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HOMOTOPY WITH m -FUNCTIONS

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1. Introduction. M -functions were introduced by G. Darbo [1] and R. Jerrard [5] as a generalization of continuous functions between topological spaces. They are weighted, finitely-valued functions with a property corresponding to that of usual continuity. In [1] and [5] it was shown that ordinary singular homology groups for compact polyhedra are actually m -homotopy type invariants. In [6] it was shown that m -homotopy type is a stronger invariance than homotopy type in the sense that two spaces may have different homotopy types but the same m -homotopy type. R. Schultz [8] has noted that m -homology differs from singular homology on some compact metric spaces. It has also been brought to our attention that in a 1975 letter, G. Bredon indicated a method of proving that m -homotopy classes of PL m -functions on finite complexes are in 1 — 1 correspondence with chain homotopy classes of chain maps. His approach is quite different from the one used in this paper. Here we define m -homotopy groups (actually R -modules) and give some of their properties. We show that for a compact polyhedron, the n th singular homology group and the n th m -homotopy group are actually isomorphic.

We show, for example, that the n th m -homotopy group has a natural definition as $m\pi_n(Y) = \text{hom}(S^n, Y)$ in a certain category of m -functions, which is an R -module under the addition of m -functions defined below. This addition turns out to be the extension to m -functions of the usual product in homotopy groups. Since $\text{hom}(X, Y)$ is always an R -module in this category, we see that m -homotopy groups (and hence singular homology groups) are special cases of the R -module $\text{hom}(X, Y)$, which is a joint m -homotopy (and topological) invariant of X and Y .

Next we show that m -homotopy theory is a homology theory by proving it satisfies the Eilenberg-Steenrod axioms [4]. The excision axiom is of special interest since it completely fails to hold for usual homotopy. It is proven to hold in m -homotopy theory by introducing several combinatorial lemmas (§4).

There is a connection between the results here and the Dold-Thom theorem [2]. They showed that $H_m(Y) \cong \pi_m(AG(Y))$ where $AG(Y)$ is the topological free abelian group on the pointed polyhedron Y . There is a natural relationship between m -functions from X to Y and functions from X to $AG(Y)$. However, we show that there are m -functions $X \rightarrow Y$ with no corresponding continuous function

$X \rightarrow AG(Y)$ and vice versa.

2. *M-functions.* We give below a brief definition of *m-functions*. For motivation we refer the reader to [5].

Let X and Y be Hausdorff spaces and R a ring with identity and without zero divisors (in most examples $R = \mathbf{Z}$ or \mathbf{R}). Suppose we are given that:

- (i) $f: X \rightarrow Y$ is a multiple-valued function such that each $f(x)$ is a finite or empty subset of Y ,
- (ii) $\bar{f}: X \times Y \rightarrow R$ is a (standard) function which defines f as a subset of $X \times Y$ by $f = \text{cl}\{(x, y) \mid \bar{f}(x, y) \neq 0\}$, and
- (iii) for any $x \in X$ and any open set $V \subset Y$ such that $\partial V \cap f(X) = \emptyset$ there exists a neighborhood U of x such that for $x' \in U$,

$$\sum_{y \in V} \bar{f}(x, y) = \sum_{y \in V} \bar{f}(x', y).$$

Then an *m-function* (denoted just by f) is f together with the weighting factor determined by the *defining function* \bar{f} . The *multiplicity* of f is $m(f) = \sum_{y \in Y} \bar{f}(x, y)$; it is independent of x if X is connected. The empty *m-function*, denoted by \emptyset is defined by $\bar{\emptyset}: X \times Y \rightarrow 0$. Any continuous function can be regarded as an *m-function* by assigning it multiplicity one.

The composition of $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is defined by $\overline{g \circ f}(x, z) = \sum_{y \in Y} \bar{f}(x, y) \bar{g}(y, z)$, so Hausdorff spaces and *m-functions* over R form a category $R\text{-}T2$, with $T2$ as a subcategory. Any two *m-functions* may be added: $f + g$ is defined by $\overline{f + g} = \bar{f} + \bar{g}$. Also, if $a \in R$ we define the *m-function* af by $\overline{af} = a\bar{f}$. Then $\text{hom}(X, Y)$ is an R -module and there are functors $\text{hom}(_, Z)$ and $\text{hom}(Z, _): R\text{-}T2 \rightarrow (R \text{ modules})$. The restriction of $f: X \rightarrow Y$ to a subset $A \subset X$ is defined by $f|A = f \circ i$ when i is the inclusion $i: A \rightarrow X$. An *m-function* $F: X \times I \rightarrow Y$ is an *m-homotopy* between $F|X \times \{0\}$ and $F|X \times \{1\}$ (denoted by \sim_m). One can form *m-homotopy* classes of *m-functions* and these preserve the ring structure, that is, $[f + g] = [f] + [g]$ and $[af] = a[f]$.

We shall work primarily in the category $R_0\text{-}phT2$ of pointed pairs of Hausdorff spaces and *m-homotopy* classes of *m-functions* over R of multiplicity zero, together with its hom-sets (they are R -modules) and its hom-functors (see [7]). An *m-function* on pointed pairs $f: (X, A, x_0) \rightarrow (Y, B, y_0)$ must satisfy $f|A: A \rightarrow B$ and $f|x_0: x_0 \rightarrow y_0$.

