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## **SUPERREGULAR SEQUENCES**

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Let  $(R, \underline{m})$  be a local ring with associated graded ring  $grR = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \cdots$ . This paper deals with the problem of finding properties of  $R$  which lead to good properties in  $grR$ . There are two main results in this paper which give techniques for recognizing when the maximal homogeneous ideal of  $grR$  contains regular elements. Applications of these results give examples of local Cohen-Macaulay rings which have Cohen-Macaulay associated graded rings.

Let  $(R, \underline{m})$  be a local ring with associated graded ring  $grR = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \cdots$ . Information about  $grR$  gives some measure of the singularity at  $R$  since properties of  $grR$  yield data about the Hilbert function of  $R$  and about monoidal transforms of  $R$ . However it is often difficult to compute  $grR$  and it is seldom true that properties of  $R$  are carried over to  $grR$ . Thus, it is important to recognize characteristics of  $R$  which lead to good properties in  $grR$ .

There are two main results in this paper which give techniques for recognizing good properties in  $grR$ . The first shows when the initial forms of a regular sequence in  $\underline{m}/\underline{m}^2$  form a regular sequence in  $grR$ ; the given regular sequence is then called super-regular. (All our applications use homogeneous regular sequences of degree one in  $grR$ , so we have avoided some complications by primarily considering this case. If  $R/\underline{m}$  is infinite and if  $gr\underline{m}$  contains a regular sequence of length  $t$ , it contains a homogeneous regular sequence of degree one of length  $t$ .) The second result shows that for certain local Cohen-Macaulay rings the question of whether  $grR$  is Cohen-Macaulay can be reduced to the same question for such Cohen-Macaulay rings of dimension one. The paper concludes with several applications of these results. The applications give examples of local Cohen-Macaulay rings which have Cohen-Macaulay associated graded rings.

We begin with some definitions. For any nonzero element  $x$  in the local ring  $(R, \underline{m})$ , let  $\bar{x}$  denote the initial form of  $x$  in  $grR$ , i.e., if  $x \in \underline{m}^s \setminus \underline{m}^{s+1}$ ,  $\bar{x} = x + \underline{m}^{s+1} \in \underline{m}^s/\underline{m}^{s+1}$ . We will say that  $x$  has order  $s$  and that  $\bar{x}$  has degree  $s$ . Let  $\bar{0}$  be the zero element in  $grR$ .  $0$  has infinite order and  $\bar{0}$  has infinite degree. A system of elements of  $R$  (of  $grR$ ) has order  $s$  (degree  $s$ ) if each element has order  $s$  (degree  $s$ ).



Recall that if  $S$  is any commutative ring, a sequence of elements  $x_1, \dots, x_t$  of  $S$  is a regular sequence if  $(x_1, \dots, x_t)S \neq S$  and if, for  $i = 1, \dots, t$ ,  $x_i$  is not a zero divisor on  $S/(x_1, \dots, x_{i-1})S$ .

For the case of a local ring  $(R, \underline{m})$  it is important to know when  $gr\underline{m}$ , the maximal homogeneous ideal of  $grR$ , contains a regular element. For the existence of a regular element in  $gr\underline{m}$  permits a change of rings which reduces dimension. This is described in the following lemma.

**LEMMA 0.1.** *Let  $(R, \underline{m})$  be a local ring with associated graded ring  $grR$ . Suppose that  $gr\underline{m}$ , the maximal homogeneous ideal of  $grR$ , contains a regular element. Then  $gr\underline{m}$  contains a homogeneous regular element  $\bar{x}$  and, if  $x$  is any element of  $R$  with initial form  $\bar{x}$ , there is an isomorphism of graded rings  $grR/\bar{x}grR \cong gr(R/xR)$  induced by the natural map of local rings  $R \rightarrow R/xR$ .*

*Proof.* By [8, Ch. VII, §2], the prime ideals  $P_1, \dots, P_h$  in  $\text{Ass } grR$  are homogeneous. By [8, pg. 286, footnote],  $gr\underline{m} \not\subseteq \bigcup P_i$  implies that there is a homogeneous element  $\bar{x}$  of positive degree  $s$ , say, such that  $\bar{x} \notin P_i$  for  $i = 1, \dots, h$ . Let  $x$  be any element of  $R$  with initial form  $\bar{x}$ . Consider the natural homomorphism  $\nu: R \rightarrow R/xR$ . Since  $\nu(\underline{m}^n) \subseteq (\underline{m}/xR)^n$ , the induced homomorphism  $gr\nu: grR \rightarrow gr(R/xR)$  given by  $(gr\nu)_n: \underline{m}^n/\underline{m}^{n+1} \rightarrow (\underline{m}/xR)^n/(\underline{m}/xR)^{n+1}$  is a homomorphism of graded rings, cf. [1, Ch. 3, §2, no. 4]. To show that  $gr\nu$  is surjective with kernel  $\bar{x}grR$ , it is enough to show that  $(gr\nu)_n$  is surjective with kernel  $(\bar{x}grR)_n$ . It is clear that  $(gr\nu)_n$  is surjective. If  $\bar{w} \in \underline{m}^n/\underline{m}^{n+1}$  and  $(gr\nu)_n \bar{w} = 0$ , then  $w \in (\underline{m}^{n+1} + xR) \cap \underline{m}^n = \underline{m}^{n+1} + (xR \cap \underline{m}^n)$  for any  $w \in \bar{w}$ . Since  $\bar{x}$  is regular in  $grR$ ,  $xR \cap \underline{m}^n = x\underline{m}^{n-1}$ , where a nonpositive power of  $\underline{m}$  is understood to be  $R$ . Thus  $w = z + xy$  with  $z \in \underline{m}^{n+1}$  and  $y \in \underline{m}^{n-s}$ . Again using that  $\bar{x}$  is regular in  $grR$ , we have that  $\bar{x}\bar{y} = \overline{xy} = \overline{z + xy} = \bar{w}$ . This proves that  $\bar{w} \in \bar{x}grR$  and completes the proof of the lemma.

We will say that an element  $x$  in  $\underline{m}$ , the maximal ideal of a local ring  $(R, \underline{m})$ , is *super-regular* if  $\bar{x}$  is a regular element in  $grR$ . A sequence of elements  $x_1, \dots, x_t$  in  $R$  will be called a *super-regular sequence* if  $\bar{x}_1, \dots, \bar{x}_t$  is a regular sequence in  $grR$ .

Several properties of super-regular sequences follow easily. A super-regular sequence  $x_1, \dots, x_t$  is a regular sequence. For if  $\bar{x}_1$  is a regular element in  $gr\underline{m}$ ,  $x_1$  is a regular element in  $R$  and, by (0.1),  $gr(R/x_1R) \cong grR/\bar{x}_1grR$ . By induction, the images of  $x_2, \dots, x_t$  form a regular sequence in  $R/x_1R$  so  $x_1, x_2, \dots, x_t$  is a regular sequence in  $R$ . A similar isomorphism shows that a sequence  $x_1, \dots, x_t$  of elements of  $R$  is super-regular if and only if there is some  $j$ ,  $1 \leq j \leq t$ , such that  $x_1, \dots, x_j$  is super-regular in  $R$  and



the image of  $x_{j+1}, \dots, x_i$  is super-regular in  $R/(x_1, \dots, x_j)R$ . It is also true that any permutation of a super-regular sequence is super-regular.

In addition to the notation and definitions introduced above, the following notation will be used.  $\lambda(A) = \lambda_R(A)$  denotes the length of an  $R$ -module  $A$ . If  $I$  is an ideal in a local ring  $(R, \underline{m})$ ,  $v(I)$  denotes the number of generators in a minimal basis of  $I$ .  $v(\underline{m})$  is the embedding dimension of  $R$ .  $e = e(R)$  is the multiplicity of  $R$ . We will often use  $\underline{x}$  to denote a system of elements  $x_1, \dots, x_t$  in  $R$ .

Let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence in  $(R, \underline{m})$ . If we say that  $f(x_1, \dots, x_t)$  is a form of degree  $s$  in  $x_1, \dots, x_t$ , we mean that  $f(X_1, \dots, X_t)$  is a degree  $s$  homogeneous polynomial in the polynomial ring  $R[X_1, \dots, X_t]$  and  $f(x_1, \dots, x_t)$  is the image of  $f(X_1, \dots, X_t)$  in  $R$  under the homomorphism sending  $X_i$  to  $x_i$ .

We will use the notion of minimal reduction of the maximal ideal of a local ring  $(R, \underline{m})$ , cf. [3], but we wish to take a more restrictive definition than in [3]. If  $(R, \underline{m})$  is a  $d$ -dimensional local ring, a system  $\underline{x} = x_1, \dots, x_d$  of  $d$  elements of  $R$  is a *minimal reduction* of  $\underline{m}$  if there is a positive integer  $r$  such that  $\underline{m}^{r+1} = \underline{x}\underline{m}^r$ . If  $R/\underline{m}$  is infinite, minimal reductions exist, as a minimal reduction of  $\underline{m}$  is the preimage in  $R$  of a degree one homogeneous system of parameters in  $grR$ . The existence of a minimal reduction is hardly ever a troublesome hypothesis because the change of rings  $R \rightarrow R(U) = R[U]_{\underline{m}R[U]}$ ,  $U$  an indeterminate, is faithfully flat and  $\underline{m}R(U)$  has minimal reductions.

