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Four theorem are proved concerning the annihilation of finitely generated ideals contained in the set of zero divisors of a commutative ring.

1. Introduction. An important theorem in commutative ring theory is that if I is an ideal in a Noetherian ring and if I consists entirely of zero divisors, then the annihilator of I is nonzero. This result fails for some non-Noetherian rings, even if the ideal I is finitely generated. We say that a commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero divisors has nonzero annihilator. Property (A) was originally studied by Y. Quentel in [7]. (Our Property (A) is Quentel's Condition (C).) Theorem 1 shows that all nontrivial graded rings have Property (A). (For our purposes a *nontrivial graded ring* is a ring R graded over the integers such that R contains an element x, not a zero divisor, of positive homogenous degree.) Theorem 2 completely characterizes those reduced rings with Property (A).

Property (A) is closely connected with two other conditions on a reduced ring. One is the *annihilator condition* (a.c.): If (a, b) is an ideal of R, then there exists $c \in R$ such that $\mathrm{Ann}(a, b) = \mathrm{Ann}(c)$. The other condition is that $\mathrm{MIN}(R)$, the space of minimal prime ideals of R, is compact. Our Theorem 3 shows that for a reduced coherent ring R Property (A), (a.c.), and the total quotient ring of R being a von Neumann regular ring are equivalent conditions; and that each (and hence all) of these conditions imply that $\mathrm{MIN}(R)$ is compact. Finally, in Theorem 4, we prove that every reduced nontrivial graded ring satisfies (a.c.).

We assume that all rings are commutative with identity. If R is such a ring, let $T(R)$ be the total quotient ring of R, let $Z(R)$ be the set of zero divisors of R, and let $Q(R)$ denote the complete ring of quotients of R as defined in [5]. Elements of R that are not zero divisors are called *regular elements*.

2. Graded rings.. Y. Quentel, [7, p. 269], proved that if R is a reduced ring, then the polynomial ring $R[X]$ satisfies Property (A). We generalize this to arbitrary nontrivial graded rings, and hence to polynomial rings that are not necessarily reduced.

THEOREM 1. *If R is nontrivial graded ring, then R satisfies Property* (A).

Proof. Let $I = (a_1, \cdots, a_p)$ be an ideal of R contained in $Z(R)$. For $i = 1, \cdots, p$, let $a_i = \sum_{k=m_i}^{n_i} b_k^{(i)}$ be the homogeneous decomposition of a_i, where $\deg b_k^{(i)} = k$. Let x be a regular homogeneous element in R of degree $t > 0$. Construct an element a as follows:

$$a = a_1 + a_2 x^{s_2} + \cdots + a_p x^{s_p} ,$$

where the s_i are integers such that $ts_2 + m_2 > n_1$, and $ts_i + m_i > n_{i-1} + ts_{i-1}$; $i = 3, \cdots, p$. There exists a nonzero homogeneous element c such that $ca = 0$. (The proof of this is identical to the proof of McCoy's Theorem: If f is a zero divisor in $R[X]$, then there is a nonzero $b \in R$ such that $bf = 0$.)

Since $deg[b_k^{(i)} x^{s_i}] \neq \deg[b_h^{(j)} x^{s_j}]$ unless $i = j$ and $k = h$, the homogeneous compontets of a are $\{b_k^{(i)} x^{s_i}\}_{i=1,\cdots p}^{k=m_i,\cdots,n_i}$. Thus, by the unique representation in terms of the homogeneous components $cb_k^{(i)} x^{s_i} = 0$ for all i, k. Since $x \notin Z(R)$, $cb_k^{(i)} = 0$ for all i, k. Therefore, $c \in \mathrm{Ann}(I)$.

COROLLARY 1. *If R is any ring, then the polynomial ring $R[X]$ satisfies Property* (A).

3. Reduced rings. In this section all rings are assumed to be reduced.

THEOREM 2. *For a reduced ring R, the following statements are equivalent*:

(1) *R has Property* (A);

(2) *$T(R)$ has property* (A);

(3) *If I is a finitely generated ideal of R contained in $Z(R)$, then I is contained in a minimal prime ideal of R;*

(4) *Every finitely generated ideal of R contained in $Z(R)$, extends to a proper ideal in $Q(R)$.*

Proof. $(1) \leftrightarrow (2)$ is clear.

