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# SIMILARITY ORBITS OF APPROXIMATELY FINITE C\*-ALGEBRAS

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Let H denote a Hilbert space, and let B(H) denote the algebra of all bounded linear operators on H. In this note, an intrinsic characterization of those Banach subalgebras of B(H) which are similar to approximately finite  $C^*$ -subalgebras of B(H) is obtained.

This can be viewed as a noncommutative analog of theorems of Mackey ([9], p. 131) and Wermer ([10], Theorem 1). These authors gave conditions on certain families of idempotents  $\{E_{\alpha}\}_{\alpha\in A}$  in B(H)which insured the existence of an invertible T in B(H) such that  $TE_{\alpha}T^{-1}$  is a projection for all  $\alpha$  in A. The main idea of the present paper involves finding conditions on certain families of matrix units  $\{e(i, j)\}$  in B(H) which guarantee the existence of an invertible T in B(H) for which  $\{Te(i, j)T^{-1}\}$  spans a  $C^*$ -algebra. This technique also has interesting applications to the orthogonalization of continuous representations of  $C^*$ -algebras (cf. [11]).

2. Preliminary definitions and lemmas. We begin by recalling the definition of an approximately finite  $C^*$ -algebra. A  $C^*$ -algebra  $\mathscr{C}$  is approximately finite if there is an increasing sequence of finite dimensional  $C^*$ -subalgebras of  $\mathscr{C}$  whose union is norm dense in  $\mathscr{C}$ . These algebras were defined and studied by Ola Bratteli in 1972 ([1]) as a generalization of the UHF algebras of Glimm ([7]), and have become popular objects of study among  $C^*$ -algebra enthusiasts (cf. [2], [3], [4], [5], and [8]).

The definition of approximate finiteness can be extended slightly to the context of Banach algebras as follows:

DEFINITION 2.1. A Banach algebra  $\mathscr{S}$  is approximately finite if there is an increasing sequence  $\{\mathscr{S}_n\}_{n=1}^{\infty}$  of finite dimensional, semisimple subalgebras such that  $\mathscr{S} = (\bigcup_n \mathscr{S}_n)^-$ , where - denotes norm closure.

Note that the most natural definition of approximate finiteness for Banach algebras would not include the hypothesis of semisimplicity on the  $\mathscr{M}_n$ 's. It is included here primarily to simplify the statement of Theorem 3.1 below.

Consider then an approximately finite Banach algebra  $\mathcal{M} = (\bigcup_n \mathcal{M}_n)^-$ . Since each  $\mathcal{M}_n$  is by definition finite dimensional and semisimple, it has a Wedderburn decomposition

$$\mathscr{M}_n = + \{ \mathscr{M}_k^{(n)} \colon k = 1, \cdots, r_n \}$$
,

where  $\mathscr{M}_{k}^{(n)}$  is isomorphic to the full complex matrix algebra  $M_{[n,k]}$  of order [n, k]. (This notation is the same as [1].) One may hence select matrix units  $\{e_{k}^{(n)}(i, j): i, j=1, \dots, [n, k]\}$  for each  $\mathscr{M}_{k}^{(n)}$  for which

$$\mathcal{M}_{k}^{(n)} = \text{linear span of } \{e_{k}^{(n)}(i, j): i, j = 1, \dots, [n, k]\}$$

Now  $\mathscr{N}_n \subseteq \mathscr{N}_{n+1}$ , and the selection of matrix units  $\{e_k^{(n)}(i, j)\}$  can be made to reflect this inclusion. This is accomplished in the following proposition, whose proof, left to the reader, is a straightforward modification of the proof of Proposition 1.7 of [1].

