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# LONG WALKS IN THE PLANE WITH FEW COLLINEAR POINTS

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### LONG WALKS IN THE PLANE WITH FEW COLLINEAR POINTS

#### JOSEPH L. GERVER

Let S be a set of vectors in  $\mathbb{R}^n$ . An S-walk is any (finite or infinite) sequence  $\{z_i\}$  of vectors in  $\mathbb{R}^n$  such that  $z_{i+1}-z_i \in S$  for all *i*. We will show that if the elements of S do not all lie on the same line through the origin, then for each integer  $K \geq 2$ , there exists an S-walk  $W_{\mathbb{K}} = \{z_i\}_{i=1}^{N(K)}$ such that no K+1 elements of  $W_{\mathbb{K}}$  are collinear and N(K)grows faster than any polynomial function of K.

Specifically, we will prove that

$$\log_2 N(K) > \frac{1}{9} (\log_2 K - 1)^2 - \frac{1}{6} (\log_2 K - 1)$$
.

We will then show that if the elements of S lie on at least L distinct lines through the origin, then there exists an S-walk of length N(K, L) with no K+1 elements collinear, such that  $N(K, L) \ge (1/4)L^*N(K-1)$ , where  $L-2 \le L^* \le L+1$  and  $L^* \equiv 0 \mod 4$ . In [3] it was shown that if  $S \subset Z^2$ , and for all  $s \in S$  we have  $||s|| \le M$ , then there does not exist an S-walk  $W = \{z_i\}_{i=1}^{N(K,M)}$  such that no K+1 elements of W are collinear and

$$\log_2 N(K, M) > 2^{13}M^4K^4 + \log_2 K$$
.

Before proving these theorems we introduce some notation. If  $A = (a_1, \dots, a_n)$  and  $B = (b_1, \dots, b_m)$  are ordered sets of vectors, we let  $RA = (a_n, \dots, a_1)$  and we let  $(A, B) = (a_1, \dots, a_n, b_1, \dots, b_m)$ . We let 2A = (A, A) and, for every positive integer k, we let (k+1)A = (kA, A). If J is a vector operator, we let  $JA = (Ja_1, \dots, Ja_n)$ .

THEOREM 1. Let S contain two vectors independent over R, and let K be an integer greater than or exual to 2. There exists an S-walk  $W_{\kappa} = \{z_{p}\}_{p=1}^{N(K)}$  such that no K + 1 elements of  $W_{\kappa}$  are collinear and such that

$$\log_2 N(K) > \frac{1}{9} (\log_2 K - 1)^2 - \frac{1}{6} (\log_2 K - 1)$$
.

*Proof.* If we let  $(\log_2 K - 1)^2/9 - (\log_2 K - 1)/6 = \log_2 K$ , then  $\log_2 K = (25 + 3\sqrt{65})/4 > 12$  or  $(25 - 3\sqrt{65})/4 < 1$ . Therefore if  $1 \le \log_2 K \le 12$ , and  $2 \le K \le 4096$ , then

$$\frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) < \log_2 K .$$

Since  $W_{\kappa}$  cannot have more than N(K) collinear points, we need only consider K > 4096.

We may let  $S = \{i, j\}$  without loss of generality, where *i* and *j* are orthonormal unit vectors.

For every positive integer m and nonnegative integer n, let  $A_0^m = i$ , and let

$$A_{n+1}^{m} = (mA_{n}^{m}, 2^{n}RJA_{n}^{m})$$
,

where Ji = j and Jj = i. Let  $V = \{v_p\}_{p=1}^N = \mu A_{\nu}^{\mu}$ , where  $\mu$  is the greatest integer less than or equal to  $((7/9)K)^{1/3}$ , and  $\nu$  is the least integer greater than or equal to  $\log_2 \mu - 3/2$ . Note that since K > 4096, we have  $\mu \ge 14$ , and  $\nu \ge 3$ . Let  $z_p = \sum_{q=1}^p v_q$  for each p, and let  $W = \{z_p\}_{p=1}^N$ . We maintain that W has no more than K collinear points and that  $\log_2 N > (\log_2 K - 1)^2/9 - (\log_2 K - 1)/6$ .

