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The identities valid in the ring of real quaternions are defined in more general classes of rings with involution. With one exception, these classes of rings satisfy the standard polynomial of degree 4 and form a chain under inclusion. There are examples which show that these inclusions are proper. An example of an exterior algebra shows that a ring with involution whose symmetric elements commute does not necessarily satisfy the standard polynomial of degree 4.

We assume that a ring has a unit although some of the proofs which follow do not require the existence of a unit in general.

Throughout the paper, R will be a ring equipped with an involution $*$, i.e., a map $R \to R$ such that for all $x, y \in R$, $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $x^{**} = x$.

The sets S and K of *symmetric* and *skew-symmetric* elements of R consist respectively of elements x of R such that $x^* = x$ and $x^* = -x$.

The *trace* and *norm* of an element x in R are respectively $T(x) = x + x^*$ and $N(x) = xx^*$.

As usual, $[x, y] = xy - yx$ denotes the commutator, i.e., the standard polynomial of degree 2 of $x, y \in R$ and the symbol Z denotes the center of R.

We shall require that a ring R be subject to certain identities, all of which are valid in the case of real quaternions. In other words, we are extending the properties of real quaternions to more general classes of rings.

R will be called a *scalar product ring* if for all $x, y \in R$,

$$(1) \qquad T(xy) = T(yx) .$$

This definition follows from Dyson [1].

R is called a *normal ring* if for all $x \in R$, $xx^* = x^*x$.

R is called a *central trace ring* if for all $x \in R$, $T(x) \in Z$.

R is called a *central norm ring* if for all $x \in R$, $N(x) \in Z$.

R is called a *central symmetric ring* if the symmetric elements of R are central i.e., $S \subset Z$.

R is called a *commuting symmetric ring* if the symmetric elements of R commute.

Except for commuting symmetric rings, all the above classes of rings satisfy S_4, the standard polynomial of degree 4. A commuting symmetric ring does not necessarily satisfy S_4 even though its symmetric elements satisfy a polynomial of degree 2.

Clearly, a central symmetric ring is a central trace ring; a 2-torsion free central trace ring is a central symmetric ring.

We shall first study scalar product rings.

From (1) we see that a scalar product ring satisfies

$$(2)\qquad\qquad [x, y]^* = -[x, y]$$

or equivalently,

$$(3)\qquad\qquad [x^*, y^*] = [x, y] \ .$$

A scalar product ring satisfies

$$(4)\qquad\qquad T([x, y]z) = T(x[y, z])$$

because T is additive and $T(y \cdot xz) = T(xz \cdot y)$. In particular, (4) implies $T([x, y]y) = 0$ and hence

$$(5)\qquad\qquad [x, y]y = y^*[x, y] \ .$$

THEOREM 1. *A sacalar product ring R stisfies the standard polynomial of degree 4.*

Proof. Let a, b, c, d denote any four elements in R. The standard polynomial of degree 4 can be written in terms of commutators as follows:

$$(6)\qquad \begin{aligned} S_4[a, b, c, d] &= [a, b][c, d] + [b, c][a, d] + [c, a][b, d] \\ &\quad + [c, d][a, b] + [a, d][b, c] + [b, d][c, a] \ . \end{aligned}$$

By identities (2), (4) and the Jacobi identity respectively,

$$\begin{aligned} S_4[a, b, c, d] &= T([a, b][c, d]) + T([b, c][a, d]) + T([c, a][b, d]) \\ &= T([[a, b], c]d + [[b, c], a]d + [[c, a], b]d) \\ &= 0 \ . \end{aligned}$$

As shown in [5], a normal ring is a scalar product ring and hence satisfies S_4. On the other hand, a 2-torsion free scalar product ring is a normal ring. In [5], there is an example of a scalar product ring which is neither 2-torsion free nor normal.

THEOREM 2. *A central trace ring is a normal ring. Conversely, a normal ring with $2 = 0$ is a central trace ring.*

Proof. If R is a central trace ring then for all $x \in R$, $xT(x) = T(x)x$ and hence $xx^* = x^*x$.

