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THE FULL C*-ALGEBRA OF THE FREE GROUP ON TWO GENERATORS

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 $C^*(F_2)$ is a primitive C^* -algebra with no nontrivial projection. $C^*(F_2)$ has a separating family of finite-dimensional representations.

1. Introduction. We present a "generic" C^* -algebra in illustration of several peculiar phenomena that may occur in the theory of representations.

Let F_2 denote the free group on two generators. If π is the universal unitary representation of F_2 on a Hilbert space \mathscr{H} , then the *full group* C^* -algebra $C^*(F_2)$ is the C^* -subalgebra of $\mathscr{B}(\mathscr{H})$ generated by the set $\{\pi(g): g \in F_2\}$ (see [4, §13.9]). Alternatively, we can re-define $C^*(F_2)$, in an operator-theoretical setting, as follows:

DEFINITION. Let U, V be two unitary operators on a Hilbert space \mathscr{H} . We say that (U, V) is a universal pair of unitaries iff for each pair of unitary operators (U_1, V_1) on a Hilbert space \mathscr{H} , the assignment

$$\begin{cases} U\longmapsto U_1\\ V\longmapsto V_1 \end{cases}$$

extends to a *-homomorphism from $C^*(U, V)$ onto $C^*(U_1, V_1)$.

DEFINITION. We let $C^*(F_2)$ denote the abstract C^* -algebra which is *-isomorphic with the C^* -algebra generated by a universal pair of unitaries.

Obviously, the universal pairs of unitaries are *unique* up to algebraic *-isomorphic equivalence. To see the existence of a universal pair of unitaries, we may simply let

$$(*) U = \bigoplus U_{\nu}, \quad V = \bigoplus V_{\nu}$$

where (U_{ν}, V_{ν}) runs through all possible pairs of unitary operators on a fixed separable Hilbert space. By some judicious selection, it suffices to let ν run through only a countable index set. [In fact, for a general separable C^* -algebra $\mathfrak{A} \subseteq \mathscr{B}(\mathscr{H})$. There is always a projection P of countable dimension, such that $A \mapsto PAP$ is a *-isomorphism from \mathfrak{A} onto $P\mathfrak{A}P \subseteq \mathscr{B}(P\mathscr{H})$.]

The main result of this paper is concerned with various expres-

sions for the universal pairs of unitaries. On one hand, we can just take the expression (*) above, such that each (U_{ν}, V_{ν}) is a pair of finite-dimensional unitary matrices (Theorem 7). On the other hand, there exist a universal pair of unitary operators that do not have a common nontrivial reducing subspace (Theorem 6). These two apparently opposite constructions induce two important representations of $C^*(F_2)$.

Now, we examine $C^*(F_2)$ in regard to its operator-algebraic structure. First and foremost, $C^*(F_2)$ is a primitive C^* -algebra; i.e., $C^*(F_2)$ has a faithful irreducible representation (Theorem 6). This yields the key information that is indispensable to the study of Prim (F_2) (cf. [7, Proposition 6.1]). Furthermore, by the universal property of $C^*(F_2)$, all C^* -algebras generated by two unitaries (including all C^* -algebras generated by single operators) are *-homomorphic images of $C^*(F_2)$. It may be surprising to see that such a "tremendous" C^* -algebra has no nontrivial projection (Theorem 1), and no nonnormal hyponormal element (Corollary 8); indeed, $C^*(F_2)$ even has a faithful tracial state (Corollary 9).

In short, $C^*(F_2)$, being faithfully irreducibly represented, serves as an example for each of the following unusual conditions:

(i) an irreducible C^* -algebra with (the most) abundant ideals.

(ii) an irreducible C^* -algebra with no nontrivial projection (cf. another example given by Philip Green [5]).

(iii) an irreducible C^* -algebra that admits a separating family of finite-dimensional representations.

Finally, we remark that all results of this paper can be extended to $C^*(F_n)$ (for n > 1) and $C^*(F_\infty)$. Readers are also referred to [2, Lemma 4.4-Theorem 4.5, pp. 1108-1109; 9, Proposition 2.7, p. 250; 10, § 3; 11, Theorem 12] for some other unusual aspects of $C^*(F_2)$.

