

Pacific Journal of Mathematics

GOOD CHAINS WITH BAD CONTRACTIONS

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Let $R \subset T$ be commutative rings with T integral over R . In the study of chains of prime ideals, it is often of interest to know about primes $q \subset q'$ of T such that $\text{height}(q'/q) < \text{height}(q' \cap R/q \cap R)$. In this paper we will consider a chain of primes $q_1 \subset q_2 \subset \cdots \subset q_m$ in T which is well behaved in that $\text{height}(q_m/q_1) = \sum_{i=2}^m \text{height}(q_i/q_{i-1})$, but which suffers the pathology that $\text{height}(q_i \cap R/q_{i-1} \cap R) > \text{height}(q_i/q_{i-1})$ for each $i=2, \dots, m$. Our goal is to find a bound on how large m can be.

Our main result is that if T is generated as an R -module by n elements, then there is a bound b_n such that $m \leq b_n$; moreover $b_2=2$ and in general $b_n \leq b_{n-1}^{n-2} + b_{n-1}^{n-3} + \cdots + b_{n-1} + 2$. Let us quickly add that we do not claim that this formula gives the best bound possible. (We rather suspect not.) If $c=b_{n-1}+2$, we also have, as part of our main result, that $m \leq \text{height}(q_c/q_1) + b_{n-1}$. (If $m > b_{n-1}$, so that q_c exists.) Finally, if we have the added assumption that $\text{height}(q_i/q_{i-1}) \leq r$ for $i=2, \dots, m$, then $m \leq 2(r+1)^{n-2}$.

The bulk of our effort is needed to discuss the case that $T=R[u]$ is a simple integral extension of R . This is done in § 3. That section also introduces a new "going down" technique of some interest. Section 2 treats a highly special situation in which we obtain a much sharper bound. This case has some interest in its own right and also starts an induction needed in § 3. The fourth section gives the main result mentioned above. Lastly, in § 5, we present some examples. These illustrate the point that there is no bound in general, even in the case of Noetherian domains, on m which is independent of the size of the integral extension $R \subset T$. Specifically, we show that $b_n \rightarrow \infty$ as $n \rightarrow \infty$. Thus our bounds, while presumably not sharp, have the proper form.

DEFINITION. The chain of primes $P_1 \subset P_2 \subset \cdots \subset P_m$ is *taut* if $\text{height}(P_m/P_1) = \sum_{i=2}^m \text{height}(P_i/P_{i-1})$.

NOTATION. The following notation will be standard throughout except when specifically indicated otherwise. $R \subset T$ will be an integral extension of domains, $q_1 \subset \cdots \subset q_m$ will be a taut chain of primes in T lying over $p_1 \subset \cdots \subset p_m$ in R . $\text{Height}(p_m/p_1)$ will be finite and $\text{height}(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$, $i = 2, \dots, m$. Finally, x will be an indeterminate.

2. **Split simple extensions.** In this section, as well as the next, we will assume, in addition to the standard assumptions mentioned in the introduction, that T is a simple integral extension of R . In order to be more specific, we make a definition.

DEFINITION. Let the domain $T = R[u]$ be a simple integral extension of R with u a root of a monic polynomial $f(x) \in R[x]$. We will say that T is a *simple integral extension of R via $f(x)$* . Throughout §§ 2 and 3, without further mention, we will assume that $T = R[u]$ is a simple integral extension of R via $f(x)$ with $f(x)$ having degree n and $f(u) = 0$. Furthermore, in the present section we add one more assumption, namely that $f(x)$ is split.

DEFINITION. The polynomial $f(x) \in R[x]$ is said to be *split* if $R[u] = R[u']$ for any two roots u and u' of $f(x)$.

Notice that if $f(x) = x^2 + ax + b = (x - u)(x - u') \in R[x]$, then $-u - u' = a \in R$ so that $R[u] = R[u']$. Thus if $n = 2$, $f(x)$ is split. We will show in this section that when $f(x)$ is split, m is bounded by $\deg f(x)$. Our first lemma is well known. We state it explicitly because it is frequently used in what follows.

LEMMA 2.1. (a) *Let p be prime in a ring A . Let $g(x)$ be a monic polynomial in $A[x]$ with $\deg g(x) = d$. Then there are at most d primes of $A[x]$ which lie over p and contain $g(x)$.*

(b) *Let $T = R[u]$ be a simple integral extension of R via $f(x)$ with $\deg f(x) = n$. Let p be prime in R . Then at most n primes in T lie over p .*

Proof. (a) follows from standard facts such as [3, §§ 1-5] and the fact that taken modulo p , $g(x)$ has at most d irreducible factors. (b) follows from (a) by considering preimages under the map $R[x] \rightarrow R[u] = T$.

THEOREM 2.2. *Let $f(x)$ be split. Let q be prime in T with $p = q \cap R$. In $R[x]$, let P be prime with $P \cap R = p$ and suppose that $f(x) \in P$. Then for some root u of $f(x)$, q is the image of P under the homomorphism $R[x] \rightarrow R[u] = T$.*

Proof. As is well known, there is a $g(x) \in P$ such that $P = \{h(x) \in R[x] / sh(x) \in (p, g(x))R[x] \text{ for some } s \in R - p\}$. Since $R[x] \subset T[x]$ is integral and $qT[x] \cap R[x] = pR[x]$, by going up we can find a prime Q of $T[x]$ with $Q \cap T = q$ and $Q \cap R[x] = P$. Thus $f(x) \in P \subset Q$ and as $f(x)$ splits in $T[x]$, for some root u of $f(x)$ we have

$x - u \in Q$. Now $g(x) \in P \subset Q$ and as $x \equiv u \pmod{Q}$, $g(u) \in Q \cap T = q$. Thus the preimage of q under the map $R[x] \rightarrow R[u] = T$ contains $g(x)$, and so is easily seen to be P .

