A UNIFIED THEOREM ON CONTINUOUS SELECTIONS

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A selection theorem is proved which unifies and gene-
ralizes some known results.

1. **Introduction.** The purpose of this note is to prove the follow-
ing theorem, which unifies and generalizes previously known results.

THEOREM 1.1. *Let X be paracompact, Y a Banach space, $Z \subset X$
with $\dim_X Z \leqq 0$, and $\phi: X \to \mathscr{F}(Y)$ l.s.c. with $\phi(x)$ convex for all
$x \in X - Z$. Then ϕ admits a selection.*

Recall that a map $\phi: X \to \mathscr{F}(Y)$, where $\mathscr{F}(Y)$ denotes $\{S \subset Y:
S \neq \varnothing, S$ closed in $Y\}$, is *lower semi-continuous*, or l.s.c., if $\{x \in X:
\phi(x) \cap V \neq \varnothing\}$ is open in X for every open V in Y. A *selection* for
a map $\phi: X \to \mathscr{F}(Y)$ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$
for all $x \in X$. Finally, if $Z \subset X$ then $\dim_X Z \leqq 0$ means that
$\dim E \leqq 0$ for every set $E \subset Z$ which is closed in X (where $\dim E$
denotes the covering dimension of E)[1].

Theorem 1.1 incorporates several known results: The case
$Z = \varnothing$ is [1, Theorem 1], the case $Z = X$ implies [1, Theorem 2],
and the case where Z is open in X and $\phi(x)$ is a singleton for
all $x \in X - Z$ implies [3, Theorem 1.2][2].

The conclusion of Theorem 1.1 can be strengthened to assert
that, if $A \subset X$ is closed, then every selection g for $\phi | A$ extends to
a selection f for ϕ: In fact, if we define $\phi_g: X \to \mathscr{F}(Y)$ by $\phi_g(x) =
\phi(x)$ for $x \notin A$ and $\phi_g(x) = \{g(x)\}$ for $x \in A$, then ϕ_g is l.s.c. by [2,
Example 1.3], so ϕ_g has a selection f by Theorem 1.1, and this f is
a selection for ϕ which extends g.

2. **Proof of Theorem 1.1.** As in the proofs of the special
cases of Theorem 1.1 which were obtained in [1], it will suffice to
show that for each $\varepsilon > 0$ there exists a continuous $f: X \to Y$ such
that $f(x) \in B_\varepsilon(\phi(x))$[3] for all $x \in X$. Once that is done, one can obtain
the required selection for ϕ as the limit of a uniformly Cauchy
sequence of continuous functions $f_n: X \to Y$ such that $f_n(x) \in B_{1/n}(\phi(x))$
for all $x \in X$.

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[1] Observe that, for normal X, $\dim_X Z \leqq 0$ is valid if either $\dim Z \leqq 0$ or $\dim X \leqq 0$.

[2] In the latter two cases, Theorem 1.1 is valid if Y is any complete metric space,
since such a space is always homeomorphic to a closed subset of a Banach space.

[3] $B_\varepsilon(S)$ denotes the open ε-neighborhood of S.

So let $\varepsilon > 0$ be given. For each $y \in Y$, let $U_y = \{x \in X: y \in B_\varepsilon(\phi(x))\}$. Then $\{U_y: y \in Y\}$ is an open cover of X because ϕ is l.s.c., so there exists a locally finite, open cover $\{V_y: y \in Y\}$ of X with $\bar{V}_y \subset U_y$ for all $y \in Y$. For each $x \in X$, let $F_x = \{y \in Y: x \in \bar{V}_y\}$; then F_x is finite, and $F_x \subset B_\varepsilon(\phi(x))$. Let $S = X - Z$, and for each $s \in S$ define

$$G_s = \{x \in X: \operatorname{conv} F_s \subset B_\varepsilon(\phi(x))\} - \bigcup\{\bar{V}_y: y \notin F_s\} \ .$$

Then $s \in G_s$ because $B_\varepsilon(\phi(s))$ is convex, and G_s is open in X because ϕ is l.s.c. and conv F_s is compact (see [3, Lemma 11.3]). For later use, let us also note that $F_x \subset F_s$ for all $x \in G_s$.

Let $G = \bigcup\{G_s: s \in S\}$, and let $E = X - G$. Then E is closed in X and $E \subset Z$, so dim $E \leq 0$. Hence the relatively open cover $\{V_y \cap E: y \in Y\}$ of E has a relatively open, disjoint refinement $\{D_y: y \in Y\}$[4].

Let $W_y = V_y \cap (D_y \cup G)$. The $\{W_y: y \in Y\}$ is a locally finite, open cover of X, and thus has a partition of unity $\{p_y: y \in Y\}$ subordinated to it. Define

$$f(x) = \sum_{y \in Y} (p_y(x))y \ .$$

Clearly f is continuous, so we need only check that $f(x) \in B_\varepsilon(\phi(x))$ for all $x \in X$.

If $x \in E$, the $f(x) = y \in B_\varepsilon(\phi(x))$ for the unique $y \in Y$ such that $x \in D_y$. So suppose that $x \in G$. Then $x \in G_s$ for some $s \in S$, so

$$f(x) \in \operatorname{conv} F_x \subset \operatorname{conv} F_s \subset B_\varepsilon(\phi(x)) \ .$$

That completes the proof.

REMARK. The above proof implies that X need only be assumed normal and countably paracompact if Y is separable, and that X need only be normal if $\bigcup_{x \in X} \phi(x)$ is contained in a compact subset of Y.

REFERENCES

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[4] This follows, for instance, from [1, Proposition 2].

PACIFIC JOURNAL OF MATHEMATICS

Pacific Journal of Mathematics

Vol. 87, No. 1   January, 1980