2. The author is indebted to Robert Powers for helpful communication leading to the following theorem. (The idea of the proof is actually originated by Joel Cohen [3].)

THEOREM 1. $C^*(F_2)$ has no nontrivial projection.

Proof. By a faithful representation, we may write $C^*(F_2) = C^*(U, V)$, where U, V are a universal pair of unitary operators on a Hilbert space \mathcal{H} . Let

 $\mathfrak{A} = \left\{ egin{all norm-continuous functions \varPhi: [0, 1] \longrightarrow \mathscr{B}(\mathscr{H})$} \\ \mathrm{such that $\varPhi(0)$ are scalar operators} \end{array}
ight\} \,.$

Then \mathfrak{A} is a C^* -algebra with no nontrivial projections. In fact, if $\varphi \in \mathfrak{A}$ is a projection, then $\varphi(0)$ is 0 or I and by continuity, the

projections $(\varPhi(t))_{t \in [0,1]}$ must be all 0 or all *I*. Now we claim that $C^*(F_2)$ can be imbedded into \mathfrak{A} as a C^* -subalgebra and consequently, $C^*(F_2)$ has no nontrivial projection either.

To see the claim, we first choose, by the spectral theorem, two hermitian operators $A, B \in \mathscr{B}(\mathscr{H})$ such that $U = e^{iA}, V = e^{iB}$. Next, define two unitary elements $\Phi_{U}, \Phi_{V} \in \mathfrak{A}$ by

$$arPsi_{\scriptscriptstyle U}(t)=e^{itA}$$
, $arPsi_{\scriptscriptstyle V}(t)=e^{itB}$.

Then obviously, the evaluation map $\Phi \mapsto \Phi(1)$ is a *-homomorphism from $C^*(\Phi_U, \Phi_V)$ onto $C^*(F_2)$. On the other hand, by the universal property of $C^*(F_2)$, the assignment $U \mapsto \Phi_U$, $V \mapsto \Phi_V$ determines a *-homomorphism from $C^*(F_2)$ onto $C^*(\Phi_U, \Phi_V)$. Hence, the two *-homomorphisms above, being inverse to each other, must be *-isomorphisms. Therefore, $C^*(F_2)$ can be imbedded into \mathfrak{A} as claimed.

COROLLARY 2. If π is a faithful representation of $C^*(\mathbf{F}_2)$ on a Hilbert space \mathcal{H} , then $\pi(C^*(\mathbf{F}_2))$ contains no nonzero compact operator.

Proof. Any C^* -algebra, containing a nonzero compact operator K, must also contain K^*K and, thus, the finite-rank spectral projections of K^*K . Since $\pi(C^*(F_2)) \simeq C^*(F_2)$ contains no nontrivial projection, we have that $\pi(C^*(F_2))$ contains no nonzero compact operator, either.

We proceed to construct a universal pair of unitary operators that do not have a common nontrivial reducing subspace. The main technique below is a variant of Radjavi-Rosenthal's treatment on nonexistence of common invariant subspace [8, Theorem 7.10, p. 121; Theorem 8.30, p. 162].

LEMMA 3. Let A, B be two infinite matrices standing for operators on a separable Hilbert space \mathscr{H} endowed with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. If A is a diagonal operator with all distinct diagonal entries, and if all first-column entries of B are nonzero, then A, B do not have a common nontrivial reducing subspace.

Proof. By simple evaluation on infinite matrices, we deduce that the commutant of A consists of diagonal operators only, and, diagonal operators commuting with B must be scalar operators. Hence, the projections commuting with both A and B are trivial projections. Therefore, A, B do not have a common nontrivial reducing subspace. In the following two lemmas, we deal with the compact perturbations of unitary operators.

LEMMA 4. Let U be a unitary operator on a separable Hilbert space \mathscr{H} . Then there exists a compact operator K, and a unitary diagonal operator D with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, such that U = D + K and all diagonal entries of D are distinct.