COROLLARY 2.3. *Let $f(x)$ be split. Let p be prime in R .*

(a) *If P_1 and P_2 are prime in $R[x]$ with $P_1 \cap R[x] = p = P_2 \cap R[x]$ and $f(x) \in P_1 \cap P_2$ then $R[x]/P_1 \approx R[x]/P_2$, this isomorphism fixing R/p .*

(b) *Let q_1 and q_2 be primes in T both lying over p . Then $T/q_1 \approx T/q_2$, this isomorphism fixing R/p .*

Proof. (a) Let q be a prime of T lying over p . By Theorem 2.2, for roots u_1 and u_2 of $f(x)$, q is the image of P_i under $R[x] \rightarrow R[u_i] = T$, $i = 1, 2$. Thus $R[x]/P_1 \approx R[u_1]/q = R[u_2]/q \approx R[x]/P_2$.

(b) If P is prime in $R[x]$ with $P \cap R = p$ and $f(x) \in P$, and if q is any prime in T lying over p , then the proof of (a) shows that $T/q \approx R[x]/P$. Thus $T/q_1 \approx R[x]/P \approx T/q_2$.

THEOREM 2.4. *Let $f(x)$ be split. Then $m \leq \deg f(x)$.*

Proof. We first claim that there are distinct primes Q_1, \dots, Q_m lying over p_m satisfying $q_1 \subset Q_j$ and $\text{height}(Q_j/q_1) \geq \text{height}(q_m/q_1)$, $j = 1, \dots, m$. To do this, we induct on m . For $m = 2$, by going up there is a prime q'_2 of T with $q_1 \subset q'_2$, $q'_2 \cap R = p_2$ and $\text{height}(q'_2/q_1) = \text{height}(p_2/p_1) > \text{height}(q_2/q_1)$. Let $Q_1 = q_2$ and $Q_2 = q'_2$.

For $m > 2$ take q'_2 as above. The isomorphism in Corollary 2.3 between T/q_2 and T/q'_2 carries $q_2 \subset \dots \subset q_m$ isomorphically to a chain $q'_2 \subset \dots \subset q'_m$ which also lies over $p_2 \subset \dots \subset p_m$ (since R/p_2 is fixed). By induction there are distinct primes Q_1, \dots, Q_{m-1} of T lying over p_m with $q'_2 \subset Q_j$ and $\text{height}(Q_j/q'_2) \geq \text{height}(q'_m/q'_2)$, $j = 1, \dots, m-1$. Since $q_2 \subset \dots \subset q_m$ and $q'_2 \subset \dots \subset q'_m$ are "isomorphic", $\text{height}(q'_m/q'_2) = \text{height}(q_m/q_2)$. Recall also $\text{height}(q'_2/q_1) > \text{height}(q_2/q_1)$. By the tautness of $q_1 \subset \dots \subset q_m$ we have for $j = 1, \dots, m-1$, $\text{height}(Q_j/q_1) \geq \text{height}(Q_j/q'_2) + \text{height}(q'_2/q_1) \geq \text{height}(q'_m/q'_2) + \text{height}(q'_2/q_1) > \text{height}(q_m/q_2) + \text{height}(q_2/q_1) = \text{height}(q_m/q_1)$. That is, $\text{height}(Q_j/q_1) > \text{height}(q_m/q_1)$, for $j = 1, \dots, m-1$. Letting $Q_m = q_m$ proves our claim.

Finally, as the number of primes in T contracting to any given prime in R cannot exceed $\deg f(x)$, the existence of Q_1, \dots, Q_m shows that $m \leq \deg f(x)$.

The final result in this section discusses the situation when the bound given by Theorem 2.4 is obtained.

PROPOSITION 2.5. *Let $f(x)$ be split and let $m = \deg f(x)$. Suppose that $p \subseteq p_1 \subset p_m \subseteq p'$ with p, p' primes in R and that $q \cap R =$*

$p, q' \cap R = p'$ with q, q' primes in T . Then $q \subset q'$.

Proof. The proof of Theorem 2.4 shows that there are primes $Q_1, \dots, Q_m = q_m$ lying over p_m , each of which contains q_1 . By going up, find a prime q'_1 of T with $q \subset q'_1$ and $q'_1 \cap R = p_1$. Now q_1 is contained in m primes lying over p_m (namely Q_1, \dots, Q_m) and so by Corollary 2.3 q'_1 is also contained in m primes lying over p_m . However, since $\deg f(x) = m$, Q_1, \dots, Q_m are the only primes lying over p_m and so $q \subset q'_1 \subset Q_1 \cap \dots \cap Q_m$.

Now consider $R[x] \rightarrow R[u] = T$ and let Q^*, Q_1^*, \dots, Q_m^* be the preimages of q', Q_1, \dots, Q_m respectively. Obviously $Q^* \cap R = p'$, $Q_j^* \cap R = p_m$, $j = 1, \dots, m$ and $f(x) \in Q^* \cap Q_1^* \cap \dots \cap Q_m^*$ since $f(u) = 0$. By [4, Lemma 3] (applied to R/p_m) we easily see that there is a prime P of $R[x]$ with $P \cap R = p_m$, and $f(x) \in P \subset Q^*$. However since $\deg f(x) = m$, at most m primes in $R[x]$ can contain $f(x)$ and also contract to P_m . As each of Q_1^*, \dots, Q_m^* do just that, obviously $P = Q_j^*$ for some $j = 1, \dots, m$. Thus $Q_j^* = P \subset Q^*$ from which we see that $Q_j \subset q'$. Thus $q \subset Q_1 \cap \dots \cap Q_m \subset Q_j \subset q'$ and we are done.

3. Arbitrary simple extensions. We now drop the "split" assumption and just assume that T is a simple integral extension of R via $f(x)$ with $\deg f(x) = n$. We will show that there is a number b_n such that $m \leq b_n$. We do not identify the best such bound although we do give an inequality limiting the size of the best such bound. To be explicit, let us use b_n to denote the smallest number such that $m \leq b_n$ for all such m .

We have already seen at the start of §2 that if $n = 2$ then $f(x)$ is split, and so by Theorem 2.4 we have $b_2 = 2$. (This is best possible, [5, Example 2, pp. 203-205].) We will now assume inductively that b_{n-1} exists.