1. When are regular sequences super-regular? If  $x$  is a regular element of order one in the local ring  $(R, \underline{m})$ , it is clear that  $x$  is super-regular if and only if  $(x) \cap \underline{m}^{i+1} = x\underline{m}^i$  for all  $i \geq 0$ . The assumption that  $x$  is regular cannot be dropped as the example  $k[[x, y]]/(y^2, xy)$ ,  $k$  a field, shows. An analogous characterization holds for a regular sequence  $\underline{x} = x_1, \dots, x_t$  of order one. It will be useful to prove a little more. We will see that  $l$  intersection equalities  $(x) \cap \underline{m}^{i+1} = x\underline{m}^i$  for  $i \leq l$  means that  $\underline{x}$  is super-regular "up to  $\underline{m}^{l+1}$ ."

**THEOREM 1.1.** *Let  $(R, \underline{m})$  be a local ring and let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence of order one. Then,*

$$(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i,$$

for all positive integers  $i \leq$  some positive integer  $l$  if and only if

$$((\bar{x}_1, \dots, \bar{x}_{j-1}): \bar{x}_j) \subseteq (\bar{x}_1, \dots, \bar{x}_{j-1}) + (gr \underline{m})^l$$

for  $1 \leq j \leq t$ .



We first note the following.

**LEMMA 1.2.** *Let  $(R, \underline{m})$  be a local ring and let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence of order one. If for all positive integers  $i \leq$  some positive integer  $l$ ,  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ , then*

$$(\underline{x})^j \cap \underline{m}^{i+1} = (\underline{x})^j \underline{m}^{i+1-j},$$

for  $1 \leq j \leq i+1$  and  $i \leq l$ .

*Proof.* We prove the lemma by induction on  $j$ . Let  $j > 1$ . Let  $f_j(x_1, \dots, x_t)$ , a form of degree  $j$  in  $x_1, \dots, x_t$ , be in  $(\underline{x})^j \cap \underline{m}^{i+1} \subseteq (\underline{x})^{j-1} \cap \underline{m}^{i+1} = \underline{x}^{j-1} \underline{m}^{i+1-j+1}$ .  $f_j(x_1, \dots, x_t) = g_{j-1}(x_1, \dots, x_t)$ , where  $g_{j-1}$  is a form of degree  $j-1$  in  $x_1, \dots, x_t$  with coefficients in  $\underline{m}^{i+1-j+1}$ . Since  $(x_1, \dots, x_t)^{j-1}/(x_1, \dots, x_t)^j$  is free over  $R/(x_1, \dots, x_t)R$ , the coefficients of  $g_{j-1}$  are in  $(\underline{x}) \cap \underline{m}^{i+1-j+1} = \underline{x}\underline{m}^{i+1-j}$ , so  $f_j(x_1, \dots, x_t) \in (\underline{x})\underline{m}^{j+i-1-j}$ .

*Proof of Theorem 1.1.* Assume that  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq l$ . We will prove that  $((\bar{x}_1, \dots, \bar{x}_{j-1}); \bar{x}_j) \subseteq (\bar{x}_1, \dots, \bar{x}_{j-1}) + (gr\underline{m})^i$  for  $1 \leq j \leq t$ , by induction on  $t$ . The proof for  $t=1$  is immediate so we will assume that  $t > 1$ . First, we show that  $(0; \bar{x}_1) \subseteq (gr\underline{m})^i$ , i.e., that  $x_1 z \in \underline{m}^{i+1}$  implies that  $z \in \underline{m}^i$  for  $i \leq l$ . If  $x_1 z \in \underline{m}^2$ , then  $x_1 z = x_1 a_1 + \dots + x_t a_t$  with  $a_1, \dots, a_t \in \underline{m}$ . Thus  $z - a_1 \in (x_2, \dots, x_t)$  and  $z \in \underline{m}$ . We claim that the following relation holds:

$$(*) \quad (\underline{m}^{i+1}; x_1) \cap (\underline{x})^j \underline{m} \subseteq (\underline{x})^j \underline{m}^{i+1-j-1} + (\underline{x})^{j+1} \underline{m},$$

for  $2 \leq i \leq l$  and  $0 \leq j \leq i-2$ . Suppose  $(*)$  has been proved. Let  $x_1 z_0 \in \underline{m}^{i+1}$  for  $i \geq 2$ . Then, with  $j=0$  in  $(*)$ , we get  $z_0 \in \underline{m}^i + \underline{x}\underline{m}$ .  $z_0 = \mu_i + z_1$  with  $\mu_i \in \underline{m}^i$  and  $z_1 \in (\underline{m}^{i+1}; x_1) \cap \underline{x}\underline{m}$ . Now we apply  $(*)$  with  $j=1$  and continue in this way until we have  $z_0 = \tilde{\mu}_i + z_{i-2}$  with  $\tilde{\mu}_i \in \underline{m}^i$  and  $z_{i-2} \in (\underline{m}^{i+1}; x_1) \cap (\underline{x})^{i-2} \underline{m}$ . Finally, we apply  $(*)$  with  $j=i-2$ , to get  $z_{i-2} \in (\underline{x})^{i-2} \underline{m}^2 + (\underline{x})^{i-1} \underline{m} \subseteq \underline{m}^i$ .

Now we prove  $(*)$ . Let  $j=0$ . Let  $x_1 z \in \underline{m}^{i+1}$  with  $i \geq 2$ .  $x_1 z \in \underline{m}^{i+1} \cap (\underline{x}) = \underline{x}\underline{m}^i$ , so  $x_1 z = a_1 x_1 + \dots + a_t x_t$  with  $a_1, \dots, a_t \in \underline{m}^i$ .  $z - a_1 \in (x_2, \dots, x_t)$  since  $x_1, \dots, x_t$  is a regular sequence. In fact,  $z - a_1 \in (x_2, \dots, x_t)\underline{m}$ . For  $(z - a_1)x_1 \in (x_2, \dots, x_t)x_1 \cap \underline{m}^{i+1} \subseteq (\underline{x})^2 \cap \underline{m}^{i+1} = (\underline{x})^2 \underline{m}^{i-1}$  by (1.2).

Assume that  $j > 0$ . Let  $z \in (\underline{m}^{i+1}; x_1) \cap (\underline{x})^j \underline{m}$ .  $z = g_j(x_1, \dots, x_t)$  is a homogeneous polynomial in  $x_1, \dots, x_t$  of degree  $j$  with coefficients in  $\underline{m}$ .  $zx_1 \in \underline{m}^{i+1} \cap (\underline{x})^{j+1} = (\underline{x})^{j+1} \underline{m}^{i+1-j-1}$  by (1.2), so

$$(**) \quad zx_1 = g_j(x_1, \dots, x_t)x_1 = h_{j+1}(x_1, \dots, x_t)$$

with  $h_{j+1}$  a homogeneous polynomial in  $x_1, \dots, x_t$  of degree  $j+1$



and coefficients in  $\underline{m}^{i+1-j-1}$ . Equating coefficients of like monomials of degree  $j+1$  in  $(*)$ , we see that the coefficients of  $g_j(x_1, \dots, x_t)$  are in  $\underline{m}^{i+1-j-1} + (\underline{x})$ . Thus  $g_j(x_1, \dots, x_t) = \xi + f_{j+1}(x_1, \dots, x_t)$  with  $\xi \in (\underline{x})^j \underline{m}^{i+1-j-1}$  and  $f_{j+1}(x_1, \dots, x_t) \in (\underline{x})^{j+1}$ . But  $x_1(\underline{x})^{j+1} \cap \underline{m}^{i+1} \subseteq (\underline{x})^{j+2} \cap \underline{m}^{i+1} = (\underline{x})^{j+2} \underline{m}^{i+1-j-2}$  by (1.2). Consequently,  $(g_j(x_1, \dots, x_t) - \xi)x_1 = f_{j+1}(x_1, \dots, x_t)x_1 \in (\underline{x})^{j+2} \underline{m}^{i+1-j-2}$  and  $f_{j+1}(x_1, \dots, x_t) \in (\underline{x})^{j+1} \underline{m}$ . Thus  $g_j(x_1, \dots, x_t) \in (\underline{x})^j \underline{m}^{i+1-j-1} + (\underline{x})^{j+1} \underline{m}$ . This completes the proof that  $(0: \bar{x}_1) \subseteq (gr \underline{m})^l$ .