Conversely, if R is a normal ring with $2 = 0$ then $xx^* + x^*x = 0$ for all $x \in R$. We have for all $x, y \in R$,

$$
\begin{aligned}
0 &= [x, yy^* + y^*y] \\
&= y[x, y^*] + [x, y]y^* + y^*[x, y] + [x, y^*]y \\
&= [x, y^*]y^* + [x, y]y^* + [x, y]y + [x, y^*]y , \quad \text{by (5)} \\
&= [x, y^* + y](y^* + y) \\
&= [x, t]t \quad \text{where} \quad t = y^* + y \\
&= t[x, t] , \quad \text{by (5)} .
\end{aligned}
$$

The last two equalities imply

$$ xt^2 = txt = t^2x $$

that is, $t^2 \in Z$. Moreover,

$$ xt^3 = t^2 \cdot xt = t \cdot txt = t \cdot t^2x = t^3x $$

and hence $t^3 \in Z$.

As $(1 + t)^3 = 1 + t + t^2 + t^3 \in Z$, we have $t \in Z$. Hence, R is a central trace ring.

A normal ring is not a central trace ring in general since we have the following:

EXAMPLE 1. Let F be a field of char $\neq 2$ and R be the F-algebra of triangular matrices of the form:

$$
x =
\begin{bmatrix}
a & b & c & d \\
 & a & 0 & c \\
 & & a & -b \\
 & & & a
\end{bmatrix}
$$

with

$$
x^* =
\begin{bmatrix}
a & b & c & -d \\
 & a & 0 & c \\
 & & a & -b \\
 & & & a
\end{bmatrix} .
$$

It is easy to verify that $*$ is an involution on R and that R is a normal ring. Clearly, R is not a central trace ring.

THEOREM 3. *R is a central norm ring if and only if R is a central trace ring.*

Proof. If R is a central norm ring then for all $x \in R$ we have $(1 + x)(1 + x)^* \in Z$ which implies $x + x^* \in Z$.

Conversely, if R is a central trace ring and hence a scalar product ring, we have for all $x, y \in R$,

$$
\begin{aligned}
0 &= [x, y + y^*]y \\
&= [x, y]y + [x, y^*]y \\
&= y^*[x, y] + [x, y^*]y , \quad \text{by } (5) \\
&= [x, y^*y] .
\end{aligned}
$$

Hence $y^*y \in Z$ for all $y \in R$ and so R is a central norm ring.

THEOREM 4. *A 2-torsion free scalar product ring R with no nonzero central nilpotent elements is a central symmetric ring.*

Proof. For all $a \in S$, $x \in K$, we have by (3) that $[a, x] = [a^*, x^*] = -[a, x]$. Since R is 2-torsion free, $[a, x] = 0$; that is, S and K commute elementwise. Consequently, $ax = xa = -(ax)^* \in K$.

Hence for all $a, b \in S$, $x \in K$, we have $bax = axb = abx$ which yields $[a, b]x = 0$. Similarly, $x[a, b] = 0$. From (2), $[a, b] \in K$ and it follows that $[a, b]^2 = 0$.

Moreover for all $r \in R$, $2r[a, b] = (r + r^* + r - r^*)[a, b] = T(r)[a, b] = [a, b]T(r) = 2[a, b]r$. Therefore $[a, b] \in Z$.

As $[a, b]$ is central and nilpotent, we have by hypothesis that $[a, b] = 0$ for all $a, b \in S$, i.e., S is commutative.

Therefore $2[a, r] = [a, r + r^* + r - r^*] = 0$ for all $a \in S$, $r \in R$ which implies $[a, r] = 0$. That is, R is a central symmetric ring.

Example 1 also shows that a 2-torsion free scalar product ring needs not be a central symmetric ring in general.

A central symmetric ring is of course a commuting symmetric ring. On the other hand, we have the following converse:

THEOREM 5. *A semiprime commuting symmetric ring R is a central symmetric ring.*

Proof. Let $a \in S$ and $x \in R$. Since $[a, x + x^*] = 0$, we have $[a, x] = -[a, x^*] = [a, x]^*$ and hence $[a, x] \in S$. Since $0 = [a, xx^*] = x[a, x^*] + [a, x]x^*$, we have $x[a, x] = -x[a, x^*] = [a, x]x^* = (x[a, x])^*$ and hence $x[a, x] \in S$.

Therefore $a \cdot x[a, x] = x[a, x] \cdot a = xa[a, x]$ which yields $[a, x]^2 = 0$ for all $a \in S$, $x \in R$.