In our next lemma we start a chain at P_2 rather than P_1 , since that will be the situation when we apply the lemma.

LEMMA 3.1. *Let $P_2 \subset \dots \subset P_m$ be a taut chain of primes in $R[x]$ contracting to $p_2 \subset \dots \subset p_m$ in R . Let $P'_2 \neq P_2$ with $P'_2 \cap R = p_2$. Let $f(x)$ be a monic polynomial of degree n with $f(x) \in P_2 \cap P'_2$. Let $s > 0$ be an integer with $m > b_{n-1}(s-1) + 1$. Then for some $i \in \{1, \dots, m-s\}$ there is a taut chain $P'_{i+1} \subset \dots \subset P'_{i+s}$ in $R[x]$ lying over $p_{i+1} \subset \dots \subset p_{i+s}$ with $\text{height}(P'_{i+j}/P'_{i+j-1}) = \text{height}(P_{i+j}/P_{i+j-1})$, $j = 2, \dots, s$ and with $P'_2 \subseteq P'_{i+1}$ and $\text{height}(P'_{i+1}/P'_2) \geq \text{height}(P_{i+1}/P_2)$.*

Proof. Obviously we may work modulo p_2 ; so assume that $p_2 = 0$. Since $f(x) \in P_2 \cap P'_2$, $R[x]/P_2$ and $R[x]/P'_2$ are simple integral extensions of R via $f(x)$. Let $R[x]/P_2 \approx R[u]$ and $R[x]/P'_2 \approx R[u']$

with u and u' distinct roots of $f(x)$ (distinct since $P_2 \neq P'_2$). Taken modulo P_2 , $P_2 \subset \cdots \subset P_m$ becomes a taut chain $0 = q_2 \subset \cdots \subset q_m$ in $R[u]$ lying over $0 = p_2 \subset \cdots \subset p_m$. As $R[u] \subset R[u, u']$ is integral, we lift $0 = q_2 \subset \cdots \subset q_m$ to a taut chain $0 = q_2^* \subset \cdots \subset q_m^*$ in $R[u, u']$, with height $q_m^* = \text{height } q_m$.

Since $f(u') = 0$, $f(x) = (x - u')g(x)$ with $g(x)$ monic in $R[u']$. As $u \neq u'$, we have $g(u) = 0$ so that $R[u, u']$ is a simple integral extension of $R[u']$ via $g(x)$. Since $\deg g(x) = n - 1$, the induction assumption concerning the existence of b_{n-1} applies to $R[u'] \subset R[u, u']$.

Let $b = b_{n-1}$ and consider a subchain of $q_2^* \subset \cdots \subset q_m^*$, namely $q_s^* \subset q_{2+(s-1)}^* \subset q_{2+2(s-1)}^* \subset \cdots \subset q_{2+b(s-1)}^*$, which, being a subchain of a taut chain, is taut. (Note $q_{2+b(s-1)}^*$ exists since $m > b(s-1) + 1$.) Because this taut (sub)-chain contains $b + 1$ primes, by the induction assumption for some $l = 1, \dots, b$ we must have $\text{height}(q_{2+l(s-1)}^* \cap R[u'] / q_{2+(l-1)(s-1)}^* \cap R[u']) = \text{height}(q_{2+l(s-1)}^* / q_{2+(l-1)(s-1)}^*)$. Thus letting $i = 1 + (l-1)(s-1)$ we see that the tautness of $q_{i+1}^* \subset \cdots \subset q_{i+s}^*$ implies that $q_{i+1}^* \cap R[u'] \subset \cdots \subset q_{i+s}^* \cap R[u']$ is taut, and that $\text{height}(q_{i+j}^* \cap R[u'] / q_{i+j-1}^* \cap R[u']) = \text{height}(q_{i+j}^* / q_{i+j-1}^*)$ which in turn equals $\text{height}(q_{i+j} / q_{i+j-1})$, $j = 2, \dots, s$ by the manner in which $q_2^* \subset \cdots \subset q_s^*$ was constructed. Also $\text{height}(q_{i+1}^* \cap R[u']) \geq \text{height } q_{i+1}$ since $\text{height } q_{i+1} = \text{height } q_{i+1}^*$.

Finally, recalling that $R[u'] \approx R[x]/P'_2$, the chain $q_{i+1}^* \cap R[u'] \subset \cdots \subset q_{i+s}^* \cap R[u']$ gives rise to a chain $P'_{i+1} \subset \cdots \subset P'_{i+s}$ in $R[x]$ with $P'_2 \subseteq P'_{i+1}$. That this chain satisfies the lemma follows easily from what we know about $q_{i+1}^* \cap R[u'] \subset \cdots \subset q_{i+s}^* \cap R[u']$.

COROLLARY 3.2. *Let the domain T be a simple integral extension of R via $f(x)$ with $\deg f(x) = n$. Let $q_2 \subset \cdots \subset q_m$ be a taut chain in T lying over $p_2 \subset \cdots \subset p_m$ in R . Let $q'_2 \neq q_2$ be prime in T with $q'_2 \cap R = p_2$. Let $s > 0$ be an integer with $m > b_{n-1}(s-1) + 1$. Then for some $i \in \{1, \dots, m-s\}$, there is a taut chain $q'_{i+1} \subset \cdots \subset q'_{i+s}$ in T lying over $p_{i+1} \subset \cdots \subset p_{i+s}$ with $\text{height}(q'_{i+j} / q'_{i+j-1}) = \text{height}(q_{i+j} / q_{i+j-1})$, $j = 2, \dots, s$, and with $q'_2 \subseteq q'_{i+1}$ and $\text{height}(q'_{i+1} / q'_2) \geq \text{height}(q_{i+1} / q_2)$.*

Proof. Let $P_2 \subset \cdots \subset P_m$ and P'_2 be, respectively, the preimages of $q_2 \subset \cdots \subset q_m$ and q'_2 under $R[x] \rightarrow R[u] = T$. Then, since $f(x) \in P_2 \cap P'_2$, the hypothesis of Lemma 3.1 is satisfied. We complete the proof by letting $q'_{i+1} \subset \cdots \subset q'_{i+s}$ be the images of $P'_{i+1} \subset \cdots \subset P'_{i+s}$ given by Lemma 3.1.