Pass to  $(\tilde{R}, \tilde{m}) = (R/x_1 R, \underline{m}/x_1 R)$ . The images  $\tilde{x}_2, \dots, \tilde{x}_t$  form a regular sequence in  $\tilde{R}$  and  $\tilde{m}^{i+1} \cap (\tilde{x}_2, \dots, \tilde{x}_t) \tilde{R} = (\tilde{x}_2, \dots, \tilde{x}_t) \tilde{m}^i$  for  $i \leq l$ . By induction on  $t$ , we have that  $((\tilde{x}_2, \dots, \tilde{x}_{j-1}): \tilde{x}_j) \subseteq (\tilde{x}_2, \dots, \tilde{x}_{j-1}) + (gr \tilde{m})^l$ . Thus,  $((\bar{x}_1, \dots, \bar{x}_{j-1}): \bar{x}_j) \subseteq (\bar{x}_1, \dots, \bar{x}_{j-1}) + (gr \underline{m})^l$ .

For the converse we assume that  $x_1, \dots, x_t$  is a regular sequence in  $R$  with  $((\bar{x}_1, \dots, \bar{x}_{j-1}): \bar{x}_j) \subseteq (\bar{x}_1, \dots, \bar{x}_{j-1}) + (gr \underline{m})^l$  for  $1 \leq j \leq t$  and again use induction on  $t$ . We have  $(0: \bar{x}_1) \subseteq (gr \underline{m})^l$  so  $(x_1) \cap \underline{m}^{i+1} = x_1 \underline{m}^i$  for  $i \leq l$ . The proof is finished if  $t = 1$  so we take  $t > 1$ . We have  $(gr R/\bar{x}_1 gr R)_j \cong (gr(R/x_1 R))_j$  for  $j = 0, \dots, l+1$  since

$$(gr R/\bar{x}_1 gr R)_j = \underline{m}^j/x_1 \underline{m}^{j-1} + \underline{m}^{j+1}$$

and  $(gr(R/x_1 R))_j = (\underline{m}^j, x_1)/(\underline{m}^{j+1}, x_1) \cong \underline{m}^j/\underline{m}^{j+1} + (x_1) \cap \underline{m}^j$ . This means that the required hypotheses are satisfied by  $\tilde{x}_2, \dots, \tilde{x}_t$ , the images of  $x_2, \dots, x_t$  in  $(\tilde{R}, \tilde{m}) = (R/x_1 R, \underline{m}/x_1 R)$ . By induction on  $t$  we have  $(\tilde{x}_2, \dots, \tilde{x}_t) \cap \tilde{m}^{i+1} = (\tilde{x}_2, \dots, \tilde{x}_t) \tilde{m}^i$  for  $i \leq l$ , so that  $(x_1, \dots, x_t) \cap (\underline{m}^{i+1}, x_1) = (x_1, \dots, x_t) \underline{m}^i + x_1 R$ . Let  $w \in (x_1, \dots, x_t) \cap \underline{m}^{i+1}$ .  $w = ax_1 + \mu_2 x_2 + \dots + \mu_t x_t$  with  $\mu_2, \dots, \mu_t \in \underline{m}^i$ . Then  $ax_1 \in \underline{m}^{i+1}$  so  $a \in \underline{m}^i$  and  $w \in (x_1, \dots, x_t) \underline{m}^i$ .

**COROLLARY 1.3.** *Let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence in a local ring  $(R, \underline{m})$  such that  $(x_1, \dots, x_t) \cap \underline{m}^{i+1} = (x_1, \dots, x_t) \underline{m}^i$  for  $i \leq l$ . Then  $(x_1, \dots, x_s) \cap \underline{m}^{i+1} = (x_1, \dots, x_s) \underline{m}^i$  for  $1 \leq s \leq t$  and  $0 \leq i \leq l$ .*

**COROLLARY 1.4.<sup>1</sup>** *Let  $(R, \underline{m})$  be a local ring and let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence.  $x_1, \dots, x_t$  is super-regular if and only if  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x} \underline{m}^i$  for all  $i \geq 0$ .*

*Proof.* If  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x} \underline{m}^i$  for all  $i \geq 0$ , then, by (1.1),

$$((\bar{x}_1, \dots, \bar{x}_{j-1}): \bar{x}_j) \subseteq \bigcap_{i=0}^{\infty} ((\bar{x}_1, \dots, \bar{x}_{j-1}) + (gr \underline{m})^i) = (\bar{x}_1, \dots, \bar{x}_{j-1}).$$

The converse also follows immediately from (1.1).

<sup>1</sup> Added in proof. Corollary 1.4 is a special case of Corollary 2.7 of the paper "Form rings and regular sequences" by P. Valabrega and G. Valla which recently appeared in Nagaya Math. J., **72** (1978), 93-101.



**COROLLARY 1.5.** *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring.  $grR$  is Cohen-Macaulay if there is a minimal reduction  $\underline{x} = x_1, \dots, x_d$  of  $\underline{m}$  such that  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for all  $i \geq 0$ .*

*Proof.* By (1.4),  $\bar{x}_1, \dots, \bar{x}_d$  is a homogeneous system of parameters which is a regular sequence. By [2],  $grR$  is Cohen-Macaulay.

**REMARK.** (1.5) is similar to a special case of Theorem 4.4 in [2].

We take a moment to note the connection between super-regularity and the notion of analytic independence in  $\underline{m}$ . Recall, [3], that a system of elements  $\underline{x} = x_1, \dots, x_t$  in a local ring  $(R, \underline{m})$  is analytically independent in  $\underline{m}$  if given  $f_s(X_1, \dots, X_t)$ , a homogeneous polynomial in  $R[X_1, \dots, X_t]$  of (arbitrary) degree  $s$ , such that  $f_s(x_1, \dots, x_t) \in \underline{m}^{s+1}$  then all the coefficients of  $f_s$  are in  $\underline{m}$ . Stated differently,  $\underline{x} = x_1, \dots, x_t$  is analytically independent in  $\underline{m}$  if  $(x_1, \dots, x_t)^s \cap \underline{m}^{s+1} = (x_1, \dots, x_t)^s \underline{m}$  for all positive integers  $s$ . Note also that if a regular sequence  $\underline{x} = x_1, \dots, x_t$  in  $(R, \underline{m})$  satisfies  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq l$  and if  $f_s(X_1, \dots, X_t)$  is a form of degree  $s$  in  $R[X_1, \dots, X_t]$  with  $f_s(x_1, \dots, x_t) \in \underline{m}^{i+1}$  for  $i \leq l$ , then all one can say about the coefficients of  $f_s$  is that they are in  $(\underline{m}^{i+1-s}, x_1, \dots, x_t)$  as the example  $f(X_1, X_2) = x_2X_1 - x_1X_2$  shows.

Next we give a direct proof that intersection equalities as in (1.1) give information about minimal bases for powers of  $\underline{m}$ . This can also be done using the Hilbert sum transform as in [7]. The idea is that  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq l$  implies that multiplication of  $\underline{m}^i/\underline{m}^{i+1}$  by each  $\bar{x}_j$ , for  $x_j$  in the regular sequence  $\underline{x}$ , is a monomorphism for  $i \leq l$ .

We introduce some notation. Let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence in the local ring  $(R, \underline{m})$ . Let  $\mathcal{H}_i(\underline{x})$  denote the set of monomials of degree  $i$  in  $x_1, \dots, x_t$ . If  $J$  is a set of elements of  $R$ ,  $\mathcal{H}_i(\underline{x})J$  denotes the set of products  $\{\xi j \mid \xi \in \mathcal{H}_i(\underline{x}), j \in J\}$ .

**THEOREM 1.6.** *Let  $(R, \underline{m})$  be a local ring and let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence such that  $x_1, \dots, x_t, z_{t+1}, \dots, z_v$  is a minimal basis for  $\underline{m}$ . Suppose that  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq l$ . Then, for each  $i \leq l$ , the set  $\mathcal{H}_i(\underline{x}) \cup \mathcal{H}_{i-1}(\underline{x})\mathcal{Z}_1 \cup \dots \cup \mathcal{H}_1(\underline{x})\mathcal{Z}_{i-1}$  is a subset of a minimal basis for  $\underline{m}^i$ , where  $\mathcal{Z}_j$  is the set of monomials in  $z_{t+1}, \dots, z_v$  needed to complete the set  $\mathcal{H}_j(\underline{x}) \cup \dots \mathcal{H}_1(\underline{x})\mathcal{Z}_{j-1}$  to a minimal basis of  $\underline{m}^j$ .*

