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# ON UNIVERSAL EXTENSIONS OF DIFFERENTIAL FIELDS

E. R. KOLCHIN

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# ON UNIVERSAL EXTENSIONS OF DIFFERENTIAL FIELDS

## E. R. KOLCHIN

Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

The main result of this paper is the following:

THEOREM: Let  $\mathscr{U}$  be a universal extension of the differential field  $\mathscr{F}$  of characteristic zero and let  $\mathscr{G}$  be a strongly normal extension of  $\mathscr{F}$  in  $\mathscr{U}$ . Then  $\mathscr{U}$  is a universal extension of  $\mathscr{G}$ .

Introduction. We deal with differential fields, always of characteristic zero, relative to a nonempty finite set of commuting derivation operators. By an extension of a differential field, we always mean a differential field extension. An extension  $\mathcal{F}'$  of a differential field  $\mathcal{F}$  is said to be finitely generated if  $\mathcal{F}'$  has a finite subset  $\Phi$  such that  $\mathcal{F}' = \mathcal{F} \langle \Phi \rangle =$  the smallest extension of  $\mathcal{F}$  in  $\mathcal{F}'$  that contains  $\Phi$ .

Let  $\mathscr{F}$  be a differential field. Recall that an extension  $\mathscr{U}$  of  $\mathscr{F}$  is called *universal* if, for any finitely generated extension  $\mathscr{F}_1$ of  $\mathscr{F}$  in  $\mathscr{U}$  and any finitely generated extension  $\mathscr{G}$  of  $\mathscr{F}_1$  not necessarily in  $\mathscr{U}$ ,  $\mathscr{G}$  can be embedded in  $\mathscr{U}$  over  $\mathscr{F}_1$ , i.e., there exists an extension of  $\mathscr{F}_1$  in  $\mathscr{U}$  that is isomorphic (in the sense of differential fields) to  $\mathscr{G}$  over  $\mathscr{F}_1$ . Such a universal extension of  $\mathscr{F}$ always exists ([2] p. 132, Th. 2). It is not unique, but if  $\mathscr{U}$  and  $\mathscr{V}$  are two universal extensions of  $\mathscr{F}$ , then there exist universal extensions  $\mathscr{U}'$  and  $\mathscr{V}'$  of  $\mathscr{F}$ , lying in  $\mathscr{U}$  and  $\mathscr{V}$ , respectively, such that  $\mathscr{U}'$  is isomorphic to  $\mathscr{V}'$  over  $\mathscr{F}$  ([2] p. 135, Exerc. 7).

Let  $\mathscr{U}$  be a universal extension of the differential field  $\mathscr{F}$  and let  $\mathscr{G}$  be an extension of  $\mathscr{F}$  in  $\mathscr{U}$ . Under favorable conditions,  $\mathscr{U}$  is then a universal extension of  $\mathscr{G}$ , too. For example, this is the case when  $\mathscr{G}$  is finitely generated over  $\mathscr{F}$  ([2] p. 133, Prop. 4), and also when  $\mathscr{G}$  is algebraic over  $\mathscr{F}$  ([2] p. 134, Exerc. 1). The main purpose of the present note is to point out another such favorable condition. We shall show (§1) that when  $\mathscr{G}$  is a strongly normal extension of  $\mathscr{F}$ , in the general sense of Kovacic [4] (i.e., not necessarily finitely generated), then  $\mathscr{U}$  is universal over  $\mathscr{G}$ . This result shows that, in the study of strongly normal extensions, it is not necessary to replace  $\mathscr{U}$  by a larger universal extension of  $\mathscr{F}$  (see Kovacic [4] p. 518).

Every strongly normal extension of  $\mathscr{F}$  in  $\mathscr{U}$  is embeddable over  $\mathscr{F}$  in a constrained closure of  $\mathscr{F}$  in  $\mathscr{U}$  ([3] p. 162, Th. 3 or Blum [1] p. 42 (15)) and hence, in particular, is constrained over  $\mathscr{F}$  ([3] p. 148, Th. 1). It is tempting to conjecture that the above result generalizes to constrained extensions of  $\mathscr{F}$  in  $\mathscr{U}$ . We shall show (§2) by a counterexample that  $\mathscr{U}$  can fail to be universal over a constrained closure of  $\mathscr{F}$  in  $\mathscr{U}$ .