PROPOSITION 3.3. *Let $b = b_{n-1}$. Let $l \geq 0$ be an integer and let $m \geq b^l + b^{l-1} + \cdots + b + 2$. Then for some $r = 1, \dots, m$, p_r has lying over it distinct primes Q_1, \dots, Q_{l+1} in T such that $q_1 \subset Q_1 \cap \cdots \cap Q_{l+1}$*

and $\text{height}(Q_j/q_1) > \text{height}(q_r/q_1)$ for $j = 1, \dots, l + 1$.

Proof. We induct on l . First, since $\text{height}(p_s/p_1) > \text{height}(q_2/q_1)$, by going up there is a prime q'_2 of T with $q_1 \subset q'_2$ and $\text{height}(q'_2/q_1) = \text{height}(p_s/p_1)$. If $l = 0$ then $r = 2$ and $Q_1 = q'_2$ satisfy the proposition.

For $l > 0$, we apply Corollary 3.2 with $s = b^{l-1} + b^{l-2} + \dots + b + 2$. Since $m > b(s - 1) + 1$ we have for some $i \in \{1, \dots, m - s\}$ a taut chain $q'_{i+1} \subset \dots \subset q'_{i+s}$ in T lying over $p_{i+1} \subset \dots \subset p_{i+s}$ with $\text{height}(q'_{i+j}/q'_{i+j-1}) = \text{height}(q_{i+j}/q_{i+j-1})$ which is less than $\text{height}(p_{i+j}/p_{i+j-1})$ for $j = 2, \dots, s$.

We apply the case $l - 1$ of the induction assumption to the chain $q'_{i+1} \subset \dots \subset q'_{i+s}$ (recalling that $s = b^{l-1} + b^{l-2} + \dots + b + 2$), to produce an $r \in \{i + 1, \dots, i + s\}$ and distinct primes Q_1, \dots, Q_l of T lying over p_r , with $q'_{i+1} \subset Q_1 \cap \dots \cap Q_l$ and $\text{height}(Q_j/q'_{i+1}) > \text{height}(q'_r/q'_{i+1})$ for $j = 1, \dots, l$. If we now let $Q_{l+1} = q'_r$, obviously Q_{l+1} is distinct from Q_1, \dots, Q_l and we now have $q'_{i+1} \subset Q_1 \cap \dots \cap Q_{l+1}$ and $\text{height}(Q_j/q'_{i+1}) \geq \text{height}(q'_r/q'_{i+1})$ for $j = 1, \dots, l + 1$.

We have $q_1 \subseteq q'_2 \subseteq q'_{i+1}$ by Corollary 3.2. To complete the proof, we must only show that $\text{height}(Q_j/q_1) > \text{height}(q_r/q_1)$ for $j = 1, \dots, l + 1$. To do this, we collect various facts.

(i) $\text{height}(q'_r/q'_{i+1}) = \text{height}(q_r/q_{i+1})$. This follows from the fact that $\text{height}(q'_{i+j}/q'_{i+j-1}) = \text{height}(q_{i+j}/q_{i+j-1})$ $j = 2, \dots, s$ by Corollary 3.2 and the tautness of $q_{i+1} \subset \dots \subset q_{i+s}$ and $q'_{i+1} \subset \dots \subset q'_{i+s}$.

(ii) $\text{height}(Q_j/q'_{i+1}) \geq \text{height}(q_r/q_{i+1})$. This follows from (i) and the previously noted fact that $\text{height}(Q_j/q'_{i+1}) \geq \text{height}(q'_r/q'_{i+1})$.

(iii) $\text{height}(q'_{i+1}/q'_2) \geq \text{height}(q_{i+1}/q_2)$ by Corollary 3.2.

(iv) $\text{height}(q'_2/q_1) > \text{height}(q_2/q_1)$ by choice of q'_2 .

Finally, from the tautness of $q_1 \subset \dots \subset q_r$ and (ii), (iii), and (iv), we have $\text{height}(q_r/q_1) = \text{height}(q_r/q_{i+1}) + \text{height}(q_{i+1}/q_2) + \text{height}(q_2/q_1) < \text{height}(Q_j/q'_{i+1}) + \text{height}(q'_{i+1}/q'_2) + \text{height}(q'_2/q_1) \leq \text{height}(Q_j/q_1)$ for $j = 1, \dots, l + 1$ to complete the proof.

At this point we can prove that b_n exists and show that $b_n \leq b^{n-1} + b^{n-2} + \dots + b + 1$ with $b = b_{n-1}$. To see this, with the notation of Proposition 3.3, if $m > b^{n-1} + b^{n-2} + \dots + b + 1$ we would have primes q_r, Q_1, \dots, Q_n lying over p_r which are distinct (by the inequality in that proposition). However, as $\deg f(x) = n$, at most n primes can lie over p_r , a contradiction. Thus $m \leq b^{n-1} + \dots + b + 1$.

We wish to introduce a "going down" technique which will let us improve this inequality somewhat, giving $b_n \leq b^{n-2} + b^{n-3} + \dots + b + 2$, $b = b_{n-1}$, and which, in certain circumstances, allows us to give a more substantial improvement on the bound on b_n .

DEFINITION. Let p be a prime in the ring R . Let I be an

ideal in $R[x]$. Define $k(p, I) = n$ if $IR_p[x]$ contains a monic polynomial of degree n but no monic polynomial of lesser degree. (If $IR_p[x]$ contains no monic polynomial let $k(p, I) = \infty$.)