*Proof.* We prove the theorem by induction on  $i$  and  $t$ . We may assume  $i > 1$ . Let  $t = 1$ . If

$$ax_1^i + bx_1^{i-1}\xi_{11} + \dots + cx_1\xi_{i-1n_{i-1}} \in \underline{m}^{i+1}$$



where  $\xi_{pq}$ ,  $1 \leq p \leq i-1$ ,  $1 \leq q \leq n_p$ , are the monomials  $z_{t+1}, \dots, z_v$  completing the basis for  $\underline{m}^p$ , then  $x_1(ax_1^{i-1} + bx_1^{i-2}\xi_{11} + \dots + c\xi_{i-1n_{i-1}}) \in \underline{m}^{i+1}$ . Thus,  $ax_1^{i-1} + bx_1^{i-2}\xi_{11} + \dots + c\xi_{i-1n_{i-1}} \in \underline{m}^i$  so, by induction on  $i$ , the coefficients  $a, b, \dots, c$  are in  $\underline{m}$ . Assume that  $t > 1$ . Let  $\tilde{x}_2, \dots, \tilde{x}_i, \tilde{z}_{t+1}, \dots, \tilde{z}_v$  be the images of  $x_2, \dots, x_i, z_{t+1}, \dots, z_v$  in  $(\tilde{R}, \tilde{\underline{m}}) = (R/x_1R, \underline{m}/x_1R)$ . Note that for  $j < i$  we have that

$$\mathcal{H}_j(\tilde{x}_2, \dots, \tilde{x}_i) \cup \mathcal{H}_{j-1}(\tilde{x}_2, \dots, \tilde{x}_i)\tilde{\mathcal{X}}_1 \cup \dots \cup \mathcal{H}_1(\tilde{x}_2, \dots, \tilde{x}_i)\tilde{\mathcal{X}}_{j-1} \cup \tilde{\mathcal{X}}_j$$

is a minimal basis for  $\tilde{\underline{m}}^j$ . This set is the image in  $\tilde{R}$  of all the monomials in the chosen minimal basis for  $\underline{m}^j$  which do not contain  $x_1$ . It is clear that this set is a generating set for  $\tilde{\underline{m}}^j$ . It is minimal because

$$\begin{aligned} \dim_{R/\underline{m}} \tilde{\underline{m}}^j / \tilde{\underline{m}}^{j+1} &= \dim_{R/\underline{m}} \underline{m}^j / \underline{m}^{j+1} + x_1 \underline{m}^{j-1} \\ &= \dim_{R/\underline{m}} \underline{m}^j / \underline{m}^{j+1} - \dim_{R/\underline{m}} x_1 \underline{m}^{j-1} + \underline{m}^{j+1} / \underline{m}^{j+1} \end{aligned}$$

and  $\dim_{R/\underline{m}} x_1 \underline{m}^{j-1} + \underline{m}^{j+1} / \underline{m}^{j+1}$  is the number of monomials in the chosen basis for  $\underline{m}^j$  which contain  $x_1$ .

Suppose that

$$(*) \quad ax_1^{\alpha_1} \dots x_t^{\alpha_t} + \dots + bx_1^{\beta_1} \dots x_t^{\beta_t} \xi_{11} + \dots + cx_1 \xi_{i-1n_{i-1}} \in \underline{m}^{i+1}$$

is a relation mod  $\underline{m}^{i+1}$  among the elements of the set  $\mathcal{H}_i(\underline{x}) \cup \mathcal{H}_{i-1}(\underline{x})\mathcal{X}_1 \cup \dots \cup \mathcal{H}_1(\underline{x})\mathcal{X}_{i-1}$ . By passing to  $\tilde{R}$  and using induction we get that all the coefficients of monomials in  $(*)$  which do not contain  $x_1$  are in  $\underline{m}$ . So we have a relation

$$(**) \quad ax_1^{\alpha_1} \dots x_t^{\alpha_t} + \dots + bx_1^{\beta_1} \dots x_t^{\beta_t} \xi_{11} + \dots + cx_1 \xi_{i-1n_{i-1}} \in \underline{m}^{i+1}$$

with  $x_1$  appearing in each monomial. Thus,

$$x_1(ax_1^{\alpha_1-1} \dots x_t^{\alpha_t} + \dots + bx_1^{\beta_1-1} \dots x_t^{\beta_t} \xi_{11} + \dots + c\xi_{i-1n_{i-1}}) \in \underline{m}^{i+1}.$$

By (1.3),  $(x_1) \cap \underline{m}^{i+1} = x_1 \underline{m}^i$  for  $i \leq l$ , so

$$ax_1^{\alpha_1-1} \dots x_t^{\alpha_t} + \dots + bx_1^{\beta_1-1} \dots x_t^{\beta_t} \xi_{11} + \dots + c\xi_{i-1n_{i-1}} \in \underline{m}^i.$$

By induction, all the coefficients  $a, b, \dots, c$  are in  $\underline{m}$ .

**EXAMPLE.** We illustrate (1.6) with an example. Let  $R$  be the ring  $k[[t^7, t^8, t^{13}, t^{19}]]$  with  $k$  a field.  $\underline{m} = (t^7, t^8, t^{13}, t^{19})$ .  $(t^7) \cap \underline{m}^{i+1} = t^7 \underline{m}^i$  for  $i = 1, 2, 3$  but  $(t^7) \cap \underline{m}^5 \supsetneq t^7 \underline{m}^4$  as  $t^{40} = t^7 \cdot t^7(t^{13})^2$  is in  $\underline{m}^5$  but  $t^7(t^{13})^2$  is not in  $\underline{m}^4$ . We can apply (1.6) to get that  $t^7\{t^7, t^8, t^{13}, t^{19}\}$  is part of a minimal basis for  $\underline{m}^2 = (t^7 \underline{m}, t^{16})$  and  $t^7\{t^{14}, t^{15}, t^{20}, t^{26}, t^{16}\}$  is part of a minimal basis for  $\underline{m}^3 = (t^7 \underline{m}^2, t^{24})$ .

**2. Reduction to lower dimension.** Let  $(R, \underline{m})$  be a  $d$ -dimen-



sional local Cohen-Macaulay ring of multiplicity  $e$  and embedding dimension  $v$ . If  $v = d$ ,  $d + 1$  or  $e + d - 1$  then  $grR$  is Cohen-Macaulay, cf. [4]. The property  $\mathcal{P}_k$  for a local Cohen-Macaulay ring to have embedding dimension  $e + d - k$  is preserved under passage to  $R(U) = R[U]_{\mathfrak{m}_R[U]}$ ,  $U$  an indeterminate, and is preserved under reduction modulo the ideal generated by any element of minimal reduction of  $\underline{m}$ . Given a local Cohen-Macaulay ring with a property  $\mathcal{P}$  which is preserved under both types of change of rings mentioned above, (2.4) below shows that if every 1-dimensional local Cohen-Macaulay ring with  $\mathcal{P}$  has Cohen-Macaulay associated graded ring then so does every  $d$ -dimensional local Cohen-Macaulay ring with  $\mathcal{P}$ .

Thus to show that Cohen-Macaulay local rings of embedding dimension  $d$ ,  $d + 1$  or  $e + d - 1$  have Cohen-Macaulay associated graded rings, it sufficient to show that 1-dimensional local Cohen-Macaulay rings of embedding dimension 1, 2 or  $e$  have Cohen-Macaulay associated graded rings. (However, the usual proof for embedding dimension  $d$  or  $d + 1$  is more direct.) Another application of (2.4) follows in §3.

As the techniques in the proof of (2.4) have other uses, we put this result in a more general setting. In §1 we saw that "super-regularity" can be expressed in terms of certain intersection equalities. What we need are conditions which allow intersection equalities in dimension  $d - 1$  to be lifted to dimension  $d$ .

**PROPOSITION 2.1.** *Let  $(R, \underline{m})$  be a local ring and  $\underline{x} = x_1, \dots, x_t$  be a regular sequence. The following statements are equivalent.*

1.  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq$  some positive integer  $l$ .
2. (a)  $(\underline{m}/x_1R)^{i+1} \cap (\tilde{x}_2, \dots, \tilde{x}_t)R/x_1R = (\tilde{x}_2, \dots, \tilde{x}_t)(\underline{m}/x_1R)^i$  for  $i \leq l$ , where  $\sim$  denotes image in  $R/x_1R$ ;  
 (b) If  $f_s$  is a homogeneous polynomial in the polynomial ring  $R[X_1, \dots, X_t]$  of (arbitrary) degree  $s$  with some coefficient a unit such that  $rf_s(x_1, \dots, x_t) \in \underline{m}^{i+1}$  for some  $r \in R$  and any  $i \leq l$ , then  $r \in \underline{m}^{i+1-s}$ .
3. (a)  $(\underline{m}/x_1R)^{i+1} \cap (\tilde{x}_2, \dots, \tilde{x}_t)R/x_1R = (\tilde{x}_2, \dots, \tilde{x}_t)(\underline{m}/x_1R)^i$  for  $i \leq l$ , where  $\sim$  denotes image in  $R/x_1R$ ;  
 (b)  $(x_1) \cap \underline{m}^{i+1} = x_1\underline{m}^i$  for  $i \leq l$ .

**EXAMPLE.** Let  $k$  be a field and let  $R = k[[X, Y, Z]]/(Y^3 - XZ) = k[[x, y, z]]$  with  $\underline{m} = (x, y, z)$ .  $R$  is Cohen-Macaulay and  $x, z$  is a regular sequence. In  $R/xR \cong k[[Y, Z]]/(Y^3)$ , we have  $(\underline{m}/xR)^{i+1} \cap zR/xR = z(\underline{m}/xR)^i$  for  $i \geq 0$ . Similarly, in  $R/zR$  we have  $(\underline{m}/zR)^{i+1} \cap xR/zR = x(\underline{m}/zR)^i$  for  $i \geq 0$ . But  $\underline{m}^3 \cap (x, z) \neq (x, z)\underline{m}^2$  as  $xz \in \underline{m}^3$ . Thus the image of  $z$  is super-regular in  $R/xR$ , the image of  $x$  is super-regular in  $R/zR$  but neither  $x$  nor  $z$  is super-regular in  $R$ .



