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**A DUAL RELATIONSHIP BETWEEN GENERALIZED
ABEL-GONČAROV BASES AND CERTAIN PINCHERLE BASES**

FRIEDRICH HASLINGER

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F. HASLINGER

Recent results on Abel-Gončarov polynomial expansions are applied to study the representability of holomorphic functions as infinite series in a given Pincherle sequence. As a generalization of the ordinary derivative we consider the so-called Gel'fond-Leont'ev derivative \mathcal{D} . We take the exponential function with respect to the derivative \mathcal{D} and use a duality principle in order to investigate the completeness of the system $E_n(z) = z^n E(\lambda_n z)$ in the space \mathcal{F}_r of functions holomorphic on the interior of the disc of radius $r \leq \infty$. Finally we study the uniqueness of the representability of holomorphic functions as infinite series in the system E_n .

1. Basic facts and definitions. Let $0 < r \leq \infty$. We shall be interested in the nuclear Fréchet space \mathcal{F}_r consisting of all functions holomorphic on the open disk of radius r , equipped with the topology of uniform convergence on compact sets (see [15]). For the topology in the space \mathcal{F}_r , we can take the norms $\|\cdot\|_{r'}$, $0 < r' < r$ given by $\|f\|_{r'} = \max\{|f(z)|: |z| = r'\}$, $f \in \mathcal{F}_r$. It is easily seen that by Cauchy's estimates that system of norms $\{\|\cdot\|_{r'}, 0 < r' < r\}$ is equivalent to the system of norms $\{\|\cdot\|_{r'}, 0 < r' < r\}$, where

$$\|\cdot\|_{r'} = \sup_{0 \leq k < \infty} |a_k| r'^k,$$

for $f \in \mathcal{F}_r$ with Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

We recall that two systems of seminorms $\{\|\cdot\|_p, p \in P\}$ and $\{\|\cdot\|_q, q \in P\}$ are equivalent, if for each $p \in P$ there exists a constant K_p depending on p and $q \in P$ such that $\|\cdot\|_p \leq K_p \|\cdot\|_q$, and if for each $p' \in P$ there exists a constant $K_{p'}$ depending on p' and $q' \in P$ such that $\|\cdot\|_{p'} \leq K_{p'} \|\cdot\|_{q'}$.

A sequence $(f_n)_{n=0}^{\infty}$ in \mathcal{F}_r is complete if the set of all finite linear combinations of the functions f_n is dense in \mathcal{F}_r . And $(f_n)_{n=0}^{\infty}$ is a basis in \mathcal{F}_r if each $f \in \mathcal{F}_r$ has a representation

$$f = \sum_{n=0}^{\infty} c_n f_n,$$

where $(c_n)_{n=0}^{\infty}$ is a sequence of scalars uniquely determined by f and

the infinite series converges in the topology of \mathcal{F}_r . Two bases $(f_n)_{n=0}^\infty$ and $(g_n)_{n=0}^\infty$ are equivalent if $\sum_{n=0}^\infty c_n f_n$ converges in \mathcal{F}_r if and only if $\sum_{n=0}^\infty c_n g_n$ converges in \mathcal{F}_r . As is well known, the sequence of the functions $(z^n)_{n=0}^\infty$ constitutes a basis for any space $\mathcal{F}_r (0 < r \leq \infty)$. A basis $(f_n)_{n=0}^\infty$ is called proper if it is equivalent to $(z^n)_{n=0}^\infty$ (see [1], [4]).