Let  $b_0 = 1$  and let  $b_{n+1} = (\mu + 2^n)b_n$ . Then  $b_n$  is the cardinality of  $A_n^{\mu}$ , and  $N = \mu b_{\nu}$ . Clearly  $b_n \ge \mu^n$ , so  $N \ge \mu^{\nu+1}$  and  $\log_2 N \ge (\nu + 1) \log_2 \mu \ge (\log_2 \mu - 1/2) \log_2 \mu$ . Since  $\mu$  is the greatest integer less than or equal to  $((7/9)K)^{1/3}$ , and  $((7/9)K)^{1/3} > 14$ , we have  $\mu > (14/15)((7/9)K)^{1/3} > ((1/2)K)^{1/3}$ . It follows that  $\log_2 N > 1/9[\log_2((1/2)K)]^2 - \log_2((1/2)K)/6 = (\log_2 K - 1)^2/9 - (\log_2 K - 1)/6$ .

We now prove that W has no more than K collinear points.

Let  $C_n^{\alpha} = \{z_p: \alpha b_n \leq p \leq (\alpha + 1)b_n\}$ . For each n, all  $C_n^{\alpha}$  are congruent; specifically one can get from any one to any other by a translation plus, possibly, a reflection about the major diagonal (i.e., a reflection about the line passing through the vector i + j, which interchanges i and j), followed by a rotation about the origin of 180°. This reflection plus rotation is equivalent to a reflection about the line perpendicular to the major diagonal (i.e., the line passing through the vector i - j). We will refer to this latter line as the minor diagonal. Let

$$U_n^eta = \{C_n^lpha:eta(\mu+2^n) \leq lpha < (eta+1)(\mu+2^n) \ ext{if } n 
eq 
u ext{ and } U_
u^\circ = \{C_
u^lpha: 0 \leq lpha \leq \mu\} \ .$$

Note that  $C_{n+1}^{\beta} = \{z_p: \beta(\mu + 2^n)b_n \leq p \leq (\beta + 1)(\mu + 2^n)b_n\}$ , so  $U_n^{\beta}$  is a partition of  $C_{n+1}^{\beta}$  and  $U_{\nu}^{\beta}$  is a partition of W. We now consider a line with slope m and determine for each n, the maximum number of elements of  $U_n^{\beta}$  which the line can intersect (the maximum number cannot depend on  $\beta$ , since all  $C_{n+1}^{\beta}$  are congruent). Let  $r_n$  be this maximum number. Then the line cannot intersect more than  $r = \prod_{\nu=0}^{\nu} r_n$  points of W.

Let  $s_n$  be the slope of  $z_{b_n}$ ; i.e.,  $s_n = y_n/x_n$  where  $z_{b_n} = x_n i + y_n j$ . The slope of  $z_{(\alpha+1)b_n} - z_{\alpha b_n}$  is then either  $s_n$  or  $s_n^{-1}$ , depending on whether  $C_n^{\alpha}$  is a simple translation of  $C_n^{\alpha}$ , or a translation of the reflection of  $C_n^{\alpha}$  about the minor diagonal. We wish to find a lower bound on  $s_n/s_{n-1}$ .

Now  $x_0 = 1$ ,  $y_0 = 0$ ,  $x_{n+1} = \mu x_n + 2^n y_n$ , and  $y_{n+1} = \mu y_n + 2^n x_n$ . It follows that  $x_n$ ,  $y_n$ , and  $s_n$  are strictly positive for all  $n \ge 1$ . We now prove by induction that  $s_n < 2^n/\mu$ . Clearly  $s_0 = 0 < 2^0/\mu$  and  $s_1 = 1/\mu < 2^1/\mu$ . Suppose  $s_n < 2^n/\mu$ . Let  $t_n = 2^n/s_n\mu$ . Then  $t_n > 1$ . Now

$$egin{aligned} \mathbf{s}_{n+1} &= (\mu y_n + 2^n x_n)/(\mu x_n + 2^n y_n) \ &= (\mu s_n + 2^n)/(\mu + 2^n s_n) \ &= (\mu s_n + \mu s_n t_n)/(\mu + \mu s_n^2 t_n) \ &= (s_n + s_n t_n)/(1 + s_n^2 t_n) \ . \end{aligned}$$

Thus

$$egin{aligned} t_{n+1} &= 2^{n+1} / s_{n+1} \mu = 2 s_n t_n / s_{n+1} \ &= 2 s_n t_n (1 + s_n^2 t_n) / (s_n + s_n t_n) \ &= 2 t_n (1 + s_n^2 t_n) / (t_n + 1) \;. \end{aligned}$$

We now view  $t_{n+1}$  as a function of the real variables  $t_n$  and  $s_n$ , and compute its partial derivatives:

$$\partial t_{n+1} / \partial t_n = 2(s_n^2 t_n^2 + 2s_n^2 t_n + 1) / (t_n + 1) > 0$$

and

$$\partial t_{n+1}/\partial s_n = 4t_n^2 s_n/(t_n+1) > 0$$
 .