LEMMA 3.4. *Let p be prime in a ring R and let I be an ideal in $R[x]$. Suppose that $k(p, I) = n < \infty$.*

(a) *If $g(x) \in I$ and $\deg g(x) < n$ then $g(x) \in pR[x]$.*

(b) *Let $h(x) \in I$ with $\deg h(x) = n$ and the leading coefficient of $h(x)$ outside of p . Let P be prime in $R[x]$ with $P \cap R = p$. Then $I \subseteq P$ if and only if $h(x) \in P$.*

(c) *The number of primes P in $R[x]$ satisfying $P \cap R = p$ and $I \subseteq P$ does not exceed n .*

Proof. Without loss we may localize at p .

(a) Since $k(p, I) = n < \infty$ and (R, p) is quasi-local, there is in I a monic polynomial $h(x)$ of degree n , and no monic polynomial of lesser degree. If the result is false, then for some $g(x) = a_k x^k + \cdots + a_i x^i + \cdots + a_0 \in I$ with $k < n$ we have $a_i \notin p$ for some i . Assume that $g(x)$ and i have been chosen so as to make i as large as possible. Now $a_k \in p$ since $g(x)$ is not monic. We have $a_k h(x) - x^{n-k} g(x) \in I$. Its degree is clearly less than n and its $(i + n - k)$ th coefficient is not in p . This is a contradiction since $i + n - k > i$.

(b) Since $h(x)$ (in part (a)) is monic, clearly I is generated by $h(x)$ together with those polynomials in I having degree less than n . By part (a), each of these latter polynomials is in $pR[x] \subset P$. Thus $I \subseteq P$ if and only if $h(x) \in P$.

(c) This is immediate from Lemma 2.1 and (b).

PROPOSITION 3.5. *Let $p \subset p'$ be primes in a ring R . Let I be an ideal of $R[x]$, and suppose that $k(p, I) = k(p', I) < \infty$. If P' is prime in $R[x]$ with $P' \cap R = p'$ and $I \subset P'$, then there is a prime P in $R[x]$ with $P \cap R = p$ and $I \subseteq P \subset P'$.*

Proof. We may localize at p' . If $k(p', I) = n$ then I contains a monic polynomial $h(x)$ of degree n . Thus $h(x) \in I \subset P'$. By [4, Lemma 3] (applied to R/p) there is a prime P of $R[x]$ with $P \cap R = p$ and $h(x) \in P \subset P'$. By Lemma 3.4, $I \subseteq P$.

We apply Proposition 3.5 to our special situation of $R \subset R[u] = T$ a simple integral extension of domains, u a root of the monic polynomial $f(x)$.

COROLLARY 3.6. *Let $p \subset p'$ be primes in R . Let $I = \ker(R[x] \rightarrow R[u] = T)$ and suppose that $k(p, I) = k(p', I)$. If q' is prime in T*

with $q' \cap R = p'$ then there is a prime q of T with $q \cap R = p$ and $q \subset q'$.

Proof. Since $f(x) \in I$, $k(p', I) < \infty$. Let P' be the preimage of q' under $R[x] \rightarrow R[u]$. Then $P' \cap R = p'$ and $I \subset P'$. With P as in Proposition 3.5 take q to be the image of P in T .

THEOREM 3.7. $b_n \leq b^{n-2} + b^{n-3} + \cdots + b + 2$ where $b = b_{n-1}$.

Proof. Let $B = b^{n-2} + b^{n-3} + \cdots + b + 2$ and assume that $m > B$. We will derive a contradiction. Applying Proposition 3.3 to the chain $q_1 \subset \cdots \subset q_B$ we see that for some $r \in \{1, \dots, B\}$ there are distinct primes Q_1, \dots, Q_{n-1} of T lying over p_r with $q_1 \subset Q_1 \cap \cdots \cap Q_{n-1}$ and $\text{height}(Q_j/q_1) > \text{height}(q_r/q_1)$ $j = 1, \dots, n-1$. Obviously q_r is distinct from Q_1, \dots, Q_{n-1} and if we let $Q_n = q_r$ then, since $\deg f(x) = n$, Q_1, \dots, Q_n are all of the primes of T lying over p_r and we have $\text{height}(Q_j/q_1) \geq \text{height}(q_r/q_1)$ $j = 1, \dots, n$.

We claim that if p is prime in R with $p_r \subseteq p$, then $k(p, I) = n$ where $I = \ker(R[x] \rightarrow R[u] = T)$. Since $f(x) \in I$, $k(p, I) \leq n$. Also $p_r \subseteq p$ implies $k(p_r, I) \leq k(p, I)$ and so we must only show that $k(p_r, I) \geq n$. That this is true follows from Lemma 3.4 (c) and the existence of Q_1, \dots, Q_n .

We now consider a chain of maximal length between p_r and p_m . Since $k(p, I) = n$ for each prime p in that chain, we can use Corollary 3.6 iteratively to find a prime q of T with $q \cap R = p_r$, $q \subset q_m$ and $\text{height}(q_m/q) = \text{height}(p_m/p_r)$. Since $q_r \subset \cdots \subset q_m$ is taut and $\text{height}(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$ $i = r+1, \dots, m$, obviously $\text{height}(q_m/q) = \text{height}(p_m/p_r) > \text{height}(q_m/q_r)$, (here we use $m > B \geq r$). As Q_1, \dots, Q_n are all of the primes which lie over p_r , we must have $q = Q_j$, some $j = 1, \dots, n$. Thus $\text{height}(q/q_1) = \text{height}(Q_j/q_1) \geq \text{height}(q_r/q_1)$. Thus $\text{height}(q_m/q_1) \geq \text{height}(q_m/q) + \text{height}(q/q_1) > \text{height}(q_m/q_r) + \text{height}(q_r/q_1)$ contradicting the tautness of $q_1 \subset \cdots \subset q_m$. This completes the proof.

We repeat that we doubt that equality holds in Theorem 3.7. Let us note that $b_2 \leq b_3 \leq b_4 \leq \cdots$. To see this, observe that if T is a simple integral extension of R via $f(x)$, then it is also a simple integral extension of R via $xf(x)$. The examples at the end of this paper show that $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

We now consider situations in which we can give other bounds on the size of m .

