

# Pacific Journal of Mathematics

**ANNIHILATION OF IDEALS IN COMMUTATIVE RINGS**

JAMES A. HUCKABA AND JAMES M. KELLER

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**Four theorem are proved concerning the annihilation of finitely generated ideals contained in the set of zero divisors of a commutative ring.**

1. Introduction. An important theorem in commutative ring theory is that if  $I$  is an ideal in a Noetherian ring and if  $I$  consists entirely of zero divisors, then the annihilator of  $I$  is nonzero. This result fails for some non-Noetherian rings, even if the ideal  $I$  is finitely generated. We say that a commutative ring  $R$  has *Property (A)* if every finitely generated ideal of  $R$  consisting entirely of zero divisors has nonzero annihilator. Property (A) was originally studied by Y. Quentel in [7]. (Our Property (A) is Quentel's Condition (C).) Theorem 1 shows that all nontrivial graded rings have Property (A). (For our purposes a *nontrivial graded ring* is a ring  $R$  graded over the integers such that  $R$  contains an element  $x$ , not a zero divisor, of positive homogenous degree.) Theorem 2 completely characterizes those reduced rings with Property (A).

Property (A) is closely connected with two other conditions on a reduced ring. One is the *annihilator condition* (a.c.): If  $(a, b)$  is an ideal of  $R$ , then there exists  $c \in R$  such that  $\text{Ann}(a, b) = \text{Ann}(c)$ . The other condition is that  $\text{MIN}(R)$ , the space of minimal prime ideals of  $R$ , is compact. Our Theorem 3 shows that for a reduced coherent ring  $R$  Property (A), (a.c.), and the total quotient ring of  $R$  being a von Neumann regular ring are equivalent conditions; and that each (and hence all) of these conditions imply that  $\text{MIN}(R)$  is compact. Finally, in Theorem 4, we prove that every reduced nontrivial graded ring satisfies (a.c.).

We assume that all rings are commutative with identity. If  $R$  is such a ring, let  $T(R)$  be the total quotient ring of  $R$ , let  $Z(R)$  be the set of zero divisors of  $R$ , and let  $Q(R)$  denote the complete ring of quotients of  $R$  as defined in [5]. Elements of  $R$  that are not zero divisors are called *regular elements*.

2. Graded rings.. Y. Quentel, [7, p. 269], proved that if  $R$  is a reduced ring, then the polynomial ring  $R[X]$  satisfies Property (A). We generalize this to arbitrary nontrivial graded rings, and hence to polynomial rings that are not necessarily reduced.

**THEOREM 1.** *If  $R$  is nontrivial graded ring, then  $R$  satisfies Property (A).*

*Proof.* Let  $I = (a_1, \dots, a_p)$  be an ideal of  $R$  contained in  $Z(R)$ . For  $i = 1, \dots, p$ , let  $a_i = \sum_{k=m_i}^{n_i} b_k^{(i)}$  be the homogeneous decomposition of  $a_i$ , where  $\deg b_k^{(i)} = k$ . Let  $x$  be a regular homogeneous element in  $R$  of degree  $t > 0$ . Construct an element  $a$  as follows:

$$a = a_1 + a_2x^{s_2} + \dots + a_px^{s_p},$$

where the  $s_i$  are integers such that  $ts_2 + m_2 > n_1$ , and  $ts_i + m_i > n_{i-1} + ts_{i-1}$ ;  $i = 3, \dots, p$ . There exists a nonzero homogeneous element  $c$  such that  $ca = 0$ . (The proof of this is identical to the proof of McCoy's Theorem: If  $f$  is a zero divisor in  $R[X]$ , then there is a nonzero  $b \in R$  such that  $bf = 0$ .)

Since  $\deg[b_k^{(i)}x^{s_i}] \neq \deg[b_k^{(j)}x^{s_j}]$  unless  $i = j$  and  $k = h$ , the homogeneous components of  $a$  are  $\{b_k^{(i)}x^{s_i}\}_{i=1, \dots, p}^{k=m_i, \dots, n_i}$ . Thus, by the unique representation in terms of the homogeneous components  $cb_k^{(i)}x^{s_i} = 0$  for all  $i, k$ . Since  $x \notin Z(R)$ ,  $cb_k^{(i)} = 0$  for all  $i, k$ . Therefore,  $c \in \text{Ann}(I)$ .

**COROLLARY 1.** *If  $R$  is any ring, then the polynomial ring  $R[X]$  satisfies Property (A).*

**3. Reduced rings.** In this section all rings are assumed to be reduced.

**THEOREM 2.** *For a reduced ring  $R$ , the following statements are equivalent:*

- (1)  $R$  has Property (A);
- (2)  $T(R)$  has property (A);
- (3) If  $I$  is a finitely generated ideal of  $R$  contained in  $Z(R)$ , then  $I$  is contained in a minimal prime ideal of  $R$ ;
- (4) Every finitely generated ideal of  $R$  contained in  $Z(R)$ , extends to a proper ideal in  $Q(R)$ .