For $a \in S$, suppose there exists $x \in R$ such that $b = [a, x] \neq 0$. Then $b^2 = 0$ and $[b, y]^2 = 0$ for all $y \in R$ from which we get $(by)^3 = b(yb)^2y = b[b, y]^2y = 0$. Hence bR is a right ideal of R in which the

cube of every element is 0. By Lemma 1.1 in [2], R would have a nonzero nilpotent ideal which is impossible for a semiprime ring. Therefore $[a, x] = 0$ for all $a \in S$ and $x \in R$, i.e., R is a central symmetric ring.

The above theorem has also been proved by Herstein in a different context in [3, pp. 59–61].

A commuting symmetric ring does not necessarily satisfy S_4 as the following example shows:

EXAMPLE 2. Let F be a field of characteristic zero and V be a F-vector space of dimension 4. Let $R = \bigwedge V = \bigoplus_{n=0}^{\infty} \bigwedge^n V$ be the exterior algebra over V.

To simplify the notations, we write xy, instead of the usual $x \wedge y$, as the ring multiplication in R.

Since $\bigwedge^n V = 0$ for all $n > 4$, we have

$$R = F \oplus V \oplus \bigwedge^2 V \oplus \bigwedge^3 V \oplus \bigwedge^4 V .$$

The additive homomorphism $*: R \to R$ defined by

$$x^* = x_0 - x_1 - x_2 + x_3 + x_4$$

for

$$x = x_0 + x_1 + x_2 + x_3 + x_4$$

where

$$x_i \in \bigwedge^i V, i = 0, 1, \cdots, 4 ;$$

is an involution on R.

Indeed, if $x = x_0 + x_1 + x_2 + x_3 + x_4, y = y_0 + y_1 + y_2 + y_3 + y_4$ where $x_i, y_i \in \bigwedge^i V, i = 0, 1, \cdots, 4$ then

$$xy = x_0 y_0 + (x_0 y_1 + x_1 y_0) + (x_0 y_2 + x_1 y_1 + x_2 y_0)$$
$$+ (x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0) + (x_0 y_4 + x_1 y_3 + x_2 y_2 + x_3 y_1 + x_4 y_0)$$

and

$$(xy)^* = x_0 y_0 - (x_0 y_1 + x_1 y_0) - (x_0 y_2 + x_1 y_1 + x_2 y_0)$$
$$+ (x_0 y_3 + x_1 y_2 + x_2 y_1 + x_3 y_0) + (x_0 y_4 + x_1 y_3 + x_2 y_2 + x_3 y_1 + x_4 y_0)$$
$$= y_0 x_0 - (y_1 x_0 + y_0 x_1) - (y_2 x_0 - y_1 x_1 + y_0 x_2)$$
$$+ (y_3 x_0 + y_2 x_1 + y_1 x_2 + y_0 x_3) + (y_4 x_0 - y_3 x_1 + y_2 x_2 - y_1 x_3 + y_0 x_4)$$
$$= (y_0 - y_1 - y_2 + y_3 + y_4)(x_0 - x_1 - x_2 + x_3 + x_4)$$
$$= y^* x^*.$$

It can easily be seen that

$$S = F \oplus \bigwedge^3 V \oplus \bigwedge^4 V$$
$$K = V \oplus \bigwedge^2 V$$

and that S is commutative. Hence R is a commuting symmetric ring.

R does not satisfy S_4 because if a, b, c, d is a basis of V over F, then since $[a, b] = a \wedge b - b \wedge a = 2(a \wedge b)$, it follows from (6) that

$$S_4[a, b, c, d] = 24(a \wedge b \wedge c \wedge d) \neq 0 .$$

The following example is a commuting symmetric ring which satisfies S_4 but is not a scalar product ring.

EXAMPLE 3. Let C be the field of complex numbers and R be the C-algebra of 2×2 matrices of the form:

$$\begin{bmatrix} x & y \\ 0 & \bar{x} \end{bmatrix}, \quad x, y \in C .$$

The map * defined by

$$\begin{bmatrix} x & y \\ 0 & \bar{x} \end{bmatrix}^* = \begin{bmatrix} \bar{x} & y \\ 0 & x \end{bmatrix}$$

is an involution on R. It can be verified that the symmetric elements of R commute and that R satisfies S_4. However, R is not a scalar product ring since for

$$a = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we have $T(ab) \neq T(ba)$.

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