1. Strongly normal extensions. Recall ([2] p. 393), for a finitely generated extension  $\mathcal{G}$  of  $\mathcal{F}$  in a given universal extension  $\mathcal{U}$  of  $\mathcal{F}$ , that  $\mathcal{G}$  is called strongly normal over  $\mathcal{F}$  if every isomorphism  $\sigma$  over  $\mathcal{F}$  of  $\mathcal{G}$  onto an extension of  $\mathcal{F}$  in  $\mathcal{U}$  is strong, i.e., has the property that  $\sigma c = c$  for every constant c in  $\mathcal{G}$  and  $\mathcal{G}\mathcal{K} = \sigma \mathcal{G} \cdot \mathcal{K}$ , where  $\mathcal{K}$  denotes the field of constants of  $\mathcal{U}$ . This definition is apparently a relative one, depending on the universal extension  $\mathcal{U}$  of  $\mathcal{F}$  in which  $\mathcal{G}$  is embedded. It is eary to see, however, that if  $\mathcal{G}$  is strongly normal over  $\mathcal{F}$  relative to one  $\mathcal{U}$ , then  $\mathcal{G}$  is strongly normal over  $\mathcal{F}$  relative to every  $\mathcal{U}$ , so that the notion of strongly normal finitely generated extension is an absolute one. When  $\mathcal{G}$  is not necessarily finitely generated over  $\mathcal{F}$ ,  $\mathcal{G}$  is said, following Kovacic [4] p. 518, to be strongly normal over  $\mathcal{F}$  if  $\mathcal{G}$  is the union of strongly normal finitely generated extensions. Hence, also this more general notion is absolute.

It follows from [2] pp. 402-403, Th. 5, and the definition that if  $\mathscr{G}$  is any strongly normal extension of  $\mathscr{F}$  and  $\mathscr{C}$  is any extension of  $\mathscr{F}$ , both contained in an extension of  $\mathscr{F}$  having the same field of constants as  $\mathscr{F}$ , then  $\mathscr{G}\mathscr{C}$  is a strongly normal extension of  $\mathscr{C}$ , and  $\mathscr{G}$  and  $\mathscr{C}$  are linearly disjoint over  $\mathscr{G} \cap \mathscr{C}$ .

We now prove the main theorem of this paper which was stated in the opening paragraph.

*Proof.* (a) We must show that if  $\mathscr{G}_1$  is a finitely generated extension of  $\mathscr{G}$  in  $\mathscr{U}$  and  $\mathscr{H}$  is any finitely generated extension of  $\mathscr{G}_1$  not necessarily in  $\mathscr{U}$ , then there exists an embedding  $\mathscr{H} \to \mathscr{U}$  over  $\mathscr{G}_1$ . As before, denote the field of constants of  $\mathscr{U}$  by  $\mathscr{K}$ , and put  $\mathscr{C} = \mathscr{F} \cap \mathscr{K}, \ {\mathscr{C}}_1 = \mathscr{G}_1 \cap \mathscr{K}$ . Then  $\mathscr{C} = \mathscr{G} \cap \mathscr{K}$  ([2] p. 393, Prop. 9),  $\mathscr{C}_1$  is a finitely generated field extension of  $\mathscr{C}$  ([2] p. 113, Cor. 1 to Prop. 14),  $\mathscr{U}$  is a universal extension of  $\mathscr{FC}_1$ , and  $\mathscr{CC}_1$  is a strongly normal extension of  $\mathscr{FC}_1$  ([2] p. 396, Th. 2). Thus, we may replace  $(\mathscr{F}, \mathscr{G}, \mathscr{G}_1, \mathscr{H})$  by  $(\mathscr{FC}_1, \mathscr{GC}_1, \mathscr{G}_1, \mathscr{H})$ , i.e., we may suppose that  $\mathscr{F}, \mathscr{G}, \mathscr{G}_1$  have the same field of constants  $\mathscr{C}$ .

