \|\mu_2 + \pi_2 \partial A^\perp\|$ and x is a convex combination of $y \in (U^0 \cap N)^\wedge$ and $z \in \hat{U}$ with (using 2.2 (1) and (2))

$$\begin{aligned} \|z\| &= \|\mu_2 + \partial A^\perp\| \leq \alpha \|\mu_2 + \pi_2 \partial A^\perp\| = \alpha \|\mu + N + A^\perp\| \\ &= \alpha \|z + \hat{N}\|. \end{aligned}$$

This shows that $(U^0 \cap N)^\wedge = \hat{U} \cap \hat{N}$ and that \hat{U} is decomposable by \hat{N} . Therefore E is an exact ρ -interpolation set.

If E is an M -set then $\pi_2 \partial A^\perp \subset \partial A^\perp$ so that

$$\|\pi_2 \mu + \pi_2 \partial A^\perp\| \geq \|\pi_2 \mu + \partial A^\perp\|$$

and the condition of 2.5 is automatically satisfied (for A -stable U^0). More generally, if U^0 and N are A -stable and, for some $s < 1$

$$\|\pi_1 m + A^\perp \cap N\|_\rho \leq s \|\pi_2 m\|_\rho \text{ for all } m \in \partial A^\perp$$

then a computation based on [5; 4.8] shows the condition of Theorem 2.5 holds, so that E is an exact ρ -interpolation set.

COROLLARY 2.6. *If E is an M -set for the closed separating subspace $A \subset C(X)$ then E is an exact ρ -interpolation set for A for any A -superharmonic dominator ρ .*

Proof. If E is an M -set then \hat{N} is the range of a projection in A^* so that E is an interpolation set for A . The conclusion then follows from 2.5.

3. Examples. We illustrate the results of §2 with various choices of ρ . First, let X be a compact metric space with E a closed subset and $M(E)$ A -stable for the closed separating subspace $A \subset C(X)$. Let G be the collection of compact subsets $G \subset \partial_A X \setminus E$ and let $\rho = \rho(G, k)$ be the dominator mentioned in the introduction. Then (for $k < \infty$)

$$(1) \quad \|\pi_1 m + A^\perp \cap N\| \leq k \|\pi_2 m\| \text{ for all } m \in \partial A^\perp$$

if and only if E is an approximate $\rho(G, k)$ -interpolation set for all $G \in \mathcal{G}$. To see this we note that since $G \subset \partial_A X$, U^0 is A -stable so that the second property holds if and only if

$$(2) \quad \|\pi_1 m + A^\perp \cap N\|_\rho \leq \|\pi_2 m\|_\rho \text{ for all } m \in \partial A^\perp, G \in \mathcal{G}.$$

It follows easily from [5; 4.1] that if $Y = X \setminus (E \cap G)$ then

$$\|\mu\|_\rho = \|\mu|_E\| + k \|\mu|_G\| + (1 \vee k) \|\mu|_Y\|$$

so that

$$\|\pi_1 m + A^\perp \cap N\| = \|\pi_1 m + A^\perp \cap N\|_\rho$$

and, since for boundary measures μ , the metrizability of X gives

$$|\mu|(X \setminus E) = |\mu|(\partial_A X \setminus E) = \sup\{|\mu|(G) : G \in \mathcal{G}\},$$

we have

$$k \|\pi_2 m\| = \sup\{\|\pi_2 m\|_\rho : \rho = \rho(G, k), G \in \mathcal{G}\}.$$

The equivalence of (1) and (2) is now immediate. If (1) holds for $k_0 < 1$ and $k_0 < k \leq 1$ then for $\rho = \rho(G, k)$

$$\begin{aligned} \|\pi_1 m + A^\perp \cup N\|_\rho &= \|\pi_1 m + A^\perp \cap N\| \leq k_0 (\|m|_G\| + \|m|_Y\|) \\ &\leq (k_0/k)(k \|m|_G\| + \|m|_Y\|) = (k_0/k) \|\pi_2 m\|_\rho \end{aligned}$$

so that E is an exact $\rho(G, k)$ -interpolation set for A .

The study of sufficient conditions for the A -convex hull of E to be a generalized peak set (we now assume $1 \in A$) has been shown [4] to be related to an ordering on $C_c(X)$ and $M(X)$ induced by choosing P to be a closed proper convex cone with nonempty interior in C . Let α, β be the generators (of modulus one) of the dual cone $P^* = \{z: \operatorname{re} az \geq 0 \text{ for all } a \in P\}$. We denote by e the element of P such that $\operatorname{re} e\gamma = 1$ ($\gamma = \alpha, \beta$). If $f \in C_c(X)$ we say $f \geq 0(P)$ if $f(X) \subset P$ and $\mu \geq 0(P^*)$ means $\mu(B) \in P^*$ for all Borel sets $B \subset X$. Then the function $e \equiv e$ becomes an order unit for $C(X)$ in which the order unit norm $\|\cdot\|_e$ (equivalent to the uniform norm) is given by

$$\rho(x, t) = \begin{cases} 1 & \text{for } t = \pm \gamma \\ 1/c & \text{for } t \neq \pm \gamma, \end{cases} \quad \gamma = \alpha, \beta$$

where c is a constant such that

$$cz \leq |\operatorname{re} az| \vee |\operatorname{re} \beta z|.$$

This provides an example of a ρ which is not T -invariant.