Since  $t_{n+1}$  has the value 1 when  $s_n = 0$  and  $t_n = 1$ , it follows that  $t_{n+1} > 1$  when  $s_n \ge 0$  and  $t_n > 1$ , as is the case here. Therefore  $s_{n+1} < 2^{n+1}/\mu$ .

Next, recall that  $\nu - 1 < \log_2 \mu - 3/2$ , so if  $n \leq \nu - 1$ , then  $2^n \leq 2^{\nu-1} < 2^{-3/2}\mu$ . Since  $2^n > s_n\mu$ , it follows firstly that  $s_n < 2^{-3/2}$ , and secondly that

$$egin{aligned} &s_{n+1}/s_n = (\mu s_n + 2^n)/(\mu s_n + 2^n s_n^2) \ &> 2\mu s_n/(\mu s_n + 2^{-3/2}\mu s_n^2) \ &= 2/(1 + 2^{-3/2}s_n) > 2\left/ig(1 + rac{1}{8}ig) = rac{16}{9} \end{aligned}$$

It follows that, given m, there is at most one n such that  $(3/4)s_n \leq m \leq (4/3)s_n$ . Suppose there exists  $\lambda$  such that  $(3/4)s_2 \leq m \leq (4/3)s_2$ . Then  $m < (3/4)s_{2+1}$  and  $m > (4/3)s_{2-1}$ . Moreover, for all  $n > \lambda + 1$ , we have  $m < (27/64)s_n < (1/2)s_n$ , and for all  $n < \lambda - 1$ , we

have  $m > (64/27)s_n > 2s_n$ . All of the above also holds if we replace  $s_n$  by  $s_n^{-1}$ , except that some of the inequalities are reversed and constants replaced by their reciprocals in the obvious way.

We now calculate for each of the five cases,  $n = \lambda$ ,  $n = \lambda + 1$ ,  $n = \lambda - 1, n > \lambda + 1$ , and  $n < \lambda - 1$ , the maximum number  $r_n$  of elements of  $U_n^{\beta}$  which a line of slope *m* can intersect. We can assume without loss of generality that  $C_{n+1}^{\beta}$  is a simple translation of  $C_{n+1}^{\circ}$ ; if  $C_{n+1}^{\beta}$  is a translation of the reflection of  $C_{n+1}^{\circ}$  about the minor diagonal, then we can apply the same argument, replacing  $s_n$  by  $s_n^{-1}$ . Then  $C_n^{\alpha}$  is a simple translation of  $C_n^{\alpha}$  for  $\beta(\mu + 2^n) \leq 1$  $lpha < eta(\mu+2^n)+\mu$ , and a translation of the reflection of  $C^{\scriptscriptstyle 0}_n$  for  $\beta(\mu+2^n)+\mu \leq \alpha < (\beta+1)(\mu+2^n)$ . For each  $\alpha$ , the first point of  $C_n^{\alpha+1}$  coincides with the last point of  $C_n^{\alpha}$ . It is easy to prove by induction on n that  $C_n^{\circ}$  (and therefore  $C_n^{\alpha}$  for all  $\alpha$ ) lies entirely within a right triangle, with sides  $x_n$  and  $y_n$  adjacent to the right angle, and with the first and last points of  $C_n^0$  at opposite ends of the hypotenuse. Therefore the sets  $C_n^{\alpha}$ :  $\beta(\mu + 2^n) \leq \alpha < \beta(\mu + 2^n) + \beta(\mu + 2^n)$  $\mu$  lie within congruent right triangles, whose hypotenuses are adjacent segments of a line with slope  $s_n$  (see Fig. 1). It follows