M. Arsove [1], in a series of papers, has considered Pincherle sequences $(f_n)_{n=0}^\infty$ in which f_n has the form

$$f_n(z) = z^n \psi_n(z), \quad n = 0, 1, 2, \dots$$

where each function $\psi_n \in \mathcal{F}_r$ and $\psi_n(0) = 1$. Recently, in [1] Arsove and in [4] Dubinsky studied linear Pincherle sequences (see also [9])

$$f_n(z) = z^n \left(1 - \frac{z}{z_n}\right), \quad n = 0, 1, 2, \dots$$

In this paper we investigate the problem of determining when a system

$$E_n(z) = z^n E(\lambda_n z), \quad n = 0, 1, 2, \dots$$

is complete in \mathcal{F}_r , when it is not complete and when it is a basis, even a proper basis in \mathcal{F}_r . Here $(\lambda_n)_{n=0}^\infty$ is a sequence of scalars and E is a generalized exponential function corresponding to a so-called Gel'fond-Leont'ev derivative \mathcal{D} (see [8]).

Let $(d_k)_{k=1}^\infty$ denote a nondecreasing sequence of positive numbers. The Gel'fond-Leont'ev derivative \mathcal{D} is defined by

$$\mathcal{D}f(z) = \sum_{k=1}^\infty d_k a_k z^{k-1},$$

where

$$f(z) = \sum_{k=0}^\infty a_k z^k.$$

As in [2] or [7] we suppose that the sequence $(d_k)_{k=1}^\infty$ satisfies the following condition

$$(1.1) \quad (d_{k+1}/d_k)_{k=1}^\infty \text{ is nonincreasing and has limit } 1.$$

Then it follows

$$\lim_{k \rightarrow \infty} d_k^{1/k} = 1.$$

Thus if f has radius of convergence $c(f)$ then

$$\mathcal{D}f(z) = \sum_{k=1}^{\infty} d_k a_k z^{k-1}$$

has also radius of convergence $c(f)$.

The operator \mathcal{D} corresponds to the ordinary derivative when $d_k = k$ ($k = 1, 2, \dots$) and to the shift operator \mathcal{S} when $d_k = 1$ ($k = 1, 2, \dots$). \mathcal{S} is defined by

$$\mathcal{S}f(z) = \sum_{k=1}^{\infty} a_k z^{k-1}.$$

The operators \mathcal{D}^n ($n = 1, 2, \dots$) are the successive iterates of \mathcal{D} and we have

$$\mathcal{D}^n f(z) = \sum_{k=n}^{\infty} \frac{e_{k-n}}{e_k} a_k z^{k-n},$$

where $e_0 = d_0 = 1$ and $e_n = (d_1 d_2 \dots d_n)^{-1}$ for $n \geq 1$.

We write

$$E(z) = \sum_{k=0}^{\infty} e_k z^k$$

and note that this function bears the same relationship to the operator \mathcal{D} that the exponential function bears to the ordinary differentiation. This means

$$E(0) = 1 \quad \text{and} \quad \mathcal{D}E(z) = E(z).$$

Let $R = c(E)$, then, by the monotonicity of the sequence $(d_k)_{k=1}^{\infty}$ we have (see [2])

$$R = \lim_{k \rightarrow \infty} d_k = \sup_{1 \leq k < \infty} d_k.$$

The E -type of a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is the number

$$\tau_E(f) = \limsup_{k \rightarrow \infty} |a_k / e_k|^{1/k}.$$

If $R < \infty$ then

$$(1.2) \quad \tau_E(f) = \frac{R}{c(f)}, \quad (\text{see [2], [7]}).$$

Now we define for a sequence $(\lambda_k)_{k=0}^{\infty}$ of scalars the polynomials $Q_n(z; \lambda_0, \dots, \lambda_{n-1})$ by $Q_0(z) \equiv 1$ and

$$Q_n(z; \lambda_0, \dots, \lambda_{n-1}) = e_n z^n - \sum_{k=0}^{n-1} e_{n-k} \lambda_k^{n-k} Q_k(z; \lambda_0, \dots, \lambda_{k-1}).$$

It is easily seen that

$$(1.3) \quad e_n z^n = \sum_{k=0}^n e_{n-k} \lambda_k^{n-k} Q_k(z; \lambda_0, \dots, \lambda_{k-1}).$$

The polynomials $Q_n(z; \lambda_0, \dots, \lambda_{n-1})$ are called the Gončarov polynomials belonging to the operator \mathcal{D} (see [2]). They reduce to the ordinary Gončarov polynomials if $d_k = k$ ($k = 1, 2, \dots$) and the remainder polynomials if $d_k = 1$ ($k = 1, 2, \dots$).