*Proof.* (1)  $\leftrightarrow$  (2) is clear.

(1)  $\rightarrow$  (3): Assume that  $I$  is a finitely generated ideal contained in  $Z(R)$ , but not contained in a minimal prime ideal of  $R$ . Then  $cI = 0$  implies that  $c$  is in every minimal prime ideal of  $R$ ; i.e.,  $c = 0$ .

(3)  $\rightarrow$  (1): Let  $I = (x_1, \dots, x_n) \subset P$ ,  $P$  a minimal prime ideal of  $R$ . By [2, p. 111], choose  $z_i \in \text{Ann}(x_i)$ ,  $z_i \notin P$ . Then  $z = z_1z_2 \dots z_n \neq 0$  and  $z \in \bigcap_{i=1}^n \text{Ann}(x_i) = \text{Ann}(I)$ .

(1)  $\rightarrow$  (4): If  $I$  is a finitely generated ideal contained in  $Z(R)$ , then  $IQ(R)$  has nonzero annihilator in  $Q(R)$ . Hence,  $IQ(R) \subsetneq Q(R)$ . has nonzero annihilator in  $Q(R)$ . Hence,  $IQ(R) \subsetneq Q(R)$ .

(4)  $\rightarrow$  (1): Assume that  $I$  is a finitely generated dense ideal of  $R$  such that  $I \subset Z(R)$ . (A subgroup  $H$  of a ring  $R$  is *dense*, if

$\text{Ann } H = 0$ .) Then  $I$  is dense in  $Q(R)$ , [5, p. 41], and whence  $IQ(R)$  is dense in  $Q(R)$ . But  $Q(R)$  is a von Neumann regular ring, [5, p. 42]; and von Neumann regular rings have Property (A), [3, p. 30]. By the equivalence of (1) and (3) of this theorem,  $IQ(R)$  is not contained in any minimal prime ideal of  $Q(R)$ . But in  $Q(R)$ , minimal prime ideals are maximal. Therefore,  $IQ(R) = Q(R)$ , a contradiction.

The results about the compactness of  $\text{MIN}(R)$  that we need are summarized in Theorems A and B.

**THEOREM A.** *The following conditions on a reduced ring  $R$  are equivalent:*

- (1)  $Q(R)$  is a flat  $R$ -module;
- (2)  $\text{MIN}(R)$  is compact;
- (3)  $\{M \cap R : M \in \text{Spec } Q(R)\} = \text{MIN}(R)$ ;
- (4) If  $a \in R$  and if  $U = \{M \in \text{Spec } Q(R) : a \notin M \cap R\}$ , then there exists a finitely generated ideal  $I$  such that

$$\text{Spec } Q(R) \setminus U = \{M \in \text{Spec } Q(R) : I \not\subset M \cap R\};$$

- (5) If  $X$  is an indeterminate, then  $T(R[X])$  is a von Neumann regular ring.

*Proof.* A. C. Mewburn, in [6], proved the equivalence of (1) through (4). Quentel proved that (2) and (5) are equivalent, [7].

**THEOREM B.** *The following conditions on a reduced ring  $R$  are equivalent:*

- (1)  $T(R)$  is a von Neumann regular ring;
- (2)  $R$  satisfies Property (A) and  $\text{MIN}(R)$  is compact;
- (3)  $R$  satisfies (a.c.) and  $\text{MIN}(R)$  is compact.

*Proof.* In [7], Quentel proved the equivalence of (1) and (2); while M. Henriksen and M. Jerison, [2], showed that (1) and (3) are the same.

A ring  $R$  is *coherent* in case  $I$  is a finitely generated ideal of  $R$  implies there is an exact sequence  $R^m \rightarrow R^n \rightarrow I \rightarrow 0$ .

**THEOREM 3.** *For a reduced coherent ring  $R$ , the following conditions are equivalent:*

- (1)  $R$  has Property (A);
- (2)  $R$  has (a.c.);
- (3)  $T(R)$  is a von Neumann regular ring.

*Proof.* (1)  $\rightarrow$  (3): In view of Theorem B(2) we must show that

$\text{MIN}(R)$  is compact. Let  $x \in R$ . Since  $R$  is a coherent ring,  $\text{Ann}(x) = I$  is a finitely generated ideal of  $R$ , [1, p. 462]. Let  $U = \{M \in \text{Spec } Q(R) : x \notin M \cap R\}$ . Assume that  $I \subset M \cap R$  for some  $M \in \text{Spec } Q(R) \setminus U$ , then the ideal  $(I, x) \subset M \cap R$ . It is clear that  $M \cap T(R)$  is a proper ideal of  $T(R)$  and that  $M \cap R = M \cap T(R) \cap R$ . Hence,  $(I, x) \subset M \cap R \subset Z(R)$ . By Property (A),  $\text{Ann}(I, x) \neq 0$ . But this contradicts the fact that the ideal  $(I, x) = xR + \text{Ann}(x)$  is dense, [5, p. 42]. By Theorem A(4),  $\text{MIN}(R)$  is compact.