*Proof of 2.1.* Suppose (1) holds. (2) (a) follows immediately. We prove (2) (b) by induction on  $t$  and on  $i + 1 - s$ . There is no difficulty if  $t = 1$  so we assume  $t > 1$  and we may assume also that  $i + 1 - s > 0$ . Let  $f_s$  and  $r$  be as in (2) (b). Let

$$(*) \quad f_s(x_1, \dots, x_t) = ax_1^{\alpha_1} \dots x_t^{\alpha_t} + \dots + cx_1^{\gamma_1} \dots x_t^{\gamma_t},$$

with  $a$  a unit. We may assume that  $r$  is not a multiple of any  $x_i$ . If  $\alpha_1 = 0$  in  $(*)$ , by passing to  $R/x_1R$  where all the hypotheses of (1) hold for the images of  $x_2, \dots, x_t$ , we get  $r \in (\underline{m}^{i+1-s}, x_1)$ .  $r = \mu + r'x_1$  with  $\mu \in \underline{m}^{i+1-s}$  and  $r'x_1f_s(x_1, \dots, x_t) \in \underline{m}^{i+1}$ . Since  $x_1f_s$  has degree  $s + 1$  and  $i + 1 - (s + 1) < i + 1 - s$ ,  $r' \in \underline{m}^{i+1-(s+1)}$  and  $r \in \underline{m}^{i+1-s}$ . Thus we may assume that  $x_1$  appears in each monomial with unit coefficient in  $(*)$ . Pass to the ring  $R(U) = R[U]_{\underline{m}R(U)}$ ,  $U$  an indeterminate and let  $x'_1 = x_1 + Ux_2$ . Since  $R \rightarrow R(U)$  is faithfully flat,  $x'_1, x_2, \dots, x_t$  is a regular sequence in  $R(U)$  and  $(x'_1, x_2, \dots, x_t) \cap (\underline{m}R(U))^{i+1} = (x'_1, x_2, \dots, x_t)(\underline{m}R(U))^i$  for  $i \leq l$ . We have

$$f_s(x_1, \dots, x_t) = a(x'_1 - Ux_2)^{\alpha_1} \dots x_t^{\alpha_t} + \dots + c(x'_1 - Ux_2)^{\gamma_1} \dots x_t^{\gamma_t},$$

with  $rf_s(x_1, \dots, x_t) \in (\underline{m}R(U))^{i+1}$ . We have

$$(**) \quad \begin{aligned} f_s(x_1, \dots, x_t) &= ax_1^{\alpha_1} \dots x_t^{\alpha_t} + \dots \\ &\quad - aU^{\alpha_1}x_2^{\alpha_1+\alpha_2} \dots x_t^{\alpha_t} + \dots - cU^{\gamma_1}x_2^{\gamma_1+\gamma_2} \dots x_t^{\gamma_t}. \end{aligned}$$

Since  $f_s$  has terms with  $x'_1$  missing, we want to consider  $f_s$  as a form in  $x'_1, \dots, x_t$  and apply the same argument as above. However, there may be some collapsing in  $(**)$  so we have to collect terms and write  $f_s$  as a sum of distinct monomials of degree  $s$  in  $x'_1, \dots, x_t$ . If all the other monomials in  $(**)$  are distinct from  $x_2^{\alpha_1+\alpha_2} \dots x_t^{\alpha_t}$  then, by passing to  $R(U)/x'_1R(U)$  and using the same reasoning as above, we see that  $rf_s \in (\underline{m}R(U))^{i+1}$  implies that  $r \in (\underline{m}R(U))^{i+1-s} \cap R = \underline{m}^{i+1-s}$ . So we assume there is some collapsing and let  $a, a_{i_1}, \dots, a_{i_g}$  be unit coefficients in  $(*)$  such that the monomials  $x_2^{\alpha_{i_j1}+\alpha_{i_j2}} \dots x_t^{\alpha_{i_jt}}$  in  $(**)$  having coefficients  $a_{i_j}U^{\alpha_{i_j}}$  are equal to  $\mathcal{H} = x_2^{\alpha_1+\alpha_2} \dots x_t^{\alpha_t}$ . Note that  $x_2^{\alpha_{i_j1}+\alpha_{i_j2}} \dots x_t^{\alpha_{i_jt}} = \mathcal{H}$  implies that  $\alpha_{i_j3} = \alpha_3, \dots, \alpha_{i_jt} = \alpha_t$  and  $\alpha_{i_j1} + \alpha_{i_j2} = \alpha_1 + \alpha_2$ . Thus  $\alpha_{i_j1} \neq \alpha_1$  since the monomials in  $(*)$  are distinct. Let  $b$  be the coefficient (in  $\underline{m}$  and possibly zero) of  $\mathcal{H}$  in  $(*)$ . The new (collected) coefficient of  $\mathcal{H}$  is

$$A_{\mathcal{H}} = -aU^{\alpha_1} - a_{i_1}U^{\alpha_{i_11}} - \dots - a_{i_g}U^{\alpha_{i_g1}} + b.$$

Since the powers of  $U$  are distinct,  $A_{\mathcal{H}}$  is a unit. Thus  $f_s$  may be written as a form of degree  $s$  in  $x'_1, x_2, \dots, x_t$  with coefficient of the monomial  $\mathcal{H}$  a unit. Since  $x'_1$  is missing from  $\mathcal{H}$ , we apply the same argument as above to show that  $r \in \underline{m}^{i+1-s}$ . This concludes the proof that (1)  $\Rightarrow$  (2).



Clearly (2)  $\Rightarrow$  (3). Suppose (3) holds. If  $t = 1$ , then  $j = 1$  and (1) holds. Assume  $t > 1$ . Let  $w = ax_1 + g(x_2, \dots, x_t) \in (x) \cap \underline{m}^{i+1}$ . With  $\sim$  denoting images in  $R/x_1R$ , we have  $\tilde{g}(\tilde{x}_2, \dots, \tilde{x}_t) \in (\tilde{x}_2, \dots, \tilde{x}_t) \cap (\underline{m}/x_1R)^{i+1} = (\tilde{x}_2, \dots, \tilde{x}_t)(\underline{m}/x_1R)^i$ . Thus  $g(x_2, \dots, x_t) = h(x_2, \dots, x_t) + x_1r$  where  $h(x_2, \dots, x_t) \in (x_2, \dots, x_t)\underline{m}^i$ . We have  $w = ax_1 + rx_1 + h(x_2, \dots, x_t) \in \underline{m}^{i+1}$  so  $(a + r)x_1 \in \underline{m}^{i+1} \cap (x_1) = x_1\underline{m}^i$ . Since  $x_1$  is a non-zero divisor;  $a + r \in \underline{m}^i$  and  $w \in \underline{xm}^i$ .

(2.2) below is the technical result needed to reduce dimension. To eliminate excess notation, homomorphic images of the regular sequence  $\underline{x} = x_1, \dots, x_t$  will be denoted by the same letters, and if  $U_1, \dots, U_p$  are indeterminates, we denote

$$R(U_1, \dots, U_p) = R[U_1, \dots, U_p]_{\underline{m}R[U_1, \dots, U_p]}$$

by  $R(U)$ .

**THEOREM 2.2.** *Let  $(R, \underline{m})$  be a local ring and let  $\underline{x} = x_1, \dots, x_t$  be a regular sequence with  $t > 1$ . Suppose that*

$$(\underline{m}/x_1R)^{i+1} \cap (x_2, \dots, x_t)R/x_1R = (x_2, \dots, x_t)(\underline{m}/x_1R)^i$$

*for  $i \leq$  some positive integer  $l$ . Suppose that for any finite set of indeterminates  $U_1, \dots, U_p$ , there are elements  $x_{i_1}, \dots, x_{i_p} \in \{x_2, \dots, x_t\}$  and a linear polynomial  $g(U_1, \dots, U_p) = x_1 + x_{i_1}U_1 + \dots + x_{i_p}U_p$  such that*

$$\begin{aligned} & (x_2, \dots, x_t)(R(U)/gR(U)) \cap (\underline{m}R(U)/gR(U))^{i+1} \\ &= (x_2, \dots, x_t)(\underline{m}R(U)/gR(U))^i, \quad \text{for } i \leq l. \end{aligned}$$

*Then,*

$$(x_1, \dots, x_t) \cap \underline{m}^{i+1} = (x_1, \dots, x_t)\underline{m}^i, \quad \text{for } i \leq l.$$

*Proof.* We prove that (2)(b) of (2.1) holds for any local ring with a regular sequence satisfying the hypotheses of the theorem by induction on  $i + 1 - s$ , where  $s, f_s$  and  $r$  are as in the statement of (2)(b). We may assume that  $i + 1 - s > 0$ . Let

$$(*) \quad f_s = ax_1^{\alpha_1} \dots x_t^{\alpha_t} + \dots + cx_1^{\gamma_1} \dots x_t^{\gamma_t}, \quad \text{with } a \text{ a unit.}$$

If  $\alpha_1 = 0$ , pass to  $R/x_1R$ . By (2.1) (1)  $\Rightarrow$  (2)(b), we can conclude that  $r \in (\underline{m}^{i+1-s}, x_1)$ .  $r = \mu + r'x_1$  with  $\mu \in \underline{m}^{i+1-s}$ . But then  $r'x_1f_s \in \underline{m}^{i+1}$  and  $x_1f_s$  has degree  $s + 1$ . By induction, we have  $r' \in \underline{m}^{i+1-(s+1)}$ , and  $r \in \underline{m}^{i+1-s}$ . Thus we may assume that  $x_1$  appears in each monomial in (\*) with unit coefficient. By hypothesis, for the indeterminate  $U$ , there is an element, say  $x_2$  in  $\{x_2, \dots, x_t\}$  such that



$$\begin{aligned} (x_2, \dots, x_i)(R(U)/x'_1R(U)) \cap (\underline{m}R(U)/x'_1R(U))^{i+1} \\ = (x_2, \dots, x_i)(\underline{m}R(U)/x'_1R(U))^i, \end{aligned}$$

for  $i \leq l$ , where  $x'_1 = x_1 + Ux_2$ .