LEMMA 3.8. Suppose that $m > b_{n-1}$. Let $c = b_{n-1} + 1$. If p is any prime of R containing p_c , then $k(p, I) = n$ where $I = \ker(R[x] \rightarrow$

$R[u] = T$).

Proof. Since $f(x) \in I$, obviously $k(p_c, I) \leq k(p, I) \leq n$. We must show $k(p_c, I) \geq n$. For this we may localize at p_c . If $k(p_c, I) < n$ then I contains a monic polynomial $g(x)$ with $\deg g(x) = d < n$. Clearly T is a simple integral extension of R via $g(x)$. However the existence of the chain $q_1 \subset \cdots \subset q_c$ with $c > b_{n-1} \geq b_d$ contradicts the definition of b_d .

LEMMA 3.9. *Suppose that $m > b_{n-1}$ and let $c = b_{n-1} + 1$. Let p be any prime of R containing p_c and let q be any prime in T lying over p . Then $q_1 \subset q$.*

Proof. Let P_1 and P be the preimages of q_1 and q , respectively, under the map $R[x] \rightarrow R[u] = T$. We claim that $k(p, P_1) = n$. The result follows, since obviously $f(x) \in P_1 \cap P$ and so by Lemma 3.4(b) (with $h(x) = f(x)$ and $I = P_1$) $P_1 \subset P$. Thus $q_1 \subset q$.

To show that $k(p, P_1) = n$, we may work modulo p_1 . That is we go to $R/p_1 \subset T/q_1$ and so assume that $p_1 = 0 = q_1$. Now $P_1 = \ker(R[x] \rightarrow T)$ and Lemma 3.8 gives $k(p, P_1) = n$.

THEOREM 3.10. *Suppose that $m > b_{n-1}$ and let $c = b_{n-1} + 1$. Then $m \leq \text{height}(q_c/q_1) + b_{n-1}$.*

Proof. Consider a chain of maximal length between p_c and p_m . By Lemma 3.8, for each prime p in that chain, $k(p, I) = n$ with $I = \ker(R[x] \rightarrow T)$. By iteration of Corollary 3.6, we can find a prime q of T with $q \cap R = p_c$, $q \subseteq q_m$ and $\text{height}(q_m/q) = \text{height}(p_m/p_c)$. By Lemma 3.9, $q_1 \subset q$. Since $q_1 \subset \cdots \subset q_m$ is taut we have $\sum_{c+1}^m \text{height}(q_i/q_{i-1}) + \text{height}(q_c/q_1) = \text{height}(q_m/q_1) \geq \text{height}(q_m/q) + \text{height}(q/q_1) = \text{height}(p_m/p_c) + \text{height}(q/q_1) \geq \sum_{c+1}^m \text{height}(p_i/p_{i-1}) + \text{height}(q/q_1)$. Thus $\text{height}(q_c/q_1) \geq \sum_{c+1}^m [\text{height}(p_i/p_{i-1}) - \text{height}(q_i/q_{i-1})] + \text{height}(q/q_1)$. By our underlying assumption concerning how $q_1 \subset \cdots \subset q_m$ contracts to $p_1 \subset \cdots \subset p_m$, each term in this last summation is at least one. Thus $\text{height}(q_c/q_1) \geq (m - c) + \text{height}(q/q_1) \geq m - c + 1 = m - b_{n-1}$. Thus $m \leq \text{height}(q_c/q_1) + b_{n-1}$.

COROLLARY 3.11. *Suppose that $m > b_{n-1}$ and that $\text{height}(q_i/q_{i-1}) \leq r$ for $j = 2, \dots, b_{n-1} + 1$. Then $m \leq (r + 1)b_{n-1}$.*

Proof. Immediate from Theorem 3.10 and the tautness of $q_1 \subset \cdots \subset q_c$.

Suppose that we fix $r > 0$ and restrict our attention to chains

$q_1 \subset \cdots \subset q_m$ with $\text{height}(q_i/q_{i-1}) \leq r$, $i = 2, \dots, m$. Let b'_n denote the best possible bound on m for such chains when $\deg f(x) = n$. Then Lemma 3.8 through Corollary 3.11 can be repeated, replacing b_{n-1} with b'_{n-1} , thus showing that $b'_n \leq (r+1)b'_{n-1}$. Since $b'_2 = 2$, by induction we get $b'_n \leq 2(r+1)^{n-2}$.

THEOREM 3.12. *If $\text{height}(q_i/q_{i-1}) \leq r$ for $i = 2, \dots, m$, then $m \leq 2(r+1)^{n-2}$.*

4. Finitely generated modules. We give our main result, assuming only that T is a finitely generated R -module.

THEOREM 4.1. *Let $R \subset T$ be domains with T a finitely generated R -module, generated by n elements. Let $q_1 \subset \cdots \subset q_m$ be a taut chain of primes in T lying over $p_1 \subset \cdots \subset p_m$ with $\text{height}(p_m/p_1)$ finite. Suppose that $\text{height}(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$ $i = 2, \dots, m$. Then m is subject to the following:*

- (i) $m \leq b_n$,
- (ii) if $m > b_{n-1}$, then $m \leq \text{height}(q_c/q_1) + b_{n-1}$ with $c = b_{n-1} + 1$,
- (iii) $m \leq 2(r+1)^{n-2}$ with $r = \max\{\text{height}(q_i/q_{i-1}) \mid i = 2, \dots, m\}$.

Proof. Since T is a finitely generated R -module only finitely many primes of T lie over p_m , and we may choose $u \in q_m$ but in no other prime lying over p_m . Obviously q_m is the only prime of T lying over $q_m \cap R[u]$ and so $\text{height}(q_m \cap R[u]/q_1 \cap R[u]) = \text{height}(q_m/q_1)$ (by going up since $\text{height}(q_m \cap R[u]/q_1 \cap R[u]) \leq \text{height}(p_m/p_1) < \infty$). Clearly we have $(q_1 \cap R[u]) \subset \cdots \subset (q_m \cap R[u])$, a taut chain in $R[u]$ with $\text{height}(q_i \cap R[u]/q_{i-1} \cap R[u]) = \text{height}(q_i/q_{i-1}) < \text{height}(p_i/p_{i-1})$ $i = 2, \dots, m$. A standard determinant argument shows that u satisfies a monic polynomial of degree n over R , and our result follows from the existence of b_n and Theorems 3.10 and 3.12.