**PROPOSITION 2.2.** Let  $\mathscr{A}_1$  and  $\mathscr{A}_2$  be finite dimensional, semisimple algebras, with Wedderburn decompositions

$$\mathscr{M}_i = + \{ \mathscr{M}_k{}^{(i)}, \, k = 1, \, \cdots, \, n_i \} \,, \ \ i = 1, \, 2 \,, \ \mathscr{M}_k{}^{(i)} \cong M_{[i], \, k]} \,, \ \ i = 1, \, 2 \,.$$

Let  $\{e_i^{(1)}(i, j): i, j = 1, \dots, [1, l]\}, l = 1, \dots, n_1$ , be matrix units for  $\mathcal{A}_1$ . If  $\mathcal{A}_1 \subseteq \mathcal{A}_2$ , then there exists unique nonnegative integers  $n_{ki}, k = 1, \dots, n_2, i = 1, \dots, n_1$ , and matrix units  $\{e_k^{(2)}(i, j): i, j = 1, \dots, [2, k]\}, k = 1, \dots, n_2$  for  $\mathcal{A}_2$  such that

$$\sum_{p=1}^{n_1} n_{kp} [1, p] \le [2, k]$$

and

(2.1) 
$$e_{l}^{(1)}(i, j) = \sum_{q=1}^{n_2} \sum_{m=1}^{n_{ql}} e_{q}^{(2)} \left( \sum_{p=1}^{l-1} n_{qp} [1, p] + (m-1)n_{ql} + i \right),$$
$$\sum_{p=1}^{l-1} n_{qp} [1, p] + (m-1)n_{ql} + j \right).$$

The matrix units for  $\mathscr{M}_n$  are now chosen inductively by applying Proposition 2.2 at the *n*th inclusion, so that for each *n* the matrix units for  $\mathscr{M}_{n+1}$  satisfy (2.1) relative to the matrix units for  $\mathscr{M}_n$ . Such a selection of matrix units will be called an *admissable selection* of matrix units for  $\mathscr{M}$ .

We turn now to the problem of orthogonalization of matrix units in B(H) for a fixed Hilbert space H. Recall that a set of bounded operators  $\{e(i, j): i, j = 1, \dots, n\}$  on H is said to form a system of matrix units on H if

(i)  $\sum_{i=1}^{n} e(i, i) = \text{identity operator on } H$ ,

(ii)  $e(i, j)e(k, l) = \delta_{jk} \cdot e(i, l), i, j, k, l = 1, \dots, n,$ 

where  $\delta_{jk}$  denotes the Kronecker delta.  $\{e(i, j): i, j = 1, \dots, n\}$  is said to form a C<sup>\*</sup>-system of matrix units if in addition to (i) and (ii), one has

(iii)  $e(i, j)^* = e(j, i), i, j = 1, \dots, n.$ 

DEFINITION 2.3. Let  $\{e(i, j): i, j = 1, \dots, n\}$  be a system of matrix units on H. An invertible operator T on H is said to orthogonalize  $\{e(i, j): i, j = 1, \dots, n\}$  if

$$(Te(i, j)T^{-1})^* = Te(j, i)T^{-1}, i, j = 1, \dots, n$$

i.e., if  $\{Te(i, j)T^{-1}: i, j = 1, \dots, n\}$  is a C\*-system of matrix units on H.

LEMMA 2.4. Let  $\{e(i, j): i, j = 1, \dots, n\}$  be a system of matrix units on H. Then there exists an invertible operator T on H which orthogonalizes  $\{e(i, j): i, j = 1, \dots, n\}$ .

Proof. Set

$$T = \left(\sum_{1 \leq i,j \leq n} e(i,j)^* e(i,j) 
ight)^{1/2}.$$

We claim that T is invertible. For x in H,

1

$$egin{aligned} |Tx||^2 &= \sum\limits_{1 \leq i, j \leq n} ||e(i, j)x||^2 \ &\geq \sum\limits_{i=1}^n ||e(i, i)x||^2 \ &\geq n^{-2} \Bigl(\sum\limits_{i=1}^n ||e(i, i)x||\Bigr)^2 \ &\geq n^{-2} \, ||x||^2 \ , \end{aligned}$$

since  $x = \sum_{i} e(i, i)x$ ; T is thus bounded below. By a theorem of T. Crimmins ([6], Theorem 2.2),

(2.2) 
$$\operatorname{range of} T = \operatorname{range of} \left(\sum_{1 \leq i, j \leq n} e(i, j)^* e(i, j)\right)^{1/2} = \sum_{1 \leq i, j \leq n} \operatorname{range of} e(i, j)^*.$$

Since  $\sum_i e(i, i)^* = I$ ,  $H = \sum_i$  range of  $e(i, i)^*$ , and therefore by (2.2), T is surjective. T is hence invertible.