$(1) \rightarrow (3)$: Assume that I is a finitely generated ideal contained in $Z(R)$, but not contained in a minimal prime ideal of R. Then $cI = 0$ implies that c is in every minimal prime ideal of R; i.e., $c=0$.

$(3) \rightarrow (1)$: Let $I = (x_1, \cdots, x_n) \subset P$, P a minimal prime ideal of R. By [2, p. 111], choose $z_i \in \mathrm{Ann}(x_i)$, $z_i \notin P$. Then $z = z_1 z_2 \cdots z_n \neq 0$ and $z \in \bigcap_{i=1}^n \mathrm{Ann}(x_i) = \mathrm{Ann}(I)$.

$(1) \rightarrow (4)$: If I is a finitely generated ideal contained in $Z(R)$, then $IQ(R)$ has nonzero annihilator in $Q(R)$. Hence, $IQ(R) \subsetneqq Q(R)$.
has nonzero annihilator in $Q(R)$. Hence, $IQ(R) \subsetneqq Q(R)$.

$(4) \rightarrow (1)$: Assume that I is a finitely generated dense ideal of R such that $I \subset Z(R)$. (A subgroup H of a ring R is *dense*, if

$\operatorname{Ann} H = 0$.) Then I is dense in $Q(R)$, [5, p. 41], and whence $IQ(R)$ is dense in $Q(R)$. But $Q(R)$ is a von Neumann regular ring, [5, p. 42]; and von Neumann regular rings have Property (A), [3, p. 30]. By the equivalence of (1) and (3) of this theorem, $IQ(R)$ is not contained in any minimal prime ideal of $Q(R)$. But in $Q(R)$, minimal prime ideals are maximal. Therefore, $IQ(R) = Q(R)$, a contradiction.

The results about the compactness of $\mathrm{MIN}(R)$ that we need are summarized in Theorems A and B.

THEOREM A. *The following conditions on a reduced ring R are equivalent*:

(1) *$Q(R)$ is a flat R-module*;

(2) *$\mathrm{MIN}(R)$ is compact*;

(3) $\{M \cap R: M \in \operatorname{Spec} Q(R)\} = \mathrm{MIN}(R)$;

(4) *If $a \in R$ and if $U = \{M \in \operatorname{Spec} Q(R): a \notin M \cap R\}$, then there exists a finitely generated ideal I such that*

$$\operatorname{Spec} Q(R) \backslash U = \{M \in \operatorname{Spec} Q(R): I \not\subset M \cap R\} ;$$

(5) *If X is an indeterminate, then $T(R[X])$ is a von Neumann regular ring.*

Proof. A. C. Mewburn, in [6], proved the equivalence of (1) through (4). Quentel proved that (2) and (5) are equivalent, [7].

THEOREM B. *The following conditions on a reduced ring R are equivalent*:

(1) *$T(R)$ is a von Neumann regular ring*;

(2) *R satisfies Property* (A) *and* $\mathrm{MIN}(R)$ *is compact*;

(3) *R satisfies* (a.c.) and $\mathrm{MIN}(R)$ *is compact.*

Proof. In [7], Quentel proved the equivalence of (1) and (2); while M. Henriksen and M. Jerison, [2], showed that (1) and (3) are the same.

A ring R is *coherent* in case I is a finitely generated ideal of R implies there is an exact sequence $R^m \to R^n \to I \to 0$.

THEOREM 3. *For a reduced coherent ring R, the following conditions are equivalent*:

(1) *R has Property* (A);

(2) *R has* (a.c.);

(3) *$T(R)$ is a von Neumann regular ring.*

Proof. (1) $\to$ (3): In view of Theorem B(2) we must show that

$\mathrm{MIN}(R)$ is compact. Let $x \in R$. Since R is a coherent ring, $\mathrm{Ann}(x) = I$ is a finitely generated ideal of R, [1, p. 462]. Let $U = \{M \in \operatorname{Spec} Q(R)\colon x \notin M \cap R\}$. Assume that $I \subset M \cap R$ for some $M \in \operatorname{Spec} Q(R)\backslash U$, then the ideal $(I, x) \subset M \cap R$. It is clear that $M \cap T(R)$ is a proper ideal of $T(R)$ and that $M \cap R = M \cap T(R) \cap R$. Hence, $(I, x) \subset M \cap R \subset Z(R)$. By Property (A), $\mathrm{Ann}(I, x) \neq 0$. But this contradicts the fact that the ideal $(I, x) = xR + \mathrm{Ann}(x)$ is dense, [5, p. 42]. By Theorem A(4), $\mathrm{MIN}(R)$ is compact.