LEMMA 2.1. *In $R_0\text{-}phT2$ the above condition for an *m-function* to be pointed is equivalent to $f|x_0 = \emptyset$; also, for*

$$f: X \longrightarrow Y, \quad f: (X, A, x_0) \longrightarrow (Y, y_0, y_0)$$

if and only if $f|A = \emptyset$. In particular the morphisms do not depend upon the choice of base point in the image space.

Proof. We know from the pointedness condition that $\tilde{f}(x_0, y) = 0$ if $y \neq y_0$, and then from the zero multiplicity that $\tilde{f}(x_0, y_0) = 0$. Thus x_0 has no image points of nonzero multiplicity and $f|x_0 = \emptyset$. A similar argument gives the second conclusion, and the converses are trivial.

Working only with m -functions of zero multiplicity entails almost no loss of generality. To any given m -function of multiplicity $a \in R$ can be added the constant m -function of multiplicity $(-a)$ and image y_0 to get an m -function of multiplicity zero which is the representative of the given m -function in $R_0\text{-ph}T2$.

3. M -homotopy groups. In this section we define m -homotopy groups and the subsidiary concepts of boundary operator and induced homomorphism. We also obtain the surprising result that the usual product $[f][g]$ of two group elements is actually m -homotopic to $[f + g]$, the addition defined in §2. Thus the group operation is addition and m -homotopy groups turn out to be hom-sets in $R_0\text{-ph}T2$, which are R -modules.

For any pair $(X, A) = (X, A, \emptyset)$ and integer $n \geq 1$ we define the n th m -homotopy group, $m\pi_n(X, A)$ to have as underlying set, the set of m -homotopy classes of m -functions (of multiplicity zero) $f: (B^n, S^{n-1}, 1) \rightarrow (X, A)$. B^n , S^{n-1} , and 1 are subsets of E^n defined by $B^n = \{x \mid |x| \leq 1\}$, $S^{n-1} = \{x \mid |x| = 1\}$, and $1 = \{(1, 0, 0, \dots, 0)\}$. In usual homotopy, $A \neq \emptyset$ and $(X, A) = (X, A, x_0)$. But by Lemma 2.1 our definition will include this one.

To define $m\pi_0(X, A)$ we let X_A be the set of path components of X not meeting A . Then $m\pi_0(X, A)$ consists of the m -homotopy classes of m -functions $f: (S^0, 1) \rightarrow (X_A)$ of arbitrary multiplicity.

Note that in the definition of $m\pi_n(X, A)$ we can replace B^n , S^{n-1} , and 1 by I^n , I^{n-1} , and 0 respectively, where $I = [0, 1]$, $I^n = (I^n)$, and 0 denotes $\{(0, 0, \dots, 0)\}$.

Before defining the group operation, we note the following implications of our above definition and Lemma 2.1:

(i) For $n \geq 1$, if $[f] \in m\pi_n(X, A)$, then f has multiplicity zero in every path component of X .

(ii) For $n \geq 2$, if $[f] \in m\pi_n(X, A)$, then $f|S^{n-1}$ has multiplicity zero in every path component of A .

(iii) $m\pi_0(X) \cong R^m$ where X has m path components.

We define, for $n \geq 1$, the product of f and g in the traditional way by $fg: (B^{n-1} \times [-1, 1])/\sim \rightarrow X$ according to:

$$\overline{fg}(b, t, x) = \begin{cases} \overline{f}(b, 2t + 1, x) & -1 \leq t \leq 0 \\ \overline{g}(b, 2t - 1, x) & 0 \leq t \leq 1. \end{cases}$$

(For $n = 1$, drop b from the above.)

THEOREM 3.1. $fg \sim_m f + g$ (where f and g represent elements of $m\pi_n(X, A)$, for $n \geq 1$).

Proof. First define the m -functions $f_1, g_1: (B^{n-1} \times [-1, 1])/\sim \rightarrow X$ by:

$$\begin{aligned} \text{for } -1 \leq t \leq 0: \overline{f}_1(b, t, x) &= \overline{f}(b, 2t + 1, x), \overline{g}_1(b, t, x) = 0 \\ \text{for } 0 \leq t \leq 1: \overline{f}_1(b, t, x) &= 0, \overline{g}(b, t, x) = \overline{g}(b, 2t - 1, x). \end{aligned}$$

Then $\overline{fg} = \overline{f}_1 + \overline{g}_1$ and so $fg = f_1 + g_1$. We need only prove that $f \sim_m f_1$ and $g \sim_m g_1$. The proofs are similar; we give the first.

Consider the family of homeomorphisms $d_\tau: B^{n-1} \times [-1, \tau] \rightarrow B^{n-1} \times [-1, 0]$ defined by $d_\tau(b, t) = (b, (t + 1)/(\tau + 1) - 1)$ ($\tau \in I$). The m -homotopy $F: ((B^{n-1} \times [-1, 1])/\sim) \times [-1, 1] \rightarrow X$ given by $F(b, t, \tau) = f \circ d_\tau$ for $t \leq \tau$ and $F(b, t, \tau) = \emptyset$ for $t > \tau$ carries $f(\tau = 0)$ to $f_1(\tau = 1)$.

We extend the group operation to dimension zero by using m -function addition as the operation there also.