(b) That being the case, fix a finite family  $\beta$  of generators of  $\mathscr{G}_1$  over  $\mathscr{G}$ . Then  $\mathscr{U}$  is a universal extension of  $\mathscr{F}\langle\beta\rangle$  and  $\mathscr{G}_1 = \mathscr{GF}\langle\beta\rangle$  is a strongly normal extension of  $\mathscr{F}\langle\beta\rangle$ . Thus, we may replace  $(\mathscr{F}, \mathscr{G}, \mathscr{G}_1, \mathscr{H})$  by  $(\mathscr{F}\langle\beta\rangle, \mathscr{G}_1, \mathscr{G}_1, \mathscr{H})$ , i.e., we may suppose that  $\mathscr{G}_1 = \mathscr{G}$ .

(c) That being the case, let  $\mathscr{D}$  denote the field of constants of  $\mathscr{M}$ . Then  $\mathscr{D}$  is a finitely generated field extension of  $\mathscr{C}$ , so that there exists an isomorphism  $\mathscr{D} \approx \mathscr{D}'$  over  $\mathscr{C}$  with  $\mathscr{D}'$  a field extension of  $\mathscr{C}$  in  $\mathscr{M}$ . Because  $\mathscr{C}$  and  $\mathscr{D}$  are linearly disjoint over  $\mathscr{C}$  ([2] p. 87, Cor. 1 to Th. 1), and likewise  $\mathscr{C}$  and  $\mathscr{D}'$ , this can be extended to an isomorphism  $\mathscr{G} \mathscr{D} \approx \mathscr{G} \mathscr{D}'$  over  $\mathscr{C}$ . This can in turn be extended to an isomorphism  $\mathscr{H} \approx \mathscr{H}'$ , where  $\mathscr{H}'$  is a finitely generated extension of  $\mathscr{G} \mathscr{D}'$  not necessarily in  $\mathscr{U}$ . Now,  $\mathscr{U}$  is a universal extension of  $\mathscr{F} \mathscr{D}'$ ,  $\mathscr{G} \mathscr{D}'$  is a strongly normal extension of  $\mathscr{F} \mathscr{D}'$  in  $\mathscr{U}$ , and  $\mathscr{H}'$  is a finitely generated extension of  $\mathscr{G} \mathscr{D}'$ with field of constants  $\mathscr{D}'$ . An embedding  $\mathscr{H}' \to \mathscr{U}$  over  $\mathscr{G} \mathscr{D}'$ would, when composed with the isomorphism  $\mathscr{H} \approx \mathscr{H}'$  over  $\mathscr{G}$ , yield an embedding  $\mathscr{H} \to \mathscr{U}$  over  $\mathscr{G}$ . Thus, we may replace  $(\mathscr{F}, \mathscr{G}, \mathscr{H})$  by  $(\mathscr{F} \mathscr{D}', \mathscr{G} \mathscr{D}', \mathscr{H}')$ , i.e., we may suppose that the field of constants of  $\mathscr{H}$  is  $\mathscr{C}$ .

(d) That being the case, fix a finite family  $\alpha$  of generators of the extension  $\mathscr{H}$  of  $\mathscr{G}$ , and put  $\mathscr{E} = \mathscr{F}\langle \alpha \rangle$ . Then  $\mathscr{G} \cap \mathscr{E}$  is a finitely generated extension of  $\mathscr{F}$  ([2] p. 112, Prop. 14), so that  $\mathscr{U}$  is universal over  $\mathscr{G} \cap \mathscr{E}$ . Thus, we may replace  $(\mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{E})$  by  $(\mathscr{G} \cap \mathscr{E}, \mathscr{G}, \mathscr{H}, \mathscr{E})$ , i.e., we may suppose that  $\mathscr{G} \cap \mathscr{E} = \mathscr{F}$ . Since  $\mathscr{T}$  is strongly normal over  $\mathscr{F}$ , then the differential field  $\mathscr{H} = \mathscr{G}\mathscr{E}$  is strongly normal over  $\mathscr{E}$  and  $\mathscr{E}$  and  $\mathscr{E}$  are linearly disjoint over  $\mathscr{F}$ .