One verifies easily that

$$(1.4) \quad \mathcal{D}^k Q_n(\lambda_k; \lambda_0, \dots, \lambda_{n-1}) = \delta_{nk} \quad (\text{see [2]}).$$

Therefore the polynomials $Q_n(z; \lambda_0, \dots, \lambda_{n-1})$ are biorthogonal to the linear functionals

$$\mathcal{L}_n(f) = \mathcal{D}^n f(\lambda_n).$$

Now we consider the problem under which conditions the polynomials $Q_n(z; \lambda_0, \dots, \lambda_{n-1})$ constitute a basis in \mathcal{F}_r , i.e.,

$$f(z) = \sum_{n=0}^{\infty} \mathcal{D}^n f(\lambda_n) Q_n(z; \lambda_0, \dots, \lambda_{n-1})$$

for each $f \in \mathcal{F}_r$ and the infinite series converges in the topology of \mathcal{F}_r .

In this connection the Whittaker constant $W(\mathcal{D})$ belonging to the operator \mathcal{D} plays an important role. We can introduce the Whittaker constant $W(\mathcal{D})$ by

$$W(\mathcal{D}) = \left(\sup_{1 \leq n < \infty} H_n^{1/n} \right)^{-1},$$

where

$$H_n = \max |Q_n(0; \lambda_0, \dots, \lambda_{n-1})| \quad (n = 1, 2, \dots)$$

and the maximum is taken over all sequences $(\lambda_k)_{k=0}^{n-1}$ whose terms lie on the unit circle (see Buckholtz and Frank [2]).

The Whittaker constant satisfies the inequality (see [2])

$$(1.5) \quad 0 < \frac{d_1}{2} \leq W(\mathcal{D}) < d_1.$$

In [7], Frank and Shaw investigated the above problem and the following theorem is an easy consequence of their Theorem A in [7]:

THEOREM A. *Let $(\lambda_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that*

$$|\lambda_n| \leq \frac{e_{n+1}}{e_n} s, \quad n = 0, 1, 2, \dots$$

for a real number $s > 0$. Then the Gončarov polynomials constitute a basis in any space \mathcal{F}_r for

$$r > \frac{s}{W(\mathcal{D})}.$$

The following theorem, which is again an easy consequence of a theorem due to Buckholtz and Frank [3], shows that Theorem A is sharp in a certain sense:

THEOREM B. *Let r and s be positive numbers such that*

$$\frac{s}{W(\mathcal{D})} > r.$$

Then there exists a holomorphic function F of radius of convergence r such that $\mathcal{D}^n F$ has a zero in $|z| \leq (e_{n+1}/e_n) s$ for all but finitely many n .

In the following we will use Theorem A and Theorem B and two duality principles for \mathcal{F}_r in order to investigate the behavior of the Pincherle sequences

$$E_n(z) = z^n E(\lambda_n z) \quad n = 0, 1, 2, \dots,$$

where E is the exponential function belonging to the operator \mathcal{D} and $(\lambda_n)_{n=0}^\infty$ is a given sequence of scalars.

2. Completeness of the system $\{z^n E(\lambda_n z)\}$. Let $E \in \mathcal{F}_R$ ($0 < R < \infty$) with the power series expansion

$$E(z) = \sum_{k=0}^\infty e_k z^k \quad \text{and} \quad \limsup_{k \rightarrow \infty} |e_k|^{1/k} = \frac{1}{R}.$$

We suppose that $e_0 = 1$ and $e_k > 0$ for $k = 1, 2, \dots$.

In the sequel, we will always require that the sequence $(e_k)_{k=0}^\infty$ satisfies the following conditions:

(2.1a) $(e_{k-1}/e_k)_{k=1}^\infty$ is nondecreasing;

(2.1b) $(e_k^2/e_{k-1}e_{k+1})_{k=1}^\infty$ is nonincreasing and has limit 1 (compare (1.1)).