COROLLARY 3.2. For $n \geq 1$ the m -homotopy group $m\pi_n(X, A)$ is the R -module $\text{hom}[(B^n, S^{n-1}, 1), (X, A)]$. Letting $A = \emptyset$, $m\pi_n(X) = \text{hom}[(B^n, S^{n-1}, 1), (X)] \cong \text{hom}[(S^n, 1), (X)]$.

The last isomorphism can be easily proven by analogy to usual homotopy.

If $f: (X, A) \rightarrow (Y, B)$ and $n \geq 1$ then $f_*: m\pi_n(X, A) \rightarrow m\pi_n(Y, B)$ is defined by $f_*[g] = [f \circ g]$. For $n = 0$, $f_*[g] = [(f|f^{-1}(Y_B)) \circ g]$. (Alternatively we could adjust the definition of $m\pi_0$ instead of that of f_* .) The remarks above on hom -functors imply that f_* and the boundary operator ∂_* defined below are well-defined on m -homotopy classes of m -functions. Let $\delta_n: B^n \rightarrow S^n$ be the natural continuous map implied by $B^n/S^{n-1} \approx S^n$, for $n \geq 1$ (δ_n collapses S^{n-1} to 1, so that $\delta_n: (B^n, S^{n-1}, 1) \rightarrow (S^n, 1, 1)$). When we use $(I^n, \cdot I^n, 0)$, δ_n becomes the map $\delta_n: I^n \rightarrow \cdot I^{n+1}$ implied by $I^n/\cdot I^n \approx \cdot I^{n+1}$. Let $\delta_0: S^0 \rightarrow S^0$ (or $\cdot I \rightarrow \cdot I$) be the identity map. We sometimes drop the subscript n as superfluous. Let $\partial_*: m\pi_{n+1}(X, A) \rightarrow m\pi_n(A)$, called the boundary operator, be defined by $\partial_*[f] = [f \circ \delta_n]$. Then for injections $i: (A) \rightarrow (X)$ and $j: (X) \rightarrow (X, A)$ we have the m -homotopy sequence:

$$\begin{aligned} \cdots \longrightarrow m\pi_{n+1}(X, A) &\xrightarrow{\partial_*} m\pi_n(A) \xrightarrow{i_*} m\pi_n(X) \xrightarrow{j_*} m\pi_n(X, A) \longrightarrow \cdots \\ \cdots \longrightarrow m\pi_0(A) &\xrightarrow{i_*} m\pi_0(X) \xrightarrow{j_*} m\pi_0(X, A) \longrightarrow 0. \end{aligned}$$

The functorial axioms for m -homotopy follow from the fact that $\text{hom}[(B^n, S^{n-1}, 1), (_, _)]$ is a functor. Also ∂_* is a natural map since $f_* \circ \partial_*[g] = [f \circ g \circ \partial] = \partial_* \circ f_*[g]$.

THEOREM 3.3. *The m -homotopy sequence is exact.*

Proof. We use the definition of m -homotopy groups which considers m -functions from $(I^n, I^n, 0)$. The proof is divided into four cases with only case d considering $n = 0$.

(a) (Exactness at $m\pi_n(A)$.) Suppose $[f] \in m\pi_{n+1}(X, A)$, so that $f: (I^{n+1}, I^{n+1}, 0) \rightarrow (X, A)$. Define $\Gamma: I^{n+1} \times I \rightarrow I^{n+1}$ by $\Gamma_t(x) = tx$. Let $(\delta_n \times 1): I^n \times I \rightarrow I^{n+1} \times I$ be the function $(\delta_n \times 1)(x, t) = (\delta_n(x), t)$. Now define $G: (I^n, I^n, 0) \times I \rightarrow (X)$ by $G = f \circ \Gamma \circ (\delta_n \times 1)$. Since $G_1 = f \circ \delta_n$ and $G_0 = \emptyset$ (a constant m -function of zero multiplicity must be empty), G shows that $i_*\partial_*[f] = 0$, and so $\text{im } \partial_* \subset \ker i_*$.

Now suppose that $[f] \in m\pi_n(A)$ (so that $f: (I^n, I^n, 0) \rightarrow (A)$) and that $i_*[f] = 0$. Then there exists $F: (I^n, I^n, 0) \times I \rightarrow (X)$ with $F_0 = f$ and $F_1 = \emptyset$. There exists a continuous family of continuous functions $D_t: (I^n, I^n) \rightarrow (I^{n+1}, I^n \times I \cup I^n \times 1)$ for $t \in I$, with $D_1 = \delta_n$ and $D_0: I^n \rightarrow I^n \times 0$ the natural injection. Then $F \circ D_t: (I^n, I^n, 0) \rightarrow (A)$, since $F|_{I^n \times I \cup I^n \times 1} = \emptyset$. So in $m\pi_n(A)$, $[f] = [F_0] = [F \circ D_0] = [F \circ D_1] = [F \circ \delta] = \partial_*[F]$. Hence $\ker i_* \subset \text{im } \partial_*$.

(b) (Exactness at $m\pi_n(X)$.) Suppose $[f] \in m\pi_n(A)$, so that $f: (I^n, I^n, 0) \rightarrow (A)$. Now for Γ as in (a), $f \circ \Gamma: (I^n, I^n, 0) \times I \rightarrow (X, A)$, $f \circ \Gamma_1 = f$ and $f \circ \Gamma_0 = \emptyset$, so $i_*j_*[f] = 0$ and $\text{im } i_* \subset \ker j_*$.