Proof. From [6], every normal operator U can be written as $D_0 + K_0$, where K_0 is a compact operator and D_0 is a diagonal operator with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be the diagonal entries of D_0 . Since $U = D_0 + K_0$ is unitary, we derive that $\lim_{n\to\infty} |\alpha_n|=1$. It is easy to choose a complex sequence $\{\beta_n\}_{n=1}^{\infty}$ such that

$$egin{cases} |eta_n| = 1 \ ext{for all } n \ \beta_i
eq eta_j \ ext{whenever} \ i
eq j \ \lim_{n o \infty} (lpha_n - eta_n) = 0 \ . \end{cases}$$

Denoting by D for the diagonal operator with the diagonal entries $\{\beta_n\}_{n=1}^{\infty}$, we have that $D_0 - D$ is a diagonal compact operator; thus $K = K_0 + D_0 - D$ is a compact operator and U = D + K as desired.

LEMMA 5. Let U be a unitary operator on a separable Hilbert space \mathscr{H} endowed with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Then there exists a compact operator $K \in \mathscr{B}(\mathscr{H})$, such that U - K has the infinite matrix expression with all first-column entries being nonzero and with U - K unitary.

Proof. Let v be a unit vector with all nonzero co-ordinates with respect to the orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and let \mathscr{S} denote the linear span of v and $Ue_1($ = the first-column vector of U). Choose any unitary operator $V \in \mathscr{B}(\mathscr{H})$ such that

$$egin{array}{ll} V(\mathscr{S}) = \mathscr{S} ext{ with } V(Ue_1) = v ext{ ,} \ V|_{\mathscr{S}^\perp} = I|_{\mathscr{S}^\perp} \ . \end{array}$$

Then V - I is an operator of rank ≤ 2 and the first column vector of VU is v; thus

$$VU = U + (V - I)U$$

is a compact perturbation of U as desired.

THEOREM 6. $C^*(\mathbf{F}_2)$ is a primitive C^* -algebra; i.e., $C^*(\mathbf{F}_2)$ has a faithful irreducible representation.

Proof. Since $C^*(F_2)$ is separable, we may write $C^*(F_2) = C^*(U, V)$, where U, V are a universal pair of unitary operators on a separable Hilbert space \mathscr{H} . Applying Lemmas 4-5 to U, V, we have that

$$U = U_0 + \text{compact}, V = V_0 + \text{compact},$$

and with respect to a suitable orthonormal basis, U_0 is a unitary diagonal operator with distinct diagonal entries, and V_0 is a unitary operator with all first-column entries nonzero. From the universal property, the assignment

$$U\longmapsto U_0, V\longmapsto V_0$$

defines a representation

$$\pi: C^*(F_2) \longrightarrow C^*(U_0, V_0) \subseteq \mathscr{B}(\mathscr{H})$$
.

By Lemma 3, U_0 , V_0 do not have a common nontrivial reducing subspace; thus $C^*(U_0, V_0)$ is an irreducible C^* -algebra, and π is an irreducible representation.

It remains to show that π is faithful. Letting $\mathcal{K}(\mathcal{H})$ be the ideal of compact operators and $\eta: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the natural quotient map, we have then

$$\eta(C^*(U_{\scriptscriptstyle 0},\ V_{\scriptscriptstyle 0}))=C^*(\eta(U_{\scriptscriptstyle 0}),\ \eta(V_{\scriptscriptstyle 0}))=C^*(\eta(U),\ \eta(V))\;.$$

But from Corollary 2, η restricted to $C^*(U, V)$ is an *-isomorphism. The composition of the canonical *-homomorphisms

$$C^*(F_2) \xrightarrow{\pi} C^*(U_0, V_0) \xrightarrow{\eta} C^*(\eta(U_0), \eta(V_0)) = C^*(\eta(U), \eta(V))$$

 $\simeq C^*(U, V) = C^*(F_2)$

leads to the identity map on $C^*(F_2)$; therefore π is a *-isomorphism as desired.

For a general C^* -algebra $\mathfrak{A} \subseteq \mathscr{B}(\mathscr{H})$ with separable \mathscr{H} , we can construct a "completely order injection" φ from \mathfrak{A} into $\bigoplus_{n=1}^{\infty} M_n$, the direct sum of full matrix algebras, by letting

$$\varphi(A) = \bigoplus_{n=1}^{\infty} P_n A P_n$$

where $\{P_n\}$ is a sequence of finite-rank projections approaching strongly to *I*. In case $\mathfrak{A} = C^*(F_2)$, we will modify φ to get actually an "algebraic *-isomorphism".

THEOREM 7. $C^*(\mathbf{F}_2)$ has a separating family of finite-dimensional representations.