(e) Because  $\mathcal{U}$  is universal over  $\mathcal{I}$ , there exists an isomorphism  $\mathcal{E} \approx \mathcal{E}_0$  over  $\mathcal{F}$  with  $\mathcal{E}_0$  an extension of  $\mathcal{F}$  in  $\mathcal{U}$ , and this isomorphism can be extended to an isomorphism  $\sigma: \mathcal{H} \approx \mathcal{H}_0$ , where  $\mathcal{H}_0$  is an extension of  $\mathcal{F}$  (and of  $\mathcal{C}_0$ ) not necessarily in  $\mathcal{U}$ . Put Then  $\mathcal{H}_0 = \mathcal{G}_0 \mathcal{C}_0$ , this differential field is a strongly  $\mathcal{G}_{v} = \sigma \mathcal{G}.$ normal extension of  $\mathcal{C}_0$ , and  $\mathcal{G}_0$  and  $\mathcal{C}_0$  are linearly disjoint over Evidently  $\mathcal{U}$  is universal over  $\mathcal{C}_0$  (because  $\mathcal{C}_0$  is finitely F. generated over  $\mathcal{F}$ ), and hence the strongly normal extension  $\mathcal{G}_0 \mathcal{E}_0$ of  $\mathcal{C}_0$  can be embedded in  $\mathcal{U}$  over  $\mathcal{C}_0$ , i.e., there exists an isomorphism  $\sigma_0: \mathcal{G}_0 \mathcal{E}_0 \approx \mathcal{G}_2 \mathcal{E}_0$  over  $\mathcal{E}_0$  with  $\sigma_0 \mathcal{G}_0 = \mathcal{G}_2 \subset \mathcal{U}$ . The field of constants of  $\mathcal{G}_2\mathcal{E}_0$ , like those of  $\mathcal{H}_0=\mathcal{G}_0\mathcal{E}_0$  and  $\mathcal{H}=\mathcal{G}\mathcal{E}$ , is  $\mathcal{C}$ , and hence  $\mathscr{G}_{2}\mathscr{C}_{0}$  and  $\mathscr{K}$  are linearly disjoint cover  $\mathscr{C}$ . Therefore  $\mathscr{G}_{2}\mathscr{E}_{0}$  and  $\mathscr{G}_{2}\mathscr{K}$  are linearly disjoint over  $\mathscr{G}_{2}$ . But by (d),  $\mathscr{E}$  and  $\mathscr{G}$ are linearly disjoint over  $\mathcal{F}$ , so that  $\mathcal{C}_0$  and  $\mathcal{G}_0$  are, too, and hence also  $\mathscr{C}_0$  and  $\mathscr{G}_2$ . Therefore  $\mathscr{C}_0$  and  $\mathscr{G}_2 \mathscr{K}$  are linearly disjoint over  $\mathscr{F}$ . But  $\mathscr{G}$  is strongly normal over  $\mathscr{F}$ , so that  $\mathscr{G} \subset \sigma_0 \sigma \mathscr{G} \cdot \mathscr{K} = \mathscr{G}_2 \mathscr{K}$ . Hence  $\mathscr{C}_0$  and  $\mathscr{G}$  are linearly disjoint over  $\mathscr{F}$ . Therefore,  $id_{\mathscr{E}_0}$  and the isomorphism  $\mathscr{G}_{2} \approx \mathscr{G}$  (restriction of  $(\sigma_{0} \circ \sigma)^{-1}$ ) extend to an isomorphism  $\tau: \mathcal{G}_2\mathcal{E}_0 \approx \mathcal{G}\mathcal{E}_0$ . The composite isomorphism  $\tau \circ \sigma_0 \circ \sigma$  is an embedding of *H* into *U* over *G*.

2. A counterexample for constrained extensions. Recall that an extension  $\mathscr{G}$  of a differential field is said to be constrained ([3] p. 144) if every finite family of elements of  $\mathscr{G}$  is constrained over  $\mathscr{F}$  in the sense of [2] p. 142, that a differential field is said to be constrainedly closed ([3] p. 145) if it has no constrained extension other than itself, and that  $\mathscr{G}$  is said to be a constrained closure of  $\mathscr{F}$  ([3] p. 147) if  $\mathscr{G}$  is constrainedly closed and is embeddable over closed  $\mathscr{F}$  in every constrainedly extension of  $\mathscr{F}$ . A constrained closure of  $\mathscr{F}$  always exists, and it is a constrained extension of  $\mathscr{F}$ .