Now suppose that $[f] \in m\pi_n(X)$ (so that $f: (I^n, I^n, 0) \rightarrow (X)$) and $j_*[f] = 0$. Then there exists $F: (I^n, I^n, 0) \times I \rightarrow (X, A)$ with $F_0 = f$ and $F_1 = 0$. There is a continuous function $H: (I^n, I^n) \times I \rightarrow (I^{n+1}, I^n \times 0)$ such that $H|_{I^n \times 1}$ is a homeomorphism onto $I^n \times 1 \cup I^n \times I$ and $H|_{I^n \times 0}$ is the identity (H "pulls" the top of I^{n+1} over the sides, while "collapsing" the sides). Then $F \circ H: (I^n, I^n, 0) \times I \rightarrow (X)$ and $F \circ H_1: (I^n, I^n, 0) \rightarrow (A)$. Hence in $m\pi_n(X)$, $i_*[F \circ H_1] = [F \circ H_1] = [F \circ H_0] = [f]$. Thus $\ker j_* \subset \text{im } i_*$.

(c) (Exactness at $m\pi_n(X, A)$.) Suppose $[f] \in m\pi_n(X)$, so that $f: (I^n, I^n, 0) \rightarrow (X)$. Since $\delta_{n-1}: I^{n-1} \rightarrow I^n$, $f \circ \delta_{n-1} = \emptyset$. So in $m\pi_n(X)$, $\partial_*j_*[f] = [f \circ \delta_{n-1}] = 0$, and $\text{im } j_* \subset \ker \partial_*$.

Now suppose that $[f] \in m\pi_n(X, A)$ (so that $f: (I^n, I^n, 0) \rightarrow (X, A)$) and $\partial_*[f] = 0$. Then there exists $F: (I^{n-1}, I^{n-1}, 0) \times I \rightarrow (A)$ with $F_0 = f \circ \delta$ and $F_1 = \emptyset$. There are continuous functions for $0 \leq \tau \leq 1$:

$$\alpha_\tau: I^{n-1} \times [0, \tau/2] \longrightarrow I^{n-1} \times I$$

with

$$\alpha_\tau(x, t) = (x, \tau - 2t)$$

and

$$\begin{aligned} \Delta_\tau: I^{n-1} \times \left[\frac{\tau}{2}, 1 \right] &\longrightarrow I^{n-1} \times I \\ \text{such that } \begin{cases} \Delta_\tau|_{I^{n-1} \times \left\{ \frac{\tau}{2} \right\}} = \delta \\ \Delta_\tau \left(\left(I^{n-1} \times \left[\frac{\tau}{2}, 1 \right] \right) \cup (I^{n-1} \times 1) \right) = 0 \\ \Delta_\tau|_{\text{int} \left(I^{n-1} \times \left[\frac{\tau}{2}, 1 \right] \right)} \text{ is a homeomorphism onto} \\ \text{int}(I^{n-1} \times I). \end{cases} \end{aligned}$$

We define $H: (I^{n-1}, I^{n-1}, 0) \times I \rightarrow (X, A)$ by

$$H(x, t, \tau) = \begin{cases} F \circ \alpha_\tau & 0 < t \leq \tau/2 \\ f \circ \Delta_\tau & \tau/2 \leq t \leq 1. \end{cases}$$

Then $H|(\tau = 1): (I^n, I^n, 0) \rightarrow (X)$, so $[H|(\tau = 1)] \in m\pi_n(X)$, and $H|(\tau = 0) = f \circ \Delta_0 \in [f]$, since $\Delta_0 \sim 1_{I^n}$. Thus $j_*[H|(\tau = 1)] = [f]$ and $\ker \partial_* \subset \text{im } j_*$.

(d) (Exactness for $n = 0$.) At $m\pi_0(A)$, $\text{im } \partial_* \subset \ker i_*$ follows from the first part of (a). Now suppose $[f] \in m\pi_0(A)$ (so $f: (\cdot I, 0) \rightarrow (A)$) and that $i_*[f] = 0$. Then there exists $F: (\cdot I, 0) \times I \rightarrow (X)$ with $F_0 = f$ and $F_1 = \emptyset$. Define $g: (I, I, 0) \rightarrow (X, A)$ by $g(t) = F_t(1)$. Then $\partial_*[g] = [g \circ \delta_0] = [(g|0) + (g|1)] = [f|1] = [f]$. So $\ker i_* \subset \text{im } \partial_*$.

At $m\pi_0(X)$ we have $\text{im } i_* \subset \ker j_*$ because if $[f] \in m\pi_0(A)$ then $f: (\cdot I, 0) \rightarrow (A)$ and hence $f|X_A = \emptyset$. Note that $j_*[g] = [(j|X_A) \circ g]$. Now suppose $[f] \in m\pi_0[X]$ and $j_*[f] = 0$. Then $f|X_A \sim_m g$ where $g: (\cdot I, 0) \rightarrow (A)$. It follows that $[g] \in m\pi_0(A)$ and $i_*[g] = [f]$.