From condition (2.1a) we have

$$\lim_{k \rightarrow \infty} (e_k/e_{k-1}) = \frac{1}{R}$$

since $E \in \mathcal{F}_R$.

THEOREM 1. *Let $(\lambda_n)_{n=0}^\infty$ be a sequence of complex numbers such that*

$$|\lambda_n| \leq \frac{e_{n+1}s}{e_n} \quad n = 0, 1, 2, \dots$$

for a real number $s > 0$. Then the system $\{z^n E(\lambda_n z)\}_{n=0}^\infty$ is complete in any space \mathcal{F}_r for $R/r \geq s/W(\mathcal{D})$.

Proof. Here we use the following well known form of the Hahn-Banach theorem: A subset $G \subseteq \mathcal{F}_r$ is dense in \mathcal{F}_r if and only if for each continuous linear functional L on \mathcal{F}_r such that $L(g) = 0$ for each $g \in G$ it follows that $L = 0$.

Let $\{(e_k)_{k=0}^\infty, r\}$ denote the space of all holomorphic functions $h(z) = \sum_{k=0}^\infty h_k z^k$ with the property

$$\limsup_{k \rightarrow \infty} |h_k/e_k|^{1/k} < r.$$

This means that the functions $h \in \{(e_k)_{k=0}^\infty, r\}$ are holomorphic on the disk $|z| \leq R/r$, since

$$\limsup_{k \rightarrow \infty} |h_k/e_k|^{1/k} = R \limsup_{k \rightarrow \infty} |h_k|^{1/k} < r,$$

and on the other hand that the function

$$h_E(z) = \sum_{k=0}^\infty \frac{h_k z^{-k-1}}{e_k}$$

is holomorphic for $|z| \geq r$.

A duality between \mathcal{F}_r and $\{(e_k)_{k=0}^\infty, r\}$ is defined by the bilinear forms

$$(2.2) \quad \langle g, h \rangle = \frac{1}{2\pi i} \int_\gamma g(z) h_E(z) dz,$$

where $g \in \mathcal{F}_r$, $h \in \{(e_k)_{k=0}^\infty, r\}$ and γ is a circle contained in the intersection of the domain of holomorphy of g with the domain of holomorphy of h_E .

Formula (2.2) gives the general form of the continuous linear functionals on \mathcal{F}_r (see [6] or [12]).

Now let $L \in \mathcal{F}'_r$ such that $L(E_n) = 0$ for $n = 0, 1, 2, \dots$, where $E_n(z) = z^n E(\lambda_n z)$. Then there exists a function $h \in \{(e_k)_{k=0}^\infty, r\}$ such that

$$\begin{aligned} L(E_n) &= \frac{1}{2\pi i} \int_\gamma z^n E(\lambda_n z) h_E(z) dz = \frac{1}{2\pi i} \int_\gamma \left(\sum_{k=0}^\infty e_k \lambda_n^k z^{k+n} \right) \left(\sum_{k=0}^\infty \frac{h_k z^{-k-1}}{e_k} \right) dz \\ &= \sum_{k=n}^\infty \frac{e_{k-n} \lambda_n^{k-n} h_k}{e_k} = \mathcal{D}^n h(\lambda_n). \end{aligned}$$

By condition (2.1a) and inequality (1.5) we have

$$|\lambda_n| \leq \frac{e_{n+1}}{e_n} s \leq e_1 s < \frac{s}{W(\mathcal{D})} \leq \frac{R}{r},$$

which implies that $E_n \in \mathcal{F}_r$ for $n = 0, 1, 2, \dots$.