$(2) \rightarrow (3)$: Let $x \in R$, then $\mathrm{Ann}(x) = (z_1, \cdots, z_n)$ and $\mathrm{Ann}\{\mathrm{Ann}(x)\} = \mathrm{Ann}(z_1, \cdots, z_n) = \mathrm{Ann}(z)$. This last condition, given in [2], implies that $\mathrm{MIN}(R)$ is compact (even if R does not have a unit).

$(3) \rightarrow (1)$ and $(3) \rightarrow (2)$ are clear.

COROLLARY 2. *Let R be a reduced coherent ring.*

(1) *If R satisfies any (and hence all) of the conditions of Theorem 3, the* $\mathrm{MIN}(R)$ *is compact.*

(2) *If R is a nontrivial graded ring, then $T(R)$ is a von Neumann regular ring.*

THEOREM 4. *If R is a reduced nontrivial graded ring, then R satisfies* (a.c.).

Proof. Let (a, b) be an ideal in R. If $(a, b) \not\subset Z(R)$, then $\mathrm{Ann}(a, b) = \mathrm{Ann}(1)$. Assume that $(a, b) \subset Z(R)$, and write a and b in terms of their homogeneous components; say, $a = a_m + \cdots + a_n$ and $b = b_h + \cdots + b_k$. Let x be a homogeneous element of R, $x \notin Z(R)$, of degree $t > 0$. Choose an integer s satisfying $h + st > n$ and let $c = a_m + \cdots + a_n + b_h x^s + \cdots + b_k x^s$.

Since R in a reduced, $\mathrm{Ann}(c) = \cap P$, where P varies over the minimal prime ideals of R not containing c. By Lemma 3 of [8, p. 153], each P is a homogeneous ideal. Hence, $\cap P = \mathrm{Ann}(c)$ is also homogeneous.

Let d be a homogeneous element in $\mathrm{Ann}(c)$. Then $da_i = 0$ and $db_j x^s = 0$ for all i, j. Then, $da = 0 = db$ and we have $\mathrm{Ann}(c) \subset \mathrm{Ann}(a, b)$. The other inclusion is obvious.

Let R be a graded ring which contains a regular homogeneous element. Define $T_q = \{a/b\colon a$ and b are homogeneous, b is regular, and $q = \text{degree } a - \text{degree } b\}$. Just as in the integral domain case, [8, p. 157], ΣT_q is a graded ring containing R as a graded subring.

COROLLARY 3. *Let R be a reduced nontrivial graded ring. The following statements are equivalent*:

(1) $\mathrm{MIN}(R)$ *is compact*;

(2) $\mathrm{MIN}(T_0)$ *is compact;*
(3) $T(R)$ *is a von Neumann regular ring.*

Proof. (1) $\leftrightarrow$ (3) by Theorem B.

(1) $\leftrightarrow$ (2): If S is the set of regular homogeneous elements of R, then $R_S = \Sigma T_q$ and $\mathrm{MIN}(R)$ is homeomorphic to $\mathrm{MIN}(R_S)$. By [4, Lemma 1], there is a one-to-one order preserving correspondence between the graded prime ideals of R_S and the graded prime ideals of T_0. It follows from [8, p. 153] that the minimal prime ideals of a graded ring are graded. Thus, $\mathrm{MIN}(R_S)$ is homeomorphic to $\mathrm{MIN}(T_0)$.

REMARKS. (1) $\mathrm{MIN}(R)$ compact $\nrightarrow$ Property A or (a.c.). This follows from an example in [6]. (2) Property (A) $\nrightarrow$ $\mathrm{MIN}(R)$ compact. By [6. p. 427], there is a ring R for which $\mathrm{MIN}(R)$ is not compact. Applying Theorem B(5), $T(R[X])$ is not von Neumann regular. But $R[X]$ has Property (A), [7, p. 269]. Thus, $\mathrm{MIN}(R[X])$ cannot be compact.

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UNIVERSITY OF MISSOURI
COLUMBIA, MO 65211

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