At $m\pi_0(X, A)$ we take $[f]_A \in m\pi_0(X, A)$. Then $f: (\cdot I, 0) \rightarrow (X)$, $[f] \in m\pi_0(X)$, and $j_*[f] = [f]_A$. Thus j_* is onto.

4. Three lemmas about boxes. In order to prove the excision axiom, we introduce several definitions and lemmas. By an n -dimensional box we mean the Cartesian product of n (orthogonal) compact line segments. By a k -face of a box, we mean any sub-box which is formed by taking the product of k of the original segments and replacing the remaining $n - k$ segments by (in each case) either endpoint. If T is a collection of boxes, we let $O(T)(E(T))$ be the

subcollection of those boxes of odd (even) dimension. $|\cdot|$ represents the cardinality of a set.

LEMMA 4.1. *Let t be a proper k -face of an n -dimensional box, V , and let T be the set of faces of V containing t . Then $|O(T)| = |E(T)|$.*

Proof. We may assume that $V = I^n$. Note that a face of V is then determined by an ordered n -tuple where the entries are chosen from among $I, 0$, and 1 .

Let t correspond to an n -tuple consisting of k entries of I and $(n - k)$ entries of a single point, 0 or 1 . Choosing an m -face containing t is equivalent to choosing $(m - k)$ of the $(n - k)$ positions consisting of a single point, to be replaced by I . So we must show that

$$\sum_{m \text{ odd}} \binom{n - k}{m - k} = \sum_{m \text{ even}} \binom{n - k}{m - k} \quad \text{where } k \leq m \leq n.$$

This is equivalent to showing that $\sum_{s \text{ odd}} \binom{r}{s} = \sum_{s \text{ even}} \binom{r}{s}$, for $r = n - k, 0 \leq s \leq r$. One sees that this is true by considering

$$(x - 1)^r = \sum_{s \text{ even}} \binom{r}{s} x^{r-s} - \sum_{s \text{ odd}} \binom{r}{s} x^{r-s}, \quad \text{for } x = 1.$$

Now suppose I^n is subdivided into finitely many boxes by subdividing each I in the product $I^n = I \times I \times \cdots \times I$ into segments. Let T be the collection of all these n -dimensional boxes and all those faces which meet the interior of I^n . For $t \in T$, we identify the box t with the function $t: I^n \rightarrow I^n$ for which $t(x)$ is the point of t closest to x (if $x \in t$, $t(x) = x$). For $t \in T$ and $f: I^n \rightarrow X$ an m -function, we let $f_t = f \circ t$.

LEMMA 4.2. *For f and T as above, $f = (\sum_{s \in E(T)} f_s - \sum_{r \in O(T)} f_r)(-1)^n$.*

Proof. $\sum_{s \in E(T)} f_s - \sum_{r \in O(T)} f_r = \sum_{s \in E(T)} f \circ s - \sum_{r \in O(T)} f \circ r = f \circ (\sum_{s \in E(T)} s - \sum_{r \in O(T)} r)$. So it suffices to show that

$$g = \left(\sum_{s \in E(T)} s - \sum_{r \in O(T)} r \right) (-1)^n$$

is the identity m -function on I^n . (The functions are added here by considering them as m -functions.) Let $\{v_i\}_{i=1}^n$ be the n -dimensional elements of T . Fix $a \in \text{int } v_k$, for some k . To each v_i , associate v_i^* , the collection of elements, t , of T , such that $t(a) = v_i(a)$. We next

show that $\{v_i^*\}$ partitions T and that each v_i^* consists of those faces of v_i containing a special face t_i .

In general, for u a set formed by Cartesian product of subsets of each coordinate axis, let u_k be the projection of u onto the k th coordinate axis. We then write $u = u_1 \times u_2 \times \cdots \times u_n$. Given $t \in T$, $t(a)_k$ is the point of t closest to a_k . ($t(a)_k$ is either a_k , one of the two endpoints of t_k , or, if t_k is a point, t_k itself. Only one of these can occur.) So v_i^* and v_j^* are disjoint for $i \neq j$.

Let $t \in T$ be fixed. Suppose we replace some of the t_k which are points by the interval in our subdivision of the k th coordinate axis with endpoint t_k between a_k and the other endpoint. Then the new Cartesian product gives us a box $s \in T$ with $s(a) = t(a)$ and t a face of s . Making all possible such replacements, we get $t \in v_i^*$ for some i , with t a face of v_i . Hence $\{v_i^*\}$ partitions T .

The elements of v_i^* can be constructed from v_i by considering each $(v_i)_k$. If $a_k \notin (v_i)_k$ we replace $(v_i)_k$ by its endpoint nearest a_k . The new Cartesian product will give us an element of v_i^* , and any element of v_i^* is of this form. By making all such replacements we get t_i , the element of v_i^* which we require.

Now, for $a \in \text{int } v_k$, we have

$$g = v_k + (-1)^n \sum_{i \neq k} \left(\sum_{s \in E(v_i^*)} s - \sum_{r \in O(v_i^*)} r \right).$$

But for $i \neq k$, $(\sum_{s \in E(v_i^*)} s - \sum_{r \in O(v_i^*)} r)$ maps a to $v_i(a)$ with multiplicity $|E(v_i^*)| - |O(v_i^*)| = 0$. So the only image point of a under g with nonzero multiplicity is a itself, which has multiplicity one. But this is true for any a in $\bigcup_{k=1}^m \text{int } v_k$, a dense open subset of I^n . Hence g is the identity m -function on I^n .