FIGURE 1

that a line with slope  $m > s_n q/(q-1)$  or  $m < s_n (q-1)/q$  can intersect at most q of the sets  $C_n^{\alpha}: \beta(\mu+2^n) \leq \alpha < \beta(\mu+2^n) + \mu$  at distinct points (i.e., assign the last point of each set  $C_n^{\alpha}$  to the set  $C_n^{\alpha+1}$ , and do not count the line as intersecting  $C_n^{\alpha}$  if it only intersects this last point). Suppose  $m \leq 1$ . Then  $m < (1/2)s_s^{-1}$ , and a line of slope *m* can intersect no more than two of the sets  $C_n^{\alpha}$ :  $\beta(\mu + 2^n) + \beta(\mu + 2^n)$  $\mu \leq \alpha < (\beta + 1)(\mu + 2^n)$ . If  $n = \lambda$ , then a line of slope m can intersect all  $\mu$  of the sets  $C_n^{\alpha}: \beta(\mu + 2^n) \leq \alpha < \beta(\mu + 2^n) + \mu$  for a total of  $\mu + 2$ . If  $n = \lambda + 1$  or  $\lambda - 1$ , the line can intersect at most 4 of the sets  $C_n^{\alpha}: \beta(\mu+2^n) \leq \alpha < \beta(\mu+2^n) + \mu$ , for a total of 6, while if  $n > \lambda + 1$  or  $n < \lambda - 1$ , the line can intersect at most two of the sets  $C_n^{\alpha}: \beta(\mu+2^n) \leq \alpha < \beta(\mu+2^n) + \mu$  for a total of 4. If m > 1, then we obtain essentially the same results by redefining  $\lambda$  so that  $(3/4)s_{\lambda}^{-1} \leq m \leq (4/3)s_{\lambda}^{-1}$ , the only difference being that  $\mu$  is replaced by 2<sup>n</sup>, which in any case is less than  $\mu$ . Therefore we have  $r_n \leq \mu + 2$  if  $n = \lambda$ ,  $r_n \leq 6$  if  $n = \lambda - 1$  or  $\lambda + 1$ , and  $r_n \leq 4$  for all other *n*. Finally, we have

$$egin{aligned} r &= \prod_{n=0}^
u r_n \leq (\mu+2) \cdot 6^2 \cdot 4^{
u-2} < 36(\mu+2) \cdot 4^{\log_2 \mu - 5/2} \ &= rac{36}{32} \mu^2 (\mu+2) \leq rac{9}{7} \mu^3 \leq K \;. \end{aligned}$$

If  $\lambda$  does not exist, then there are at most two values of *n* for which  $(27/64)s_n \leq m \leq (64/27)s_n$ , and these two values can take the place of  $\lambda - 1$  and  $\lambda + 1$  in our argument.

REMARK. We can use this method to get slightly better results as follows: The method works by partitioning W into a heiarchy of sets, each set of order n + 1 being partitioned into  $\mu + 2^n$  sets of order n, and showing that for almost all n, a given line can intersect at most four sets of order n within a given set of order n + 1. Suppose that instead of using the partition based on the sets  $C_n^a$ , we modify this partition slightly by splitting each  $C_n^a$  into two sets of order n, namely  $\{z_p: \alpha b_n \leq p \leq \alpha b_n + \mu b_{n-1}\}$  and  $\{z_p: \alpha b_n + \mu b_{n-1} \leq p \leq (\alpha + 1)b_n\}$ . Then each set of order n + 1 would have either  $2\mu$  or  $2^{n+1}$  sets of order n, and it should not be hard to show that for almost all n, a given line can intersect at most three sets of order n within a given set of order n + 1. We would then have  $r = c\mu \cdot 3^\nu = c\mu^{1+\log_2 3}$ , where c is a constant which does not depend on K, and finally

$$\log_2 N = (1 + \log_2 3)^{-2} (\log_2 K)^2 + O(\log_2 K)$$
.

However, it seems impossible to push this method any further.

THEOREM 2. Suppose that S contains L elements which are pairwise independent over R. Then there exists an S-walk  $\Omega = \{u_i\}_{i=1}^N$  containing no set of K + 1 collinear points, such that

$$\log_2 N > \frac{1}{9} [\log_2 (K-1) - 1]^2 - \frac{1}{6} [\log_2 (K-1) - 1] + \log_2 L^* - 2 ,$$

where  $L-2 \leq L^* \leq L+1$  and  $L^* \equiv 0 \mod 4$ .

**Proof.** The L elements of S with distinct arguments must include L/2 elements (if L is even) or (L+1)/2 elements (if L is odd) in the same half-plane. Label these elements  $s_1, s_2, s_3, \cdots$  in order of their arguments. For  $1 \leq n \leq (1/4)L^*$ , let  $W_n = \varphi_n W$ where W is defined as in the proof of Theorem 1, and  $\varphi_n$  is the linear vector operator which maps i to  $s_{2n-1}$  and j to  $s_{2n}$ . Let  $N_0$ be the cardinality of W and let  $w_n = xs_{2n-1} + ys_{2n}$  be the final element of  $W_n$ . For  $1 \leq i \leq N_0$ , let  $z_i$  be defined as in the proof of Theorem 1, and let  $u_i = \varphi_1 z_1$ . Let  $u_{N_0n+i} = \sum_{j=1}^n w_j + \varphi_{n+1} z_i$  for  $1 \leq n \leq (1/4)L^* - 1$ . Finally, let  $N = (1/4)L^*N_0$  and let  $\mathcal{Q} = \{\boldsymbol{u}_i\}_{i=1}^N$ . Note that  $\mathcal{Q}$  is constructed by placing the  $W_n$  end to end in sequence.