COROLLARY 4.2. *Let R be a domain with integral closure R' . Suppose that R' is a finitely generated R -module with n generators. Let the domain T be an integral extension of R . Let $0 = q_1 \subset \cdots \subset q_m$ be a taut chain of primes in T lying over $0 = p_1 \subset \cdots \subset p_m$ in R with $\text{height } p_m$ finite. Suppose that $\text{height}(p_i/p_{i-1}) > \text{height}(q_i/q_{i-1})$ $i = 2, \dots, m$. Then (i) $m \leq b_n$; (ii) if $m > b_{n-1}$, then $m \leq \text{height}(q_c/q_1) + b_{n-1}$; and (iii) $m \leq 2(r+1)^{n-1}$ with $r = \max\{\text{height}(q_i/q_{i-1}) \mid i = 2, \dots, m\}$.*

Proof. If T' is the integral closure of T , we may lift $0 = q_1 \subset \cdots \subset q_m$ to a taut chain $0 = q'_1 \subset \cdots \subset q'_m$ in T' with $\text{height } q'_m = \text{height } q_m$. By going down in $R' \subset T'$, $\text{height } q' \cap R' = \text{height } q'$ and

we see that $0 = (q'_1 \cap R') \subset \cdots \subset (q'_m \cap R')$ is taut in R' and height $(q'_i \cap R' / q'_{i-1} \cap R') = \text{height}(q'_i / q'_{i-1}) = \text{height}(q_i / q_{i-1}) < \text{height}(p_i / p_{i-1})$ $i = 2, \dots, m$. Applying Theorem 4.1 to $0 = (q'_1 \cap R') \subset \cdots \subset (q'_m \cap R')$, we are done.

5. Examples. In this section, we construct a family of examples which demonstrate that $b_n \rightarrow \infty$ as $n \rightarrow \infty$. We also show that if $R \subset T$ is an infinite integral extension, no bound need exist at all. This construction is a generalization of Nagata's Example 2 [5, pp. 203-205] and is very similar to [2]. However, except for the quotation of one key theorem, the presentation of the basic example will be self-contained.

EXAMPLE 5.1. Retaining the previous notation, we show any m can be realized in some finite integral extension $R \subset T$ (which depends on m). Moreover, our example is Noetherian.

Fix an integer $m \geq 2$. Let K be a countable field of characteristic zero and let $y_1, \dots, y_{m-1}, z_0^{(1)}, \dots, z_0^{(m-1)}$ be indeterminates. We iteratively define a sequence of Noetherian domains $K = T_1 \subset \tilde{T}_2 \subset T_2 \subset \tilde{T}_3 \subset T_3 \subset \cdots \subset T_m = T$ as follows: Set $\tilde{T}_{i+1} = T_i[y_i]$ for each $i = 1, \dots, m-1$. Suppose $Z_i \in K[[y_i]]$ is a formal power series, say $Z_i = a_1^{(i)}y_i + a_2^{(i)}y_i^2 + \cdots$. If we set $z_n^{(i)} = (z_0^{(i)} - \sum_{j=1}^n a_j^{(i)}y_i^j) / y_i^n$ for each $n \geq 0$, then

$$(*) \quad z_n^{(i)} = (z_{n+1}^{(i)} + a_{n+1}^{(i)})y_i \in \tilde{T}_{i+1}[z_{n+1}^{(1)}].$$

Thus we may define a direct union of simple transcendental extensions of \tilde{T}_{i+1} , $T_{i+1} = \lim_{n \rightarrow \infty} \tilde{T}_{i+1}[z_n^{(i)}]$, for each $i = 1, \dots, m-1$. Moreover, by [2, Corollary 1.6], we may choose the formal power series Z_i in such a way that T_{i+1} will be Noetherian.

The nature of the construction makes it very easy to determine the primes; primes in the intermediate rings extend to primes in T . Hence we easily see, for each $i = 1, \dots, m$, $q_i = (y_1, \dots, y_{i-1})T$ is prime. Also, by (*), $z_n^{(i)} \in y_i T$ for each i, n . By the Krull Altitude Theorem, $\text{height } q_i \leq i-1$. $(0) = q_1 \subset q_2 \subset \cdots \subset q_m$ is a taut chain and $\text{height}(q_{i+1}/q_i) = 1$ $i = 1, \dots, m-1$. Before leaving this chain, we make one additional observation, also apparent from the construction. The quotient T/q_i is canonically isomorphic to the subring $S_i = K[y_i, \dots, z_n^{(i)}, \dots, y_{m-1}, \dots, z_n^{(m-1)}, \dots]$ for each $i = 1, \dots, m$.

Next we iteratively define a second chain $(0) = Q_1 \subset \tilde{Q}_2 \subset Q_2 \subset \cdots \subset Q_m$. First note that, using (*) again, $z_n^{(i)} = (z_{n+1}^{(i)} + a_{n+1}^{(i)})(y_i - 1) + z_{n+1}^{(i)} + a_{n+1}^{(i)}$. Thus $z_{n+1}^{(i)} \equiv z_n^{(i)} - a_{n+1}^{(i)} \pmod{(y_i - 1)}$. So if we set, for each $i = 1, \dots, m-1$, $\tilde{Q}_{i+1} = Q_i + (y_i - 1)T$ and $Q_{i+1} = \tilde{Q}_{i+1} + z_0^{(i)}T$, we have (using equality to denote canonical isomorphism) $T/\tilde{Q}_{i+1} = S_{i+1}[z_0^{(i)}]$ and $T/Q_{i+1} = S_{i+1}$. So these ideals are prime as required

and another application of the Krull Altitude Theorem guarantees that this chain is taut.