Note that the image of f_i is the image of $f|t$. This lemma allows us to “break up” m -functions in a manner which corresponds to the subdivision operator on simplices used in simplicial homology.

LEMMA 4.3. *Given an m -function, $f: I^n \rightarrow Y$, and an open cover of Y , $\{U_\alpha\}$, there exist m -functions f_α such that $f = \Sigma f_\alpha$ and $\text{im}(f_\alpha) \subset U_\alpha$. Further, if we choose Z , a face of I^n such that $f|Z = \emptyset$ (assuming such a face exists), then we may choose $\{f_\alpha\}$ such that $f_\alpha|Z = \emptyset$ for all α .*

Proof. For $x \in I^n$, let $f(x) = \{y_i\}_{i=1}^m$ with r_i the weight at (x, y_i) . By the definition of m -functions, there exists $W_x \subset I^n$, a neighborhood of x , and $\{V_i\}_{i=1}^m$, disjoint open subsets of elements of $\{U_\alpha\}$, such that $f|W_x$ is an m -function with image in $\bigcup_{i=1}^m V_i$. Clearly each component

of $f|W_x$ has its image in some V_i . Partition I^n into cubes of mesh less than the Lebesgue number of the cover $\{W_x\}_{x \in I^n}$. Let T be the collection of these cubes together with those faces (of any dimension) which meet $\text{int } I^n$. Then $f = (\sum_{s \in E(T)} f_s - \sum_{r \in O(T)} f_r)(-1)^n$ by Lemma 4.2. For each $t \in T$, f_t equals the sum of its component m -functions, each of which has its image in some U_α . We partition the component m -functions of f_t and add so that $f_t = \sum_\alpha f_{t\alpha}$ and the image of $f_{t\alpha}$ lies in U_α . Letting $f_\alpha = \sum_t f_{t\alpha}$, the first part of the lemma is proved.

Suppose we choose Z a face of I^n such that $f|Z = \emptyset$. For $t \in T$, let \bar{t} be the set of points of t closest to Z . Let $[t] = \{r \in T | \bar{t} = \bar{r}\}$.

We may assume that each projection Z_k is either I or 0 . If $Z_k = 0$, then $(\bar{t})_k$ is a single point. Also, $[t]$ consists precisely of those $r \in T$ such that $r_k = (\bar{t})_k$ if $Z_k \neq 0$ and $r_k = [(\bar{t})_k, b]$ if $Z_k = 0$ (and b may equal $(\bar{t})_k$). It follows that $[t]$ contains an element, t' , of maximal dimension (namely, $\dim t' = (\dim \bar{t}) + (n - \dim Z)$) and that $[t]$ consists of the faces of t' containing \bar{t} .

Note that if $z \in Z$ and $\bar{r} = \bar{s}$ (i.e., $r, s \in [t]$ for some t) then $r(z) = s(z)$. If Z is p -dimensional, let Z' be the open dense subset of Z minus the $(p-1)$ -dimensional boxes in the subdivision of I^n . But now, by the same argument as in the proof of Lemma 4.2, for $z \in Z'$, $z \notin t'$, $(-1)^n(\sum_{r \in E([t])} r - \sum_{s \in O([t])} s)$ has multiplicity zero at each image point of z . Choose an equivalence class $[t]$. Then t' and all its faces lie in some single W_x . So we can write $f \circ (-1)^n(\sum_{r \in E([t])} r - \sum_{s \in O([t])} s) = \sum_\alpha f_{t'\alpha}^*$ (allowing some $f_{t'\alpha}^*$'s to be empty) so that the image of $f_{t'\alpha}^*$ lies in U_α (just by partitioning the components and summing as before). Note that for $z \in Z' - t'$, $f_{t'\alpha}^*$ has multiplicity zero at each image point of z . Let $f_\alpha^* = \sum_t f_{t'\alpha}^*$ where we let t take one value in each equivalence class.

Now $f = \sum f_\alpha^*$ so it remains to show that $f_\alpha^*|Z = \emptyset$ for each α . Fix $a \in Z'$ and let v be the single (n -dimensional) box in T containing a . Then $v' = v$ and $f_\alpha^* = f_{v\alpha}^* + \sum_{t' \neq v} f_{t'\alpha}^*$. But $f_{v\alpha}^*$ is just the sum of certain components of $\pm f_v$ and $f_v(a) = f(a)$. Since $f|Z = \emptyset$, f_α^* has multiplicity zero at each image point of a . Hence $f_\alpha^*|Z' = \emptyset$. It follows that $f_\alpha^*|Z = \emptyset$ and we are done.

5. M -homotopy theory is a homology theory. In §2 we described m -homotopy theory, $m\pi_n$. We wish to show this is a homology theory. In §3 we proved the exact sequence axiom and noted that the functorial axioms are satisfied. We also noted that the dimension axiom is satisfied, i.e., that $m\pi_0(Z) = R^m$ where m is the number of path components of Z .