Since

$$H = \left(\limsup_{k \rightarrow \infty} |h_k|^{1/k} \right)^{-1} > \frac{R}{r}$$

the assumption $R/r \geq s/W(\mathcal{D})$ implies that the corresponding Gončarov-polynomials constitute a basis in \mathcal{F}_H (see Theorem A). By the uniqueness-property of a basis we have $h \equiv 0$ if $\mathcal{D}^n h(\lambda_n) = 0$ for $n = 0, 1, 2, \dots$. Now it follows $L = 0$, which completes our proof.

In the next theorem we show that Theorem 1 is sharp in a certain sense:

THEOREM 2. *Let r and s be positive numbers such that $se_1 < R/r < s/W(\mathcal{D})$. Then there exists a sequence of complex numbers $(\lambda_n)_{n=0}^\infty$ with the property*

$$|\lambda_n| \leq \frac{e_{n+1}}{e_n} s$$

such that the functions $E_n(z) = z^n E(\lambda_n z)$ are in \mathcal{F}_r but are not complete in \mathcal{F}_r .

Proof. We have to show that there exists a continuous linear functional $L_0 \neq 0$ on \mathcal{F}_r such that $L_0(E_n) = 0$ for $n = 0, 1, 2, \dots$, where

$$E_n(z) = z^n E(\lambda_n z) \quad \text{for } n = 0, 1, 2, \dots$$

and $(\lambda_n)_{n=0}^\infty$ is a suitable sequence of complex numbers. In view of the proof of Theorem 1 it suffices to show that there exists a function

$$h_0 \in \{(e_k)_{k=0}^\infty, r\}$$

such that $\mathcal{D}^n h_0(\lambda_n) = 0$ for $n = 0, 1, 2, \dots$ and $h_0 \neq 0$.

In order to find such a function h_0 we apply Theorem B: by our assumption

$$\frac{R}{r} < \frac{s}{W(\mathcal{D})}$$

we can find a number H_0 such that

$$\frac{R}{r} < H_0 < \frac{s}{W(\mathcal{D})},$$

and by Theorem B there exists a function \tilde{h}_0 with $c(\tilde{h}_0) = H_0$ such that $\mathcal{D}^n \tilde{h}_0$ has a zero in

$$|z| \leq \frac{e_{n+1}s}{e_n}$$

for all but finitely many n .

This implies that we can find a sequence $(\lambda_n)_{n=0}^\infty$ of complex numbers with

$$|\lambda_n| \leq \frac{e_{n+1}s}{e_n}$$

for $n = 0, 1, 2, \dots$ such that $|\mathcal{D}^n \tilde{h}_0(\lambda_n)| < \infty$ for $0 \leq n \leq N$ (take for instance $\lambda_n = 0$ for $0 \leq n \leq N$) and $\mathcal{D}^n \tilde{h}_0(\lambda_n) = 0$ for $n > N$, where N is a suitable natural number. Since $e_1 s < R/r$, we have $|\lambda_n| \leq (e_{n+1}/e_n)s \leq e_1 s < R/r$, which implies that $E_n \in \mathcal{F}_r$ for $n = 0, 1, 2, \dots$.

Now define a polynomial p_0 by

$$p_0(z) = \sum_{n=0}^N \mathcal{D}^n \tilde{h}_0(\lambda_n) Q_n(z; \lambda_0, \dots, \lambda_{n-1}).$$

Then p_0 is a polynomial of degree not greater than N and has the property

$$\mathcal{D}^n p_0(\lambda_n) = \mathcal{D}^n \tilde{h}_0(\lambda_n) \quad \text{for } 0 \leq n \leq N,$$

and $\mathcal{D}^n p_0(\lambda_n) = 0$ for $n > N$ (see part 1).

We set now

$$h_0 = \tilde{h}_0 - p_0,$$

then

$$\mathcal{D}^n h_0(\lambda_n) = 0 \quad \text{for } n = 0, 1, 2, \dots$$

and $c(h_0) = H_0$.