Let k and l be fixed positive integers between 1 and n. Then

$$egin{aligned} T^2 e(k,\,l) &= \sum\limits_{i,j}{(e(i,\,j)^* e(i,\,j))e(k,\,l)} \ &= \sum\limits_{j=1}^n{e(j,\,k)^* e(j,\,l)} \;, \end{aligned}$$

so that if  $f(k, l) = Te(k, l)T^{-1}$ ,

(2.3) 
$$f(k, l) = T^{-1} \left( \sum_{j=1}^{n} e(j, k)^* e(j, l) \right) T^{-1}.$$

Since T is positive, (2.3) yields

$$\begin{split} f(k, l)^* &= (Te(k, l)T^{-1})^* \\ &= T^{-1} \Big( \sum_{j=1}^n e(j, l)^* e(j, k) \Big) T^{-1} \\ &= f(l, k) \; . \end{split}$$

The next lemma is the basic orthogonalization lemma of Mackey (see [9], p. 135).

LEMMA 2.5. Let  $\{E_1, \dots, E_n\}$  be a pairwise independent set of idempotents in B(H) (i.e.,  $E_i^2 = E_i$ ,  $E_i E_j = 0$ ,  $i \neq j$ ) such that  $\sum_{i=1}^n E_i = I$ , and let M > 0 be such that for every set  $\{\varepsilon_1, \dots, \varepsilon_n\}$  on zero's and one's,

$$\left\|\sum_{i=1}^{n} \varepsilon_{i} E_{i}\right\| \leq M$$
.

Then for all x in H,

$$rac{||x||^2}{4M^2} \leq \sum_{i=1}^n ||E_i x||^2 \leq 4M^2 ||x||^2 \;.$$

We now extend Lemma 2.5 to matrix units:

LEMMA 2.6. Let  $\{e(i, j): i, j = 1, \dots, n\}$  be a system of matrix units on H such that each e(i, i) is a projection. Let M > 0 be such that  $||e(i, j)|| \leq M$ ,  $i, j = 1, \dots, n$ . Then for all x in H,

$$rac{n}{M^2} ||x||^2 \leqq \sum_{1 \le i, j \le n} ||e(i, j)x||^2 \leqq n M^2 ||x||^2 \ .$$

*Proof.*  $\{e(i, i): i = 1, \dots, n\}$  is a set of pairwise orthogonal projections with sum *I*, so if  $\mathcal{M}_i = \text{range of } e(i, i), i = 1, \dots, n$ , then  $H = \mathcal{M}_1 \bigoplus \cdots \bigoplus \mathcal{M}_n$ .

Since e(i, j)e(k, k) = 0,  $k \neq j$ , e(i, j)e(j, i) = e(i, i), and e(i, j)e(i, j) = e(i, j), we have

(2.4) kernel of 
$$e(i, j) \supseteq \bigoplus_{\substack{1 \le k \le n \\ k \ne j}} \mathscr{M}_k$$

(2.5) range of  $e(i, j) = \mathcal{M}_i$ .

Suppose e(i, j)x = 0, with  $x = \bigoplus_{i=1}^{n} m_i, m_i \in \mathcal{M}_i$ . Then

(2.6) 
$$m_j = e(j, j)x = e(j, i)e(i, j)x = 0, i \neq j$$
.

(2.4) and (2.6) imply

(7) kernel of 
$$e(i, j) = \bigoplus_{\substack{1 \le k \le n \\ k \ne j}} \mathcal{M}_k$$
.

From (2.5) and (2.7), it follows that e(i, j) maps  $\mathcal{M}_j$  bijectively onto  $\mathcal{M}_i$ . Therefore if e(i, j) is represented as an operator matrix relative to the decomposition  $H = \mathcal{M}_1 \bigoplus \cdots \bigoplus \mathcal{M}_n$ , then there exists an invertible linear transformation  $T_{ij} \colon \mathcal{M}_j \to \mathcal{M}_i$  such that

(2.8) e(i, j) has a matrix with  $T_{ij}$  in the (i, j)th position and zeros elsewhere.

 $\mathbf{Set}$ 

$$T = \sum_{1 \leq i \ j \leq n} e(i, j)^* e(i, j)$$
 .