Proof. Since $C^*(F_2)$ is separable, we may assume that $C^*(F_2) = C^*(U, V)$ where U, V are a universal pair of unitary operators on a separable Hilbert space \mathscr{H} . Let $\{P_n\}_n$ be a sequence of increasing projections in $\mathscr{B}(\mathscr{H})$ approaching strongly to the identity operator I with rank $P_n = n$. Write

$$\begin{split} A_n &= P_n U P_n, \qquad B_n = P_n V P_n , \\ U_n &= \begin{bmatrix} A_n & (P_n - A_n A_n^*)^{1/2} \\ (P_n - A_n^* A_n)^{1/2} & -A_n^* \end{bmatrix}, \\ V_n &= \begin{bmatrix} B_n & (P_n - B_n B_n^*)^{1/2} \\ (P_n - B_n^* B_n)^{1/2} & -B_n^* \end{bmatrix} \end{split}$$

By identifying $P_n \mathcal{H} P_n$ with M_n , we may regard P_n as the identity $n \times n$ matrix, and U_n , V_n as $2n \times 2n$ unitary matrices. From the universal property of $C^*(F_2)$, the assignment

$$U\longmapsto U_n, V\longmapsto V_n$$

defines a representation $\pi_n: C^*(F_2) \to M_{2n}$. Now, we *claim* that $\{\pi_n\}_{n=1}^{\infty}$ is a separating family of representations; in other words, the *-homomorphism

$$\pi \colon C^*(F_2) \longrightarrow igoplus_{n=1}^{\infty} M_{2n}$$
 ,

defined by

$$\pi(A) = \bigoplus_{n=1}^{\infty} \pi_n(A)$$
 ,

is actually a *-isomorphism.

Note that in the strong topology, U_n , U_n^* , V_n , and V_n^* converge to

$$\begin{bmatrix} U & 0 \\ 0 & -U^* \end{bmatrix}, \begin{bmatrix} U^* & 0 \\ 0 & -U \end{bmatrix}, \begin{bmatrix} V & 0 \\ 0 & -V^* \end{bmatrix}, \text{ and } \begin{bmatrix} V^* & 0 \\ 0 & -V \end{bmatrix}$$

respectively. Hence, if $F(\ ,\)$ is a finite linear combination of words in two free variables, then $F(U_n,\ V_n)$ also converges to

$$\begin{bmatrix} F(U, V) & 0 \\ 0 & F(-U^*, -V^*) \end{bmatrix}$$

in the strong topology. Therefore, for any $\varepsilon > 0$, and given ||F(U, V)|| = 1, we have that

$$||F(U_n, V_n)|| \ge 1 - \varepsilon$$

for all sufficiently large n; thus

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$$\|\pi(F(U, V))\| \ge \|\pi_n(F(U, V))\| = \|F(U_n, V_n)\| \ge 1 - \varepsilon$$

Since ε is arbitrary, we conclude that π , restricted to the pre-C^{*}-algebra generated by U, V, is an isometry. By continuity, π is an isometry and, thus, a *-isomorphism as desired.

We say that an operator A is hyponormal iff $A^*A \ge AA^*$.

COROLLARY 8. Every hyponormal operator in $C^*(\mathbf{F}_2)$ is normal.

Proof. From the theorem above, we may imbed $C^*(F_2)$ into $\bigoplus_{n=1}^{\infty} M_{2n}$ as a C^* -subalgebra. Since every hyponormal matrix is normal, we have then for each $A = \bigoplus A_n \in \bigoplus M_{2n}$,

A is hyponormal $\Longrightarrow A_n$ is hyponormal for each n $\Longrightarrow A_n$ is normal for each n $\Longrightarrow A$ is normal

as desired.

COROLLARY 9. $C^*(F_2)$ has a faithful tracial state.

Proof. By Theorem 7, we can imbed $C^*(F_2)$ into $\bigoplus_{n=1}^{\infty} M_{2n}$ as a C^* -subalgebra. Let τ_n be the faithful tracial state of M_{2n} . Then $\tau: \bigoplus M_{2n} \to C$, defined by

$$au(\bigoplus A_n) = \sum \left(au_n(A_n)/2^n \right)$$
 ,

is a faithful tracial state as desired.

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