We are going to exhibit an ordinary differential field  $\mathcal{F}$ , a universal extension  $\mathcal{U}$  of  $\mathcal{F}$ , and an extension  $\mathcal{G}$  of  $\mathcal{F}$  in  $\mathcal{U}$  such that  $\mathcal{G}$  is a constrained closure of  $\mathcal{F}$  and  $\mathcal{U}$  is not universal over  $\mathcal{G}$ .

Let  $\mathscr{C}$  be any denumerable field of characteristic zero and put  $\mathscr{F} = \mathscr{C}(x) = \text{the field of rational fractions over } \mathscr{C}$  in an indeterminate  $x; \mathscr{F}$  has a unique structure of ordinary differential field with field of constants  $\mathscr{C}$  in which the derivative of x is 1. By [3] p. 149, Prop. 4, we may fix a denumerable universal extension  $\mathscr{U}$  of  $\mathscr{F}$ . By [3] p. 146, Cor. 1 to Prop. 3,  $\mathscr{U}$  is constrainedly closed.

The set of solutions in  $\mathcal{U}$  different from 0 and 1 of the differential equation

 $y' = y^{\scriptscriptstyle 3} - y^{\scriptscriptstyle 2}$ 

is denumerable and hence can be arranged in a sequence

$$\eta_0, \eta_1, \eta_2, \cdots$$

By [3] §8, this set is infinite and is an independent set of conjugates over  $\mathscr{F}$ , and  $\mathscr{F}\langle\eta_0, \eta_1, \eta_2, \cdots\rangle$  is constrained over  $\mathscr{F}$  (see [3] p. 144, Prop. 1). Because  $\mathscr{U}$  is constrainedly closed,  $\mathscr{F}\langle\eta_0, \eta_1, \eta_2, \cdots\rangle$  has a constrained closure  $\mathscr{G}$  in  $\mathscr{U}$ . The differential ideal  $[y' - y^3 + y^2]$ of the differential polynomial algebra  $\mathscr{G}\{y\}$  is evidently prime and does not have a generic zero in  $\mathscr{U}$  (because all its zeros in  $\mathscr{U}$  are in  $\mathscr{G}$ ). Therefore,  $\mathscr{U}$  is not universal over  $\mathscr{G}$ . (The same argument shows that  $\mathscr{U}$  is even not universal over  $\mathscr{G}\langle\eta_0, \eta_1, \eta_2, \cdots\rangle$ .) We are going to show that  $\mathscr{G}$  is a constrained closure of  $\mathscr{F}$ .

By [3] p. 144, Prop. 2(a),  $\mathscr{C}$  is constrained over  $\mathscr{F}$ . Let  $\mathscr{H}$  be any denumerable constrained closure of  $\mathscr{F}$  (e.g., any constrained closure of  $\mathscr{F}$  in  $\mathscr{H}$ ). The set of solutions in  $\mathscr{H}$  of the above differential equation can be arranged in a sequence

As before, this set is infinite and is an independent set of conjugates over  $\mathscr{T}$ . Therefore, there exists an isomorphism

$$\varphi: \mathscr{F}\langle \eta_0, \eta_1, \eta_2, \cdots \rangle \approx \mathscr{F}\langle \zeta_0, \zeta_1, \zeta_2, \cdots \rangle.$$

Now,  $\mathscr{F}\langle\zeta_0, \zeta_1, \zeta_2, \cdots\rangle$  is normal over  $\mathscr{F}$  in  $\mathscr{H}$  (see [3] §6 p. 153). Hence, by [3] p. 159, Cor. 1 to Th. 2,  $\mathscr{H}$  is a constrained closure of  $\mathscr{F}\langle\zeta_0, \zeta_1, \zeta_2, \cdots\rangle$ . Therefore, by [3] p. 158, Th. 2(b),  $\varphi$  can be extended to an isomorphism  $\mathscr{G} \approx \mathscr{H}$ , so that  $\mathscr{G}$  is a constrained closure of  $\mathscr{F}$ .

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