We have  $f_s = a(x'_1 - Ux_2)^{\alpha_1}x_2^{\alpha_2} \cdots x_i^{\alpha_i} + \cdots + c(x'_1 - Ux_2)^{\gamma_1}x_2^{\gamma_2} \cdots x_i^{\gamma_i}$ . The same argument as in the proof of (2.1) (1)  $\Rightarrow$  (2)(b) shows that  $f_s$  can be written as a homogeneous polynomial, in the regular sequence  $x'_1, x_2, \dots, x_i$ , of degree  $s$ , having a monomial with unit coefficient in which  $x'_1$  does not appear:  $f_s = A(U)x_2^{\beta_2} \cdots x_i^{\beta_i} + \cdots + B(U)x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_i^{\epsilon_i}$ , with  $A(U)$  a unit in  $R(U)$ .  $rf_s \in (\underline{m}R(U))^{i+1}$ . By passing to  $R(U)/x'_1R(U)$  and applying (2.1) (1)  $\Rightarrow$  (2)(b), we get  $r \in ((\underline{m}R(U))^{i+1-s}, x'_1)$  so  $r = \mu + r'x'_1$  with  $\mu \in (\underline{m}R(U))^{i+1-s}$  and  $r' \in R(U)$ . Now  $R(U)$  and the regular sequence  $x'_1, x_2, \dots, x_i$  satisfy the hypotheses of the theorem. Since  $r'x'_1f_s \in (\underline{m}R(U))^{i+1}$  and  $i+1 - (s+1) < i+1-s$ , we have by induction that  $r' \in (\underline{m}R(U))^{i+1-(s+1)}$ . Thus  $r \in (\underline{m}R(U))^{i+1-s} \cap R = \underline{m}^{i+1-s}$ .

We will say that a property  $\mathcal{P}$  of a local ring  $(R, \underline{m})$  is *firm* if  $\mathcal{P}$  is preserved under the change of rings  $R \rightarrow R(U)$ ,  $U$  an indeterminate. We will say that a firm property  $\mathcal{P}$  of a  $d$ -dimensional local Cohen-Macaulay ring  $(R, \underline{m})$  is *stable for the minimal reduction*  $\underline{x} = x_1, \dots, x_d$  of  $\underline{m}$  if for any finite set of indeterminates  $U_1, \dots, U_p$ ,  $\mathcal{P}$  is preserved under reduction of  $R(U_1, \dots, U_p)$  modulo the ideal generated by any subset of generators of  $(x_1, \dots, x_d)R(U_1, \dots, U_p)$ . We will say that a firm property  $\mathcal{P}$  of a  $d$ -dimensional local Cohen-Macaulay ring  $(R, \underline{m})$  is *stable* if  $\mathcal{P}$  is preserved under reduction modulo the ideal generated by any element of a minimal reduction of  $\underline{m}$ .

EXAMPLES. If  $(R, \underline{m})$  is a local Cohen-Macaulay ring, the property of being Gorenstein or regular is clearly stable. Let  $e$  be a fixed positive integer. The property  $\mathcal{P}$  for a local Cohen-Macaulay ring to have multiplicity  $e$  is a stable property. As mentioned in the first paragraph of §2, the property  $\mathcal{P}$  for a local Cohen-Macaulay ring of dimension  $d$  and multiplicity  $e$  to have embedding dimension  $e + d - k$  is stable. As example of a property stable for a particular minimal reduction  $\underline{x}$  is given in §3.

THEOREM 2.3. *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring with a property  $\mathcal{P}$  stable for the minimal reduction  $\underline{x} = x_1, \dots, x_d$  of  $\underline{m}$ . Suppose that 1-dimensional local Cohen-Macaulay rings  $(S, \underline{n})$  with  $\mathcal{P}$  satisfy  $(x) \cap \underline{n}^{i+1} = x\underline{n}^i$  for  $i \leq$  some positive integer  $l$  and any minimal reduction  $x$  of  $\underline{n}$ . Then  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq l$ .*



*Proof.* The proof is by induction on  $d$ . We take  $d > 1$ . Let  $U = U_1, \dots, U_p$  be a finite set of indeterminates. Let  $g(U) = x_1 + U_1x_2 + \dots + U_px_2$ . By hypothesis,  $R(U)/g(U)R(U)$  has  $\mathcal{P}$  and is stable for the minimal reduction  $\tilde{x}_2, \dots, \tilde{x}_d$  which is the image of  $x_2, \dots, x_d$  in  $R(U)/g(U)R(U)$ . By induction,  $(\underline{m}R(U)/g(U)R(U))^{i+1} \cap (\tilde{x}_2, \dots, \tilde{x}_d) = (\tilde{x}_2, \dots, \tilde{x}_d)(\underline{m}R(U)/g(U)R(U))^i$  for  $i \leq l$ . By (2.2),  $\underline{m}^{i+1} \cap (x_1, \dots, x_d) = (x_1, \dots, x_d)\underline{m}^i$  for  $i \leq l$ .

**COROLLARY 2.4.** *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring with a property  $\mathcal{P}$  stable for the minimal reduction  $\underline{x} = x_1, \dots, x_d$  of  $\underline{m}$ . If  $\text{gr}S$  is Cohen-Macaulay for every 1-dimensional local Cohen-Macaulay ring  $(S, \underline{n})$  with  $\mathcal{P}$ , then  $\text{gr}R$  is Cohen-Macaulay.*

For further use it will be convenient to have a variant of (2.3) which requires us to test only some of the minimal reductions in dimension 1. We need the same types of stability for properties of a minimal reduction. Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring. Let  $U$  be an indeterminate. Let  $\underline{x} = x_1, \dots, x_d$  be a minimal reduction of  $\underline{m}$ . Let  $\mathcal{Q}$  be a property of  $\underline{x}R$ . We will say that  $\mathcal{Q}$  is *firm* if  $\underline{x}R(U)$  has  $\mathcal{Q}$ . We will say that a firm property  $\mathcal{Q}$  is *stable* if, given any element  $g$  of a minimal generating set for  $\underline{x}R(U)$ , say  $\underline{x}R(U) = (g, y_2, \dots, y_d)R(U)$ , the image  $(y_2, \dots, y_d)(R(U)/gR(U))$  has  $\mathcal{Q}$ .

For example, let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring and, for positive integers  $j$ , let  $\mathcal{Q}_j$  be the property for minimal reductions  $\underline{x}$  of  $\underline{m}$  that  $\underline{m}^j \subseteq \underline{x}R$ . Then  $\mathcal{Q}_j$  is stable. Let  $\mathcal{Q}'_j$  be the property that  $\underline{m}^j \not\subseteq \underline{x}R$ . Then  $\mathcal{Q}'_j$  is also stable.

**THEOREM 2.5.** *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring with a property  $\mathcal{P}$  stable for the minimal reduction  $\underline{x} = x_1, \dots, x_d$  of  $\underline{m}$ . Assume in addition that  $\underline{x}$  has a stable property  $\mathcal{Q}$ . Suppose that 1-dimensional local Cohen-Macaulay rings  $(S, \underline{n})$  with  $\mathcal{P}$  satisfy  $(x) \cap \underline{n}^{i+1} = x\underline{n}^i$  for  $i \leq$  some positive integer  $l$  and all minimal reductions  $x$  of  $\underline{n}$  which have the stable property  $\mathcal{Q}$ . Then  $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$  for  $i \leq l$ .*

*Proof.* The proof is the same as the proof of (2.3).

### 3. Applications.

**THEOREM 3.1.** *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Gorenstein ring of multiplicity  $e$  and embedding dimension  $e + d - 3$ . If  $\underline{x} = x_1, \dots, x_d$  is any minimal reduction of  $\underline{m}$ ,  $\underline{m}^3 \not\subseteq \underline{x}R$  and  $\underline{m}^4 \subset \underline{x}\underline{m}$ .*



$grR$  is Cohen-Macaulay if and only if  $\underline{m}^4 = \underline{xm}^3$  for some minimal reduction  $x$  of  $\underline{m}$ .