If we write

$$h_0(z) = \sum_{k=0}^{\infty} h_{0,k} z^k,$$

then

$$\limsup_{k \rightarrow \infty} |h_{0,k}/e_k|^{1/k} < r,$$

since $H_0 > R/r$. This means $h_0 \in \{(e_k)_{k=0}^\infty, r\}$. So if we set

$$L_0(g) = \langle g, h_0 \rangle = \frac{1}{2\pi i} \int_r g(z)(h_0)_E(z) dz ,$$

then $L_0(E_n) = 0$ for $n = 0, 1, 2, \dots$ and $L_0 \neq 0$.

The desired conclusion now follows again from the Hahn-Banach theorem.

3. Uniqueness of the representation by the system $\{z^n E(\lambda_n z)\}$.
 The purpose of this part is to derive conditions under which the system $\{z^n E(\lambda_n z)\}$ constitutes a basis in its closed linear hull in a certain space \mathcal{F}_r . In order to do this we use a dual relationship between basis theory and interpolation theory developed by M. M. Dragilev, V. P. Zaharjuta and Ju. F. Korobeinik in 1974 (see [4]):

Let X be a nuclear Fréchet space with a topology given by a family of seminorms $\{\|\cdot\|_p, p \in P\}$; let X' be the strong dual space. We consider two sequence spaces generated by a sequence $\{x_n\}_{n=0}^\infty$ of nonzero elements of X :

$$\mathcal{E} = \left\{ c = (c_n)_{n=0}^\infty : |c|_p := \sum_{n=0}^\infty |c_n| \|x_n\|_p < \infty, \text{ for each } p \in P \right\}$$

with the topology determined by the family of seminorms $\{|c|_p, p \in P\}$, and

$$\mathcal{E}' = \left\{ c' = (c'_n)_{n=0}^\infty : \text{there exists a } p \in P \text{ with } |c'|_p := \sup_n \frac{|c'_n|}{\|x_n\|_p} < \infty \right\}$$

with the topology of the strong dual with respect to duality, given by the formula

$$\langle c, c' \rangle = \sum_{n=0}^\infty c_n c'_n .$$

THEOREM C. (See [4], [12].) *Let X be a nuclear Fréchet space. A sequence $\{x_n\}_{n=0}^\infty$ constitutes a basis in its closed linear hull in X if and only if for each sequence $(t_n)_{n=0}^\infty \in \mathcal{E}'$ there exists $x' \in X'$ such that*

$$x'(x_n) = t_n \text{ for } n = 0, 1, 2, \dots .$$

(In this case one says that the interpolation problem $(X', \{x_n\}_{n=0}^\infty)$ is solvable).

In the sequel we use Theorem C for the system $E_n(z) = z^n E(\lambda_n z)$ considered in part 2.

THEOREM 3. *Let $(\lambda_n)_{n=0}^\infty$ be a sequence of complex numbers such that*

$$|\lambda_n| \leq \frac{e_{n+1}s}{e_n} \quad n = 0, 1, 2, \dots$$

for a real number $s > 0$. Then the system $\{z^n E(\lambda_n z)\}_{n=0}^\infty$ constitutes a basis in \mathcal{F}_r for any $r > 0$ with the property $R/r \geq s/W(\mathcal{D})$.

Proof. In order to apply the above principle we remark that Theorem C says that $\{E_n\}_{n=0}^\infty$ constitutes a basis in its closed linear hull if and only if for each sequence $(t_n)_{n=0}^\infty$ with the property

$$(3.1) \quad |t_n| \leq K \|E_n\|_{r'} \quad n = 0, 1, 2, \dots,$$

where $r' < r$ and K is a constant only depending on r' , there exists a continuous linear functional $L \in \mathcal{F}'_r$ represented by a function $h \in \{(e_k)_{k=0}^\infty, r\}$ such that

$$L(E_n) = \langle E_n, h \rangle = \mathcal{D}^n h(\lambda_n) = t_n,$$

for $n = 0, 1, 2, \dots$.