Let ρ^+ and ρ^- be strictly positive *l.s.c.* functions on X and take

$$\rho(x, t) = \begin{cases} \rho^+(x) & \text{on } X \times \{1\} \\ \rho^-(x) & \text{on } X \times \{-1\} \\ +\infty & \text{otherwise.} \end{cases}$$

Then $U = \{f \in C(X): -\rho^- \leq \operatorname{re} f \leq \rho^+\}$. If $\mu \in U^0$ and f is real then $\lambda \operatorname{if} \in U$ for all real λ so that

$$1 \geq \operatorname{re} \mu(\lambda \operatorname{if}) = -\lambda \operatorname{im} \mu(f)$$

and hence $\operatorname{im} \mu(f) = 0$. Thus μ is a real measure and $U^0 \subset \operatorname{re} M(X)$.

If A_0 is a real subspace of $C(X)$ then we can apply the results of § 2 to the self-adjoint space $A_0 + iA_0 = A$. Then $\|f\|_\rho = \|\operatorname{re} f\|_\rho$ and $m \in A^\perp$ if and only if $m = m_1 + im_2$ with m_1, m_2 real measures in A^\perp . Also m is a boundary measure if and only if m_1, m_2 are boundary. Hence E is an approximate (exact) ρ -interpolation set for A if and only if it is for $A_0 = \operatorname{re} A$, and the measure conditions of § 2 need only involve real measures in $M(X)$. If X is a compact convex subset of a locally convex space and $A_0 = A(X)$ (real affine continuous functions) then ρ is A -superharmonic if and only if $\rho^+ = (\rho^+)^\wedge$ and $\rho^- = (\rho^-)^\wedge$, that is, if and only if ρ^+ and ρ^- are concave on X .

Let X be a square in R^2 with vertices denoted $\{1, 2, 3, 4\}$ with

$E = \{1, 2\}$ diagonally opposite and $A_0 = A(X)$, $\rho^+, \rho^- \equiv 1$. Then ∂A^\perp is a one-dimensional subspace of the four-dimensional space $\partial M(X)$ spanned by the point-masses $\{\delta_i\}_{i=1}^4$. A generator for ∂A^\perp is $m = \delta_1 + \delta_2 - \delta_3 - \delta_4$. Clearly $A^\perp \cap N = \{0\}$ since coE is a simplex and so

$$\|\pi_1 m + A^\perp \cap N\| = \|\pi_1 m\| = \|\pi_2 m\|.$$

This shows E is an approximate ρ -interpolation set for $A(X)$. Obviously E is in fact an exact interpolation set, but this cannot be concluded from a condition such as (3) in the introduction. Nevertheless, the condition of 2.5 holds, since if

$$\mu = \sum \lambda_i \delta_i$$

then

$$\|\mu\| = \sum |\lambda_i|$$

and

$$\|\pi_2 \mu + \pi_2 \partial A^\perp\| = \inf\{|\lambda_3 - \lambda| + |\lambda_4 - \lambda| : \lambda \in R\} = |\lambda_4 - \lambda_3|.$$

If λ_3 and λ_4 are opposite in sign then

$$\|\pi_2 \mu + \partial A^\perp\| \leq \|\pi_2 \mu\| = |\lambda_3| + |\lambda_4| = |\lambda_4 - \lambda_3| = \|\pi_2 \mu + \pi_2 \partial A^\perp\|.$$

If, say $0 \leq \lambda_3 \leq \lambda_4$, consider $\nu = \mu + \lambda_3 m$. Then $\hat{\nu} = \hat{\mu}$ and

$$\|\nu\| = \sum |\lambda_i - \lambda_3| \leq (|\lambda_1| + |\lambda_2| + 2|\lambda_3|) + |\lambda_4| - |\lambda_3| = \|\mu\|$$

and

$$\|\pi_2 \nu + \partial A^\perp\| \leq \|\pi_2 \nu\| = \lambda_4 - \lambda_3 = \|\pi_2 \mu + \pi_2 \partial A^\perp\|.$$

We conclude with an example of an approximate interpolation set which is not exact. Let X be the unit ball of the sequence space l^1 (w^* topology) and let $\rho \equiv 1$. Then take $A = c_0$, the pre-dual of l^1 , so that $\|a\|_\rho = \|a\|_\infty = \sup\{|a_n|\}$. Let E be the singleton $\{x^0\}$, $x_n^0 = 1/2^n$, $n = 1, 2, \dots$. If $(a, x^0) = 1$ then $\sum_{n=1}^\infty a_n/2^n = 1$ so that some a_n must be greater than one. Clearly we can find such an a with $\|a\| \leq 1 + \varepsilon$ for any $\varepsilon > 0$. Thus E is an approximate, but not exact, ρ -interpolation set.

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