(2)  $\rightarrow$  (3): Let  $x \in R$ , then  $\text{Ann}(x) = (z_1, \dots, z_n)$  and  $\text{Ann}\{\text{Ann}(x)\} = \text{Ann}(z_1, \dots, z_n) = \text{Ann}(z)$ . This last condition, given in [2], implies that  $\text{MIN}(R)$  is compact (even if  $R$  does not have a unit).

(3)  $\rightarrow$  (1) and (3)  $\rightarrow$  (2) are clear.

**COROLLARY 2.** *Let  $R$  be a reduced coherent ring.*

(1) *If  $R$  satisfies any (and hence all) of the conditions of Theorem 3, the  $\text{MIN}(R)$  is compact.*

(2) *If  $R$  is a nontrivial graded ring, then  $T(R)$  is a von Neumann regular ring.*

**THEOREM 4.** *If  $R$  is a reduced nontrivial graded ring, then  $R$  satisfies (a.c.).*

*Proof.* Let  $(a, b)$  be an ideal in  $R$ . If  $(a, b) \not\subset Z(R)$ , then  $\text{Ann}(a, b) = \text{Ann}(1)$ . Assume that  $(a, b) \subset Z(R)$ , and write  $a$  and  $b$  in terms of their homogeneous components; say,  $a = a_m + \dots + a_n$  and  $b = b_h + \dots + b_k$ . Let  $x$  be a homogeneous element of  $R$ ,  $x \notin Z(R)$ , of degree  $t > 0$ . Choose an integer  $s$  satisfying  $h + st > n$  and let  $c = a_m + \dots + a_n + b_h x^s + \dots + b_k x^s$ .

Since  $R$  is a reduced ring,  $\text{Ann}(c) = \bigcap P$ , where  $P$  varies over the minimal prime ideals of  $R$  not containing  $c$ . By Lemma 3 of [8, p. 153], each  $P$  is a homogeneous ideal. Hence,  $\bigcap P = \text{Ann}(c)$  is also homogeneous.

Let  $d$  be a homogeneous element in  $\text{Ann}(c)$ . Then  $da_i = 0$  and  $db_j x^s = 0$  for all  $i, j$ . Then,  $da = 0 = db$  and we have  $\text{Ann}(c) \subset \text{Ann}(a, b)$ . The other inclusion is obvious.

Let  $R$  be a graded ring which contains a regular homogeneous element. Define  $T_q = \{a/b : a \text{ and } b \text{ are homogeneous, } b \text{ is regular, and } q = \text{degree } a - \text{degree } b\}$ . Just as in the integral domain case, [8, p. 157],  $\Sigma T_q$  is a graded ring containing  $R$  as a graded subring.

**COROLLARY 3.** *Let  $R$  be a reduced nontrivial graded ring. The following statements are equivalent:*

(1)  $\text{MIN}(R)$  is compact;

- (2)  $\text{MIN}(T_0)$  is compact;  
 (3)  $T(R)$  is a von Neumann regular ring.

*Proof.* (1)  $\leftrightarrow$  (3) by Theorem B.

(1)  $\leftrightarrow$  (2): If  $S$  is the set of regular homogeneous elements of  $R$ , then  $R_S = \Sigma T_q$  and  $\text{MIN}(R)$  is homeomorphic to  $\text{MIN}(R_S)$ . By [4, Lemma 1], there is a one-to-one order preserving correspondence between the graded prime ideals of  $R_S$  and the graded prime ideals of  $T_0$ . It follows from [8, p. 153] that the minimal prime ideals of a graded ring are graded. Thus,  $\text{MIN}(R_S)$  is homeomorphic to  $\text{MIN}(T_0)$ .

REMARKS. (1)  $\text{MIN}(R)$  compact  $\rightarrow$  Property A or (a.c.). This follows from an example in [6]. (2) Property (A)  $\rightarrow$   $\text{MIN}(R)$  compact. By [6, p. 427], there is a ring  $R$  for which  $\text{MIN}(R)$  is not compact. Applying Theorem B(5),  $T(R[X])$  is not von Neumann regular. But  $R[X]$  has Property (A), [7, p. 269]. Thus,  $\text{MIN}(R[X])$  cannot be compact.

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Received October 9, 1978 and in revised form January 30, 1979.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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