By Theorem 1,

$$\log_2 N > rac{1}{9} (\log_2 K - 1)^2 - rac{1}{6} (\log_2 K - 1) + \log_2 L^* - 2 \; .$$

We will now prove that no K + 2 points of  $\Omega$  are collinear. Substituting K-1 for the bound variable K then gives us Theorem 2 for the case  $K \ge 3$ . For the case K = 2, we simply let  $u_i = \sum_{j=1}^{i} s_j$ . The resulting set  $\{u_i\}$ , which contains at least  $(1/2)L^*$ elements, is the set of vertices of a convex polygon; hence no three elements are collinear.

Let  $T_n = \{u_i\}_{i=N_0(n-1)+1}^{N_0(n-1)+1}$  and let  $t_n = \sum_{j=1}^n w_j$ , so that  $t_n$  is the final element of  $T_n$ . Let  $t_0 = 0$  and let  $r_n = t_{n-1} + xs_{2n-1}$  for  $n \ge 1$ . Note that  $t_n = r_n + ys_{2n}$ . Note also that from results proved previously, the set  $T_n$  must lie entirely on or in the interior of the triangle  $\Delta_n$  with vertices  $t_{n-1}$ ,  $r_n$ , and  $t_n$ . Consequently any line which intersects  $T_n$  must intersect  $\Delta_n$ . Now consider the polygon P with vertices  $t_0, r_1, t_1, r_2, t_2, \cdots, r_{L^*/4}, t_{L^*/4}$  in that order. The (directed) edges of this polygon are the vectors  $xs_1, ys_2, xs_3, \cdots$ ,  $ys_{L^{*/2}}$ , and  $-x\sum_{n=1}^{J^{*/4}} s_{2n-1} - y\sum_{n=1}^{L^{*/4}} s_{2n}$ . Since the vectors  $s_1, s_2, s_3, \cdots$ are listed in order of increasing argument, and the range of all their arguments is less than 180°, it follows that the interior angles of P are all less than  $180^{\circ}$ , so P is convex. Now any line intersecting  $\Delta_n$ , and in particular any line intersecting  $T_n$ , must intersect at least two sides of  $\Delta_n$  (including each vertex in its two adjacent sides), and therefore must intersect P. Since P is convex, a line can only intersect P at one or two points, or along an edge. Therefore no line can intersect more than two of the  $T_{m}$ . Unless the slope of a line is between that of  $s_{2n-1}$  and  $s_{2n}$  inclusive, it can only intersect one point of  $T_n$ . By Theorem 1, no line can intersect more than K points of  $T_n$ . Therefore, no line can contain more than K+1 points of  $\Omega$ .

REMARK. In order to compare these results with the upper bound in [3], we can consider the case where  $S = \{s \in Z^2 : ||s|| \leq M\}$ . Since the number of lattice points in a disc of radius R is  $\pi R^2 + O(R)$  [2], we know that the number of lattice points with both coordinates divisible by q, in a disc of radius M, is  $\pi M^2/q^2 + O(M/q)$ . Therefore the number L of lattice points with relatively prime coordinates is

$$\pi M^2 \sum_{n=0}^{\infty} (-1)^n \sum_{q \in Q_n} q^{-2} + O(M \sum_{q \in Q} q^{-1})$$
 ,

where Q is the set of square free positive integers less than or equal to M, and  $Q_n$  is the set of integers in Q with n distinct prime factors. It follows [1] that

$$L = 6M^2/\pi + O(M\log M)$$
 .

Finally, if we let N(K, M) be the length of the longest S-walk with no more than K collinear points, and we choose any constants  $c_1 < (9 \log 2)^{-1}$  and  $c_2 > 2^{13} \log 2$ , then we have

 $M^2 \exp [c_1 (\log K)^2] < N(K, M) < \exp [c_2 M^4 K^4]$ 

for all M and all but a finite number of K.

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