Our next step is to construct R . Again we construct a chain of rings $T = R_1 \supset R_2 \supset \cdots \supset R_m = R$. For each $i = 1, \dots, m-1$, set $R_{i+1} = S_{i+1} + (q_{i+1} \cap Q_{i+1} \cap R_i)$. Since $S_{i+1} \subset S_i \subset R_i$, $R_{i+1} \subset R_i$ as desired. We claim that R_i is an integral extension of R_{i+1} , generated by two elements as an R_{i+1} -module. To verify the claim, consider the canonical R_{i+1} -module homomorphism $\pi_i: R_i \rightarrow (R_i/q_{i+1} \cap R_i) \oplus (R_i/Q_{i+1} \cap R_i) = S_{i+1} \oplus S_{i+1}$. Note $\pi_i(1) = (1, 1)$ and $\pi_i(y_i) = (0, 1)$ together generate $S_{i+1} \oplus S_{i+1} = \text{image}(\pi_i)$ and so $R_i = (1)R_{i+1} + (y_i)R_{i+1} + \text{kernel}(\pi_i)$. However, $\text{kernel}(\pi_i) = q_{i+1} \cap Q_{i+1} \cap R_i \subset R_{i+1}$ and so $R_i = R_{i+1} + y_i R_{i+1}$, proving our claim. Therefore T is generated as an R -module by 2^{m-1} elements. Consequently, by Eakin's Theorem [1, p. 281], R is a Noetherian domain.

It now only remains to show $R \subset T$ exhibits the desired chain behavior. As $\dim \tilde{T}_{i+1} = (\dim T_i) + 1$ and $\dim T_{i+1} = (\dim \tilde{T}_{i+1}) + 1$ for each $i = 1, \dots, m-1$, $\dim T = 2(m-1)$. So, by going up, $\dim R = 2(m-1)$. Thus $(0) = Q_1 \cap R \subset \tilde{Q}_2 \cap R \subset Q_2 \cap R \subset \cdots \subset Q_m \cap R$ is taut; then $Q_1 \cap R \subset Q_2 \cap R \subset \cdots \subset Q_m \cap R$ is likewise taut and $\text{height}(Q_{i+1} \cap R)/(Q_i \cap R) = 2$ for each $i = 1, \dots, m-1$. However, by construction, $Q_i \cap R = q_i \cap R$ and so $\text{height}(q_{i+1} \cap R)/(q_i \cap R) = 2$. As $(0) = q_1 \subset \cdots \subset q_m$ is a taut chain in T and $\text{height}(q_{i+1}/q_i) = 1$, we have the desired chain.

In particular, this example shows $b_{2^{m-1}} \geq m$ and so $b_n \rightarrow \infty$ as $n \rightarrow \infty$.

EXAMPLE 5.2. There is an infinite integral extension $R \subset T$ and an infinite taut chain in T , $(0) = q_1 \subset q_2 \subset \cdots$, such that $(0) = q_1 \cap R \subset q_2 \cap R \subset \cdots$ is taut and $\text{height}(q_{i+1}/q_i) = 1 < 2 = \text{height}(q_{i+1} \cap R/q_i \cap R)$ for each i . Necessarily, R is not Noetherian.

Example 5.2 will be a direct union of domains constructed in the manner of (5.1). We begin as in (5.1) with a sequence of domains $K = T_1 \subset \tilde{T}_2 \subset T_2 \subset \cdots \subset T_m \subset \cdots$, this time choosing an infinite sequence. For each fixed m , we perform the construction in (5.1), superscripting our symbols with (m) when confusion is possible. Thus $T_m = T^{(m)}$ and we have $R^{(m)} \subset T^{(m)}$.

Noting $T^{(m)} \subset T^{(m+1)}$, $q_i^{(m)} \subset q_i^{(m+1)}$ and $Q_i^{(m)} \subset Q_i^{(m+1)}$ for each $i = 1, \dots, m$, and $S_i^{(m)} \subset S_i^{(m+1)}$ for each $i = 1, \dots, m$, we have direct unions $T = \bigcup T^{(m)}$, $q_i = \bigcup q_i^{(m)}$, $Q_i = \bigcup Q_i^{(m)}$, and $S_i = \bigcup S_i^{(m)}$ with $T/q_i = S_i = T/Q_i$. Next, using the fact that $S_m^{(m)} = K = S_{m+1}^{(m+1)}$ and some obvious containments, we have $R^{(m)} = R_m^{(m)} = S_m^{(m)} + (q_m^{(m)} \cap Q_m^{(m)} \cap R_{m-1}^{(m)}) = S_m^{(m)} + (q_m^{(m)} \cap Q_m^{(m)} \cap R_m^{(m)}) \subset S_{m+1}^{(m+1)} + (q_{m+1}^{(m+1)} \cap Q_{m+1}^{(m+1)} \cap R_m^{(m+1)}) =$

$R^{(m+1)}$. Thus, we also have a direct union $R = \bigcup R^{(m)}$. We claim $R \subset T$ is the desired extension.

If $y \in T$, $y \in T^{(m)}$ for some m and so is integral over $R^{(m)}$ and consequently R . Thus we have an integral extension. Since R and T are direct unions, the statement about the q_i 's is valid because it holds in $R^{(m)} \subset T^{(m)}$ for each m .

EXAMPLE 5.3. There is a Noetherian domain R such that, for each m , we may find an integral extension T of R and a taut chain of primes $(0) = q_1 \subset q_2 \subset \cdots \subset q_m$ in T such that $q_1 \cap R \subset \cdots \subset q_m \cap R$ is taut and $\text{height}(q_{i+1}/q_i) < \text{height}(q_{i+1} \cap R/q_i \cap R)$ for each $i = 1, \dots, m - 1$.

This example will not be formally constructed. It is obtained by combining two construction ideas. One constructs a family of local Example (5.1)'s and combines them in the manner of Nagata's Example 1 [5, p. 203] (the Noetherian ring with infinite Krull dimension). This is a useful and straightforward way of obtaining this sort of infinite bad behavior.

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Received July 12, 1978 and in revised form March 15, 1978.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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