From (2.8), we find that the operator matrix of  $e(i, j)^*e(i, j)$  has  $T_{ij}^*T_{ij}$  in the (j, j)th position and zeros elsewhere, so that T is the diagonal matrix

(2.9) 
$$T = \begin{pmatrix} I_{\mathscr{M}_1} + A_1 \\ I_{\mathscr{M}_2} + A_2 \\ \ddots \\ \vdots \\ I_{\mathscr{M}_n} + A_n \end{pmatrix}$$

where

$$A_k = \sum_{\substack{1 \leq i \leq n \ i \neq k}} T^*_{ik} T_{ik}$$
,  $k = 1, \cdots, n$ .

Let  $x \in H$ ,  $x = \bigoplus_{i=1}^{n} x_i$ ,  $x_i \in \mathcal{M}_i$ . By (2.9),

(2.10) 
$$\sum_{\substack{1 \le i, j \le n \\ i \le i, j \le n}} ||e(i, j)x||^2 = (Tx, x) \\ = \sum_{\substack{i=1 \\ j \ne i}}^n \left( ||x_i||^2 + \sum_{\substack{1 \le j \le n \\ j \ne i}} ||T_{ji}x_i||^2 \right).$$

Since  $||e(i, j)|| \leq M$ ,  $i, j = 1, \dots, n$ ,

 $(2.11) ||T_{ij}|| = ||e(i, j)|| \le M, i, j = 1, \cdots, n.$ 

Also e(i, j)e(j, i) = e(i, i) implies  $T_{ij}T_{ji} = I_{\mathscr{M}_i}$ , so that by (2.11), for x in  $\mathscr{M}_j$ ,

(2.12) 
$$||T_{ij}x||^2 = ||T_{ji}^{-1}x||^2$$
  
 $\geq \frac{||x||^2}{||T_{ji}||^2}$   
 $\geq \frac{||x||^2}{M^2}.$ 

Therefore by (2.10) and (2.12),

$$egin{aligned} &\sum_{1 \leq i,j \leq n} || \, e(i,\,j) x \, ||^2 \geq \sum_{i=1}^n \Big( \, || \, x_i \, ||^2 + rac{1}{M^2} \sum_{1 \leq j \leq n lpha \ i \neq i} || \, x_i \, ||^2 \Big) \ & \geq n M^{-2} \sum_{i=1}^n || x_i ||^2 \ &= n M^{-2} || \, x \, ||^2 \; . \end{aligned}$$

By (2.10) and (2.11),

$$egin{aligned} &\sum_{1 \leq i, j \leq n} || \, e(i, \, j) x \, ||^2 \leq \sum_{i=1}^n \left( || \, x_i \, ||^2 + M^2 \sum_{\substack{1 \leq j \leq n \ j \neq i}} || \, x_i \, ||^2 
ight) \ &\leq n M^2 \sum_{i=1}^n || \, x_i \, ||^2 \ &= n M^2 || \, x \, ||^2 \; . \end{aligned}$$

LEMMA 2.7. Let  $\{E_i\}_{i=1}^n$  be a pairwise independent set of idempotents in B(H) and M > 0 a constant satisfying the hypotheses of Lemma 2.5. Then there exists an invertible T in B(H) such that  $TE_iT^{-1}$  is self-adjoint,  $i = 1, \dots, n$ , and  $||T||^{\pm 1} \leq 2M$ .

Proof. The proof is similar to the proof of Lemma 2.4. Set

$$T = \left(\sum_{i=1}^n E_i^* E_i\right)^{1/2}$$

Then for x in H, Lemma 2.5 gives

$$rac{||x||^2}{4M^2} \leq ||\,Tx\,||^2 = \sum_{i=1}^n \,||\,E_ix\,||^2 \leq 4M^2 ||\,x\,||^2$$
 ,

whence  $||T^{\pm 1}|| < 2M$ . One shows that  $(TE_iT^{-1})^* = TE_iT^{-1}$ , i = 1,  $\cdots$ , n, as before.

3. The theorem. The following theorem can now be stated and proved.

THEOREM 3.1. Let  $\mathscr{A}$  be a Banach subalgebra of B(H). Then  $\mathscr{A}