THEOREM 5.1 (*The excision axiom*). *If $\bar{U} \subset \text{int } A$, then the inclusion map $i: (X - U, A - U) \rightarrow (X, A)$ induces an isomorphism $i_*: m\pi_n(X - U, A - U) \rightarrow m\pi_n(X, A)$ for all n .*

Proof. Suppose $[f] \in m\pi_n(X, A)$, so $f: (I^n, \cdot I^n, 0) \rightarrow (X, A)$. By Lemma 4.3, there exists $g, h: (I^n, 0) \rightarrow (X)$ such that $f = g + h$ and $\text{im}(g) \subset X - \bar{U}$ and $\text{im}(h) \subset \text{int } A$. But then in $m\pi_n(X, A)$, $[h] = 0$ (just consider $h \circ \Gamma_t$ where $\Gamma_t: I^n \rightarrow I^n$, $\Gamma_t(x) = tx$). Since f and h represent elements of $m\pi_n(X, A)$, so does $g = f - h$; in fact $[g] = [f]$. But $\text{im}(g) \subset X - U$, and $\text{im}(g|I^n) \subset A$, so $\text{im}(g|I^n) \subset A - U$. Hence g represents an element of $m\pi_n(X - U, A - U)$ and i_* maps the m -homotopy class of g to the m -homotopy class of f and so is onto.

In the present paragraph, $[\cdot]$ will represent an m -homotopy class in $m\pi_n(X - U, A - U)$. Suppose $[f] \in m\pi_n(X - U, A - U)$ and there exists $F: (I^n, I^n, 0) \times I \rightarrow (X, A)$ with $F_0 = f$, $F_1 = \emptyset$ (i.e., f is null- m -homotopic in $m\pi_n(X, A)$). Then $F: (I^{n+1}, 0 \times I) \rightarrow (X)$, so by Lemma 4.3, we write $F = G + H$ with $G: (I^{n+1}, 0 \times 1) \rightarrow (X - \bar{U})$ and $H: (I^{n+1}, 0 \times I) \rightarrow (\text{int } A)$. Since $f = G_0 + H_0$, $\text{im}(H_0) \subset \text{im}(f) \cup \text{im}(G_0) \subset X - U$. So $\text{im}(H_0) \subset A - U$, and $[H_0] = 0$ (consider $H_0 \circ \Gamma_t$). Hence, as before, $[f] = [G_0]$. Now $F_1 = \emptyset = G_1 + H_1$, so $\text{im}(G_1) = \text{im}(H_1) \subset A - U$. But then $[G_1] = [H_1] = 0$. Since $\text{im}(G) \subset X - U$, $G = F - H$, $\text{im}(F|I^n \times I) \subset A$, and $\text{im}(H) \subset A$, we can conclude that $\text{im}(G|I^n \times I) \subset A - U$. So $G: (I^n, I^n, 0) \times I \rightarrow (X - U, A - U)$ and $G: [G_0] = [G_1]$. Hence $[f] = [G_0] = [G_1] = 0$, and i_* is one-to-one.

We have neglected the case where $n = 0$. In this case, $m\pi_0(Y, B)$ is essentially the possible finite assignments of multiplicities to components of Y_B . Let X' be a component of X . If X' is disjoint from A , then X' is a component of $X - U$. If X' meets A , let X'' be a component of $X' - U$. Suppose $u \in \bar{X}'' \cap \bar{U}$. Then, since $\bar{U} \subset \text{int } A$, some neighborhood of u lies in A and also meets X'' . Hence X'' meets A . On the other hand, if $\bar{X}'' \cap \bar{U} = \emptyset$, then \bar{X}'' is a component of X and so $X' \cap U = \emptyset$ and $X'' = X'$. In either case, X'' meets $A - U$. So $X_A = (X - U)_{A - U}$. It is now easy to check that i_* is an isomorphism for $n = 0$ also.

Hence m -homotopy theory is a homology theory. By uniqueness we can conclude that $m\pi_n(X, A) = H_n(X, A)$ where (X, A) is any compact polyhedral pair and H_n is singular homology.

6. Examples; the Dold-Thom theorem. In this final section we consider the connections between m -homotopy groups, singular homology groups, and the Dold-Thom expression of homology groups as homotopy groups.

of compact supports (see [9]).

Proof. If $[f] \in m\pi_n(X, A)$, then $f: (I^n, \cdot I^n, 0) \rightarrow (X, A)$ and we let $X' = \text{im}(f)$, $A' = \text{im}(f| \cdot I^n)$. Then $[f]^* \in m\pi_n(X', A)$ (where $[\cdot]^*$ represents an equivalence class in $m\pi_n(X', A)$) and the map i_* induced by the injection $i: (X', A') \rightarrow (X, A)$ maps $[f]^*$ to $[f]$. In the case where $[f] \in mH_n(X, A)$, $f: \Delta^n \rightarrow X$ and $X' = f'(\Delta^n)$ is compact. Let $A' = X' \cap A$. Then $[f]^\wedge \in mH_n(X', A')$ ($[\cdot]^\wedge$ represents an equivalence class in $mH_n(X', A')$), and i_* maps $[f]^\wedge$ to $[f]$.