We take a sequence $(t_n)_{n=0}^\infty$ such that inequality (3.1) holds. Since the systems of norms $(\|\cdot\|_{r'}, r' < r)$ and $(\|\cdot\|_{r'}, r' < r)$ are equivalent in \mathcal{F}_r , inequality (3.1) can be replaced by

$$(3.2) \quad |t_n| \leq Kr'^n \sup_k (e_k |\lambda_n|^{k r'^k}) \quad n = 0, 1, 2, \dots$$

This follows from the fact that

$$E_n(z) = z^n \sum_{k=0}^\infty e_k \lambda_n^k z^k$$

and by the definition of the norms $\|\cdot\|_{r'} (r' < r)$. Now we obtain

$$\limsup_{n \rightarrow \infty} |t_n|^{1/n} \leq r' \limsup_{n \rightarrow \infty} \left[\sup_k (e_k |\lambda_n|^{k r'^k}) \right]^{1/n}.$$

By inequality (1.5) and the assumption $R/r \geq s/W(\mathcal{D})$ we have $|\lambda_n| < R/r$ for $n = 0, 1, 2, \dots$. This means $E_n \in \mathcal{F}_r$ for $n = 0, 1, 2, \dots$, and

$$\sup_k (e_k |\lambda_n|^{k r'^k}) \leq \sup_k \left(e_k \left(\frac{R}{r} \right)^k r'^k \right).$$

Since $(R/r)r' < R$, we have

$$\sup_k \left(e_k \left(\frac{R}{r} \right)^k r'^k \right) < K_E,$$

where K_E is a constant depending on E .

This implies

$$\limsup_{n \rightarrow \infty} |t_n|^{1/n} \leq r'.$$

Now we obtain

$$\limsup_{n \rightarrow \infty} |e_n t_n|^{1/n} \leq (\limsup_{n \rightarrow \infty} e_n^{1/n}) \left(\limsup_{n \rightarrow \infty} |t_n|^{1/n} \right) \leq \frac{r'}{R}.$$

By Theorem A the Gončarov-polynomials $Q_n(z; \lambda_0, \dots, \lambda_{n-1})$ constitute a basis in $\mathcal{F}_{R/r'}$, since

$$\frac{R}{r'} > \frac{R}{r} \geq \frac{s}{W(\mathcal{D})}.$$

Consider the polynomials

$$e_n^{-1} Q_n(z; \lambda_0, \dots, \lambda_{n-1});$$

since

$$e_n^{-1} Q_n(z; \lambda_0, \dots, \lambda_{n-1}) = z^n - \sum_{k=0}^{n-1} \frac{e_{n-k}}{e_n} \lambda_k^{n-k} Q_k(z; \lambda_0, \dots, \lambda_{k-1}),$$

(see 1.3) it follows that the bases $\{e_n^{-1} Q_n(z; \lambda_0, \dots, \lambda_{n-1})\}_{n=0}^\infty$ and $\{z^n\}_{n=0}^\infty$ are equivalent in $\mathcal{F}_{R/r'}$, i.e.,

$$\sum_{n=0}^\infty c_n e_n^{-1} Q_n(z; \lambda_0, \dots, \lambda_{n-1})$$

converges in $\mathcal{F}_{R/r'}$ if and only if $\sum_{n=0}^\infty c_n z^n$ converges in $\mathcal{F}_{R/r'}$ (see [14], pg. 188).

Now since $\limsup_{n \rightarrow \infty} |e_n t_n|^{1/n} \leq r'/R$, we have $\sum_{n=0}^\infty e_n t_n z^n$ converges in $\mathcal{F}_{R/r'}$, and therefore $\sum_{n=0}^\infty t_n Q_n(z; \lambda_0, \dots, \lambda_{n-1})$ converges in $\mathcal{F}_{R/r'}$; in other words: there exists a function $h \in \mathcal{F}_{R/r'}$ such that