*Proof.* Let  $\underline{x} = x_1, \dots, x_d$  be a minimal reduction of  $\underline{m}$ . Pass to  $(\tilde{R}, \tilde{\underline{m}}) = (R/\underline{x}R, \underline{m}/\underline{x}R)$ . Since  $v(\tilde{\underline{m}}) = e - 3$  and  $\tilde{\lambda}(\tilde{R}) = e$ , we must have that  $\tilde{\underline{m}}^4 = 0$ . Since  $\lambda(\text{socle } \tilde{R}) = 1$ ,  $\tilde{\underline{m}}^3 = \text{socle } \tilde{R}$ . Thus  $\underline{m}^3 \not\subseteq \underline{x}R$  and  $\underline{m}^4 \subset \underline{x}R$ . By the analytic independence of  $\underline{x}$ ,  $\underline{m}^4 \subset \underline{xm}$ .

If, for some minimal reduction  $\underline{x}$  of  $\underline{m}$ ,  $\underline{m}^4 \not\subseteq \underline{xm}^3$ , then  $\bar{x}_1, \dots, \bar{x}_d$  is a homogeneous system of parameters in  $grR$  that is not a regular sequence by (1.1) so that  $grR$  is not Cohen-Macaulay by [2].

To complete the proof we will show that if  $(R, \underline{m})$  is a  $d$ -dimensional local Cohen-Macaulay ring of embedding dimension  $e + d - 3$  and if  $\underline{x}$  is a minimal reduction of  $\underline{m}$  such that  $\underline{m}^3 \not\subseteq \underline{x}R$  and  $\underline{m}^4 = \underline{xm}^3$  then  $grR$  is Cohen-Macaulay. Let  $\mathcal{P}$  be the property for  $d$ -dimensional local Cohen-Macaulay rings  $(S, \underline{n})$  that  $S$  has embedding dimension  $e + d - 3$  and that there exists a minimal reduction  $\underline{y}$  of  $\underline{n}$  such that  $\underline{n}^3 \not\subseteq \underline{y}R$  and  $\underline{n}^4 = \underline{yn}^3$ .  $R$  has  $\mathcal{P}$  and  $\mathcal{P}$  is stable for the minimal reduction  $\underline{x}$  of  $\underline{m}$ . Thus by (2.4), we may assume  $d = 1$ . Let  $\underline{x} = x$  and let  $\underline{m} = (x, w_1, \dots, w_{e-3})$ . There exist  $p, q \in \{1, \dots, e-3\}$  such that  $\underline{m}^2 = (x\underline{m}, w_p w_q)$ ,  $\underline{m}^3 = (x\underline{m}^2, w_p^2 w_q)$  and  $w_p^2 w_q \notin \underline{xm}$ . This follows from the fact that  $(\underline{m}/xR)^2$  and  $(\underline{m}/xR)^3$  are non-zero principal ideals. It is clear that  $\underline{m}^{i+1} \cap (x) = x\underline{m}^i$  for  $i \geq 3$ . We must show that  $\underline{m}^3 \cap (x) = x\underline{m}^2$ . Let  $z \in \underline{m}^3 \cap (x)$ .  $z = x\mu + rw_p^2 w_q$  with  $\mu \in \underline{m}^2$  and  $r \in R$ . Since  $rw_p^2 w_q \in (x)$ ,  $r \in \underline{m}$  and  $rw_p^2 w_q \in \underline{m}^4 = \underline{xm}^3$ . Thus  $z \in x\underline{m}^2$  and  $\underline{m}^3 \cap (x) = x\underline{m}^2$ .

REMARKS. Note that, with the hypotheses of (3.1), if  $grR$  is Cohen-Macaulay it is not Gorenstein. (3.1) may also be proved directly, using just (1.5) instead of (2.4) and (1.5).

EXAMPLES. 1. Let  $k$  be a field. The rings  $k[[t^5, t^6, t^9]]$ ,  $k[[t^6, t^7, t^{10}, t^{11}]]$ ,  $k[[t^6, t^{11}, t^{13}, t^{20}]]$  are Gorenstein and satisfy  $v = e + d - 3$  and  $\underline{m}^4 = x\underline{m}^3$ . Recall that the multiplicity  $e$  of a numerical semigroup ring  $R = k[[t^{\alpha_1}, t^{\alpha_2}, \dots, t^{\alpha_n}]]$  with  $\alpha_1 < \alpha_2 < \dots < \alpha_n$  and  $\gcd(\alpha_1, \alpha_2, \dots, \alpha_n) = 1$  is just  $\alpha_1$ . This follows for example, from [8, VIII, § 10, Thm. 24].

2.  $k[[t^5, t^6, t^9]] = k[[X, Y, Z]]/(X^3 - YZ, Y^3 - Z^2)$  is a complete intersection with associated graded ring  $k[X, Y, Z]/(YZ, Z^2, Y^4 - ZX^3)$  which is Cohen-Macaulay but not a complete intersection.

3. The hypothesis that  $R$  is Gorenstein cannot be omitted from (3.1). Let  $R = k[[t^7, t^8, t^{11}, t^{13}, t^{17}]]$ , with  $k$  a field.  $R$  satisfies  $v(\underline{m}) = e + d - 3 = 7 + 1 - 3$  and  $\underline{m}^4 = t^7 \underline{m}^3$  but  $grR$  is not Cohen-Macaulay at  $t^{17} \underline{m} \subset \underline{m}^3$ .

It is possible that the hypothesis  $\underline{m}^4 = \underline{xm}^3$  is redundant in (3.1),



i.e., it is possible that all  $d$ -dimensional local Gorenstein rings of embedding dimension  $e + d - 3$  have Cohen-Macaulay associated graded rings.<sup>1</sup> By (2.4) it is sufficient to prove this for  $d = 1$ . It is true that if  $(R, \underline{m})$  is a 1-dimensional analytically irreducible local Gorenstein domain with algebraically closed residue field and with embedding dimension  $e + d - 3$ , then  $grR$  is Cohen-Macaulay.

Next, we want to show that Cohen-Macaulay local rings with certain Hilbert polynomials have Cohen-Macaulay associated graded rings. First, we recall definitions of the concepts involved. For any  $d$ -dimensional local ring  $(R, \underline{m})$ , the Hilbert function is defined for nonnegative integers  $n$  by  $H_R(n) = \dim_{R/\underline{m}}(\underline{m}^n/\underline{m}^{n+1})$ . The Hilbert sum transforms are defined inductively for nonnegative integers  $n$  by

$$H_R^0(n) = H_R(n) \quad \text{and} \quad H_R^i(n) = \sum_{j=0}^n H_R^{i-1}(j).$$

For large  $n$ ,  $H_R^i(n)$  is a polynomial  $P_R^i(n)$  of degree  $d - 1 + i$ . If  $x \in \underline{m}$ , then  $H_{R/xR}^1(n) - H_R^0(n) = \lambda((\underline{m}^{n+1}; x)/\underline{m}^n)$ , cf. [8], Lemma 3, VIII, §8. If  $x$  is super-regular,  $H_{R/xR}^1(n) = H_R^0(n)$ . If  $x$  just has the property that  $(\underline{m}^{n+1}; x) = \underline{m}^n$  for all large  $n$ , we still get that  $P_{R/xR}^1(n) = P_R^0(n)$ . This prompts the definition of superficial element, cf. [8], VIII, §8. An element  $x$  in a local ring  $(R, \underline{m})$  is superficial if there is a positive integer  $c$  such that  $(\underline{m}^{n+1}; x) \cap \underline{m}^c = \underline{m}^n$ , for all  $n \geq c$ . If  $x$  is a superficial element and a nonzero divisor then  $(\underline{m}^{n+1}; x) = \underline{m}^n$  for all large  $n$ . A superficial element is the preimage in  $R$  of an element of degree 1 in  $grR$  which does not lie in any prime belonging to 0 in  $grR$  except possibly  $gr\underline{m}$ . Superficial elements exist if  $R/\underline{m}$  is infinite. If, in addition,  $\underline{m}$  does not belong to 0 in  $R$ , there exists a superficial element which is also a nonzero divisor.

**THEOREM 3.2.** *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring with  $d \geq 2$  and multiplicity  $e$ . The Hilbert polynomial  $P_R^0(n)$  for  $R$  is*

$$P_R^0(n) = \binom{n + d - 2}{n - 1} e + \binom{n + d - 2}{n}$$

*if and only if  $R$  has embedding dimension  $e + d - 1$  in which case  $grR$  is Cohen-Macaulay.*

*Proof.* It was proved in [4] that  $v(\underline{m}) = e + d - 1$  implies that  $grR$  is Cohen-Macaulay and in [5] that  $v(\underline{m}) = e + d - 1$  implies that  $H_R^0(n) = \binom{n + d - 2}{n - 1} e + \binom{n + d - 2}{n}$ , for all  $n \geq 0$ .