We conclude (see [4]) from this corollary that if (X, A) is any polyhedral pair, $H_n(X, A) \approx mH_n(X, A) \approx m\pi_n(X, A)$. For example, suppose for some specific pair, (X, A) , we know that $z \in H_n(X, A)$ is nonzero. Then $mH_n(X, A)$ is nontrivial, but what is the m -function corresponding to z ? This question is answered by describing the two isomorphisms above. For $[z] \in H_n(X, A)$, $z = \sum_{i=1}^n r_i \sigma_i$, a formal sum where each $\sigma_i: \Delta^n \rightarrow X$ is a singular simplex. Let $f = \sum_{i=1}^n r_i \sigma_i$ be an m -function sum. Then the map $z \rightarrow f$ induces a homomorphism from $H(X, A)$ to $mH(X, A)$ which is the unique isomorphism between them. Similarly, if $[f] \in m\pi_n(X, A)$, $f: (\Delta^n, \cdot \Delta^n, 0) \rightarrow (X, A)$. In particular, $f: \Delta^n \rightarrow X$ and so f determines an element $[f]^\wedge$ in $mH_n(X, A)$. This map ($f \rightarrow f$, but with the second f we ignore the last two elements of the triple) induces a homomorphism from $m\pi(X, A)$ to $mH(X, A)$ which must be the unique isomorphism.

There is a relationship between the results here and the Dold-Thom theorem [2]: $H_m(Y) \cong \pi_m(AG(Y))$ where Y is a pointed polyhedron and $AG(Y)$ is the topological free abelian group on Y . We next define $AG(Y)$ and describe this relationship.

Regarding $Y \vee Y$ as a subset of $Y \times Y$ we use the notation $y = (y, *)$, $-y = (*, y)$, $* = (*, *)$. Starting with an element of $Y' = \sum_{q=1}^\infty \prod_{i=1}^q (Y \vee Y)$ we remove any simultaneous occurrences of y and $-y$, remove all occurrences of $*$, and identify two resulting k -tuples if one is a permutation of the other. (The summation above is free union.) This equivalence relation, R , gives us a quotient map, $\pi: Y' \rightarrow Y'/R = AG(Y)$. Addition in $AG(Y)$ is by juxtaposition of representatives of elements followed by π .

Given spaces X and Y there is a natural mapping from m -functions $f: X \rightarrow Y$ to (standard) functions $f^*: X \rightarrow AG(Y)$. Namely, if $f(x) = \{y_1, \dots, y_n\}$ then let $f^*(x) = \sum_{i=1}^n \bar{f}(x, y_i) y_i$. Although this correspondence seems to identify a class of "nice" m -functions to a class of "nice" continuous functions from X to $AG(Y)$, we show below that there are degenerate examples on each side.

EXAMPLE 6.4. f may be m -function, but f^* not continuous. The

graph in Figure 1 indicates how we might define an m -function f such that $f^*: I \rightarrow AG(I)$ fails to be continuous (at 1). It is convenient to take $*$ = 0. Thus we can identify $I \vee I$ to $[-1, 1]$ in a natural way. Let $U \subset AG(I)$ consist of those points $\langle w_1, w_2, \dots, w_n \rangle$ for which $\sum_{i=1}^n w_i < 1/10$ (the addition is in \mathbf{R} and is independent of representation of the point). It is easy to verify that $\pi^{-1}(U)$ and hence U is open. But we can define f so that $f^{*-1}(U) \cap [1/2, 1] = 1$. Along any vertical line the distances between adjacent lines, L_1 and L_2 , L_2 and L_3 , etc., can be taken to be proportional to $1, 1/4, 1/9, 1/16, \dots, 1/n^2$, etc. Given a vertical line, if we choose n large, the sum of the vertical distances from L_n to $L_i (i < n)$ can be made arbitrarily large. Define f using this information.

EXAMPLE 6.5. f^* may be continuous, but f not an m -function. In Figure 2, we indicate an example of a weighted multiple-valued function f for which we can define f^* as above. Then, although f fails to be an m -function ($f(0)$ is infinite) f^* is continuous. The only point where the continuity of f^* is questionable is at 0. There $f^*(0) = *$. If U is a neighborhood of $*$ in $AG(I)$, then $\pi^{-1}(*) \subset \pi^{-1}(U)$ and $\pi^{-1}(*) \cap ((I \vee I) \times (I \vee I)) \subset \pi^{-1}(U) \cap ((I \vee I) \times (I \vee I))$ which is open in $(I \vee I) \times (I \vee I)$. Since I is compact there exists $\varepsilon > 0$ such that if $|x - y| < \varepsilon$ then $\langle x, -y \rangle \in U$ (the brackets represent unordered pair, so $\langle x, -y \rangle = x + (-y)$ where addition is in $AG(I)$ not \mathbf{R}). So, from the figure we see that there exists $\delta > 0$ such that if $x < \delta$, $f^*(x) \in U$.

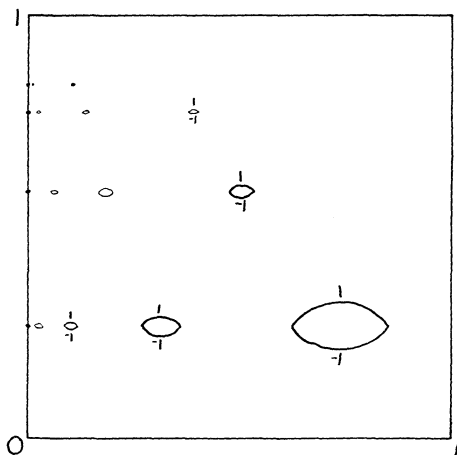


FIGURE 2

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