$$\mathcal{D}^n h(\lambda_n) = t_n \quad n = 0, 1, 2, \dots.$$

The fact that $h \in \mathcal{F}_{R/r'}$ implies that for

$$h(z) = \sum_{k=0}^\infty h_k z^k$$

we have

$$\limsup_{k \rightarrow \infty} |h_k|^{1/k} \leq \frac{r'}{R} < \frac{r}{R}$$

and hence

$$\limsup_{k \rightarrow \infty} \left| \frac{h_k}{e_k} \right|^{1/k} < r.$$

This means $h \in \{(e_k)_{k=0}^\infty, r\}$; now by the representation of the continuous linear functionals on \mathcal{F}_r we see that there exists a continuous linear functional $L \in \mathcal{F}'_r$ such that

$$L(E_n) = \langle E_n, h \rangle = \mathcal{D}^n h(\lambda_n) = t_n \quad n = 0, 1, 2, \dots$$

Theorem C implies that the system $\{E_n\}_{n=0}^\infty$ constitutes a basis in its closed linear hull in \mathcal{F}_r , by Theorem 1 the system $\{E_n\}_{n=0}^\infty$ is complete in \mathcal{F}_r , so $\{E_n\}_{n=0}^\infty$ constitutes a basis in \mathcal{F}_r and the proof of Theorem 3 is finished.

By [14] it follows that the system $\{z^n E(\lambda_n z)\}_{n=0}^\infty$ constitutes a basis in \mathcal{F}_r which is equivalent to the canonical basis $\{z^n\}_{n=0}^\infty$. We remark that under the assumptions of Theorem 2 the system $\{z^n E(\lambda_n z)\}_{n=0}^\infty$ does not constitute a basis for \mathcal{F}_r , because the system $\{z^n E(\lambda_n z)\}_{n=0}^\infty$ is even not complete in \mathcal{F}_r .

Some other results of this kind can be found in [11], [12] or [13] (see also the references in [11]). But these are all sufficient conditions for a system $\{z^n f(\lambda_n z)\}_{n=0}^\infty$ to be a basis in \mathcal{F}_r and there is no similar result to Theorem 2.

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Michael James Beeson, <i>Goodman's theorem and beyond</i>	1
Robert S. Cahn and Michael E. Taylor, <i>Asymptotic behavior of multiplicities of representations of compact groups</i>	17
Douglas Michael Campbell and Vikramaditya Singh, <i>Valence properties of the solution of a differential equation</i>	29
J.-F. Colombeau, Reinhold Meise and Bernard Perrot, <i>A density result in spaces of Silva holomorphic mappings</i>	35
Marcel Ern�, <i>On the relativization of chain topologies</i>	43
Le Baron O. Ferguson, <i>Uniform and L_p approximation for generalized integral polynomials</i>	53
Kenneth R. Goodearl and David E. Handelman, <i>Homogenization of regular rings of bounded index</i>	63
Friedrich Haslinger, <i>A dual relationship between generalized Abel-Gon�arov bases and certain Pincherle bases</i>	79
Miriam Hausman, <i>Generalization of a theorem of Landau</i>	91
Makoto Hayashi, <i>2-factorization in finite groups</i>	97
Robert Marcus, <i>Stochastic diffusion on an unbounded domain</i>	143
Isabel Dotti de Miatello, <i>Extension of actions on Stiefel manifolds</i>	155
C. David (Carl) Minda, <i>The hyperbolic metric and coverings of Riemann surfaces</i>	171
Somashekhar Amrith Nainpally and Mohan Lal Tikoo, <i>On T_1-compactifications</i>	183
Chia-Ven Pao, <i>Asymptotic stability and nonexistence of global solution for a semilinear parabolic equation</i>	191
Shigeo Segawa, <i>Harmonic majoration of quasibounded type</i>	199
Sze-Kai Tsui and Steve Wright, <i>The splitting of operator algebras</i>	201
Bruce Williams, <i>Hopf invariants, localization and embeddings of Poincar� complexes</i>	217
Leslie Wilson, <i>Nonopenness of the set of Thom-Boardman maps</i>	225
Alicia B. Winslow, <i>There are 2^c nonhomeomorphic continua in $\beta R^n - R^n$</i>	233