Suppose that  $(R, \underline{m})$  is a local Cohen-Macaulay ring with Hilbert polynomial  $P_R^0(n) = \binom{n + d - 2}{n - 1} e + \binom{n + d - 2}{n}$ . We will show

<sup>1</sup> Added in proof. The author has recently verified that this is the case.



that  $v(\underline{m}) = e + d - 1$  by induction on  $d$ . Let  $d = 2$ . We may assume that  $R/\underline{m}$  is infinite and take a superficial element  $x$  with  $x$  a nonzero divisor. Then, for  $n \geq n_0$ , some integer  $n_0$ ,  $H^1_{R/xR}(n) = H^0_R(n) = ne + 1$ .  $R/xR$  is a 1-dimensional local Cohen-Macaulay ring so, for all  $n \geq 1$ ,  $H^1_{R/xR}(n) = 1 + e - j_1 + e - j_2 + \cdots + e - j_n$  with nonnegative integers  $j_i$  having the property that if  $j_i = 0$ , then  $j_k = 0$  for  $k \geq i$ . Since for  $n \geq n_0$ ,  $H^1_{R/xR}(n) = 1 + ne - \sum_{k=1}^n j_k = 1 + ne$ , it must be true that  $\sum_{k=1}^n j_k = 0$ . Thus  $v(\underline{m}/xR) = e$  and  $v(\underline{m}) = e + 1$ .

Suppose  $d > 2$ . Again, assuming  $R/\underline{m}$  infinite, if necessary, we take a superficial element  $x$  which is also a nonzero divisor. For large  $n$ ,  $H^1_{R/xR}(n) = H^0_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-2}{n}$ . Since  $H^0_{R/xR}(n) = H^1_{R/xR}(n) - H^1_{R/xR}(n-1)$ , we have for large  $n$ ,

$$\begin{aligned} H^0_{R/xR}(n) &= \binom{n+d-2}{n-1}e + \binom{n+d-2}{n} \\ &\quad - \binom{n-1+d-2}{n-2}e - \binom{n-1+d-2}{n-1} \\ &= \binom{n+(d-1)-2}{n-1}e + \binom{n+(d-1)-2}{n}. \end{aligned}$$

Thus, by induction,  $v(\underline{m}/xR) = e + (d-1) - 1$ . Therefore,  $v(\underline{m}) = e + d - 1$ .

REMARK. Clearly, the hypothesis  $d \geq 2$  is necessary in (3.2) as every 1-dimensional local Cohen-Macaulay ring  $(R, \underline{m})$  has  $P^0_R(n) = e$ .

THEOREM 3.3. *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring with  $d \geq 2$ , and multiplicity  $e$ . The Hilbert polynomial  $P^0_R(n)$  for  $R$  is*

$$P^0_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-3}{n}$$

*if and only if  $R$  has embedding dimension  $e + d - 2$  and  $grR$  is Cohen-Macaulay.*

We need two lemmas. Lemma 3.4 was given in [6]. We give a proof below using the results of §1.

LEMMA 3.4. *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring of multiplicity  $e$  and embedding dimension  $e + d - 2$ . Let  $\underline{x} = x_1, \dots, x_d$  be a minimal reduction for  $\underline{m}$ . Then  $\underline{m}^3 \subset (\underline{x})$  and  $grR$  is Cohen-Macaulay if and only if  $\underline{m}^3 = \underline{x}\underline{m}^2$ , for some minimal*



reduction  $\underline{x}$  of  $\underline{m}$ .

*Proof.* Pass to  $(\tilde{R}, \tilde{\underline{m}}) = (R/\underline{x}R, \underline{m}/\underline{x}R)$ .  $\lambda(\tilde{R}) = e$  and  $v(\tilde{\underline{m}}) = e - 2$ . Counting lengths, we see that  $\tilde{\underline{m}}^2 \neq 0$  and  $\tilde{\underline{m}}^3 = 0$ . Thus  $\tilde{\underline{m}}^3 \subset (\underline{x})$ . Suppose that  $\underline{m}^3 = \underline{x}\underline{m}^2$ . Then  $\underline{m}^{i+1} \cap (\underline{x}) = \underline{x}\underline{m}^i$  for  $i \geq 2$ . But, by the analytic independence of  $\underline{x}$ , cf. [3], we also have  $\underline{m}^2 \cap (\underline{x}) = \underline{x}\underline{m}$ . Thus  $grR$  is Cohen-Macaulay by (1.5). If, on the other hand,  $\underline{m}^3 \not\subset \underline{x}\underline{m}^2$ , then by (1.1) there is some  $j$ ,  $1 \leq j \leq d$ , such that  $(\bar{x}_1, \dots, \bar{x}_{j-1}; \bar{x}_j) \not\subseteq (\bar{x}_1, \dots, \bar{x}_{j-1}) + (gr\underline{m})^2$  so that  $grR$  is not Cohen-Macaulay.

**LEMMA 3.5.** *Let  $(R, \underline{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring with  $d \geq 2$ . Let  $\underline{x} = x_1, \dots, x_d$  be a minimal reduction for  $\underline{m}$ . Suppose that  $(\underline{m}/x_1R)^3 \cap (x_2, \dots, x_d)R/x_1R = (x_2, \dots, x_d)(\underline{m}/x_1R)^2$  and that  $(\underline{m}/x_2R)^3 \cap (x_1, x_3, \dots, x_d)R/x_2R = (x_1, x_3, \dots, x_d)(\underline{m}/x_2R)^2$ . Then*

$$\underline{m}^3 \cap (x_1, \dots, x_d)R = (x_1, \dots, x_d)\underline{m}^2.$$

*Proof.* Let  $w = a_1x_1 + \dots + a_dx_d \in \underline{m}^3 \cap (x_1, \dots, x_d)$ . Passing to  $R/x_1R$  and  $R/x_2R$  in turn, we get that  $a_1, a_2, \dots, a_d \in (\underline{m}^2, \underline{x})$ . Thus  $w = f_2(x_1, \dots, x_d) + w'$ , with  $w' \in (x_1, \dots, x_d)\underline{m}^2$  and  $f_2(x_1, \dots, x_d)$  a form of degree 2 in  $x_1, \dots, x_d$ .  $f_2(x_1, \dots, x_d) \in \underline{m}^3$ , by the analytic independence of  $\underline{x}$ ,  $f_2(x_1, \dots, x_d) \in (\underline{x})^2\underline{m}$  and  $w \in \underline{x}\underline{m}^2$ , as desired.

*Proof of 3.3.* The proof is by induction on  $d$ . We may assume that  $R/\underline{m}$  is infinite and take a minimal reduction  $\underline{x} = x_1, \dots, x_d$  of  $\underline{m}$  having the property that each  $x_i$  is a superficial element. Let  $d = 2$ . For large  $n$  and for  $i = 1, 2$ ,  $H^1_{R/x_iR}(n) = H^0_R(n) = ne$ . Since  $R/x_iR$  is a 1-dimensional local Cohen-Macaulay ring, for every  $n \geq 1$ ,  $H^1_{R/x_iR}(n) = 1 + e - j_1 + \dots + e - j_n$ , with nonnegative integers  $j_k$  having the property that if  $j_l = 0$ , then  $j_k = 0$  for  $k \geq l$ . For large  $n$ ,  $H^1_{R/x_iR}(n) = 1 + ne - \sum_{k=1}^n j_k = ne$ . Thus,  $\sum_{k=1}^n j_k = 1$  so  $j_1 = 1$  and  $j_k = 0$  for  $k > 1$ . It follows that  $v(\underline{m}/x_iR) = e - 1$  and  $v((\underline{m}/x_iR)^l) = e$  for  $l > 1$ . Therefore,  $v(\underline{m}) = e$  and  $(\underline{m}/x_1R)^3 = x_2(\underline{m}/x_1R)^2$  and  $(\underline{m}/x_2R)^3 = x_1(\underline{m}/x_2R)^2$ . By (3.5), we have  $\underline{m}^3 \cap (x_1, x_2) = (x_1, x_2)\underline{m}^2$  and by (3.4),  $grR$  is Cohen-Macaulay.

Assume that  $d > 2$ . For each  $x_i$ ,  $i = 1, \dots, d$ , and for large  $n$ , we have  $H^1_{R/x_iR}(n) = H^0_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-2}{n}$ . Since,  $H^0_{R/x_iR}(n) = H^1_{R/x_iR}(n) - H^1_{R/x_1R}(n-1)$ , it follows that

$$H^0_{R/x_iR}(n) = \binom{n+(d-1)-2}{n-1}e + \binom{n+(d-1)-2}{n},$$

for large  $n$ . We may apply induction to  $R/x_iR$  to get  $v(\underline{m}/x_iR) =$



$e + (d - 1) - 2$  and  $v(\underline{m}) = e + d - 2$ . Also, by induction,  $(\underline{m}/x_i R)^3 = (x_1, \dots, x_i, \dots, x_d)(\underline{m}/x_i R)^2$ , so, by (3.5) and (3.4),  $\underline{m}^3 = (x_1, \dots, x_d)\underline{m}^2$  and  $grR$  is Cohen-Macaulay.

REMARK. (3.3) answers a question D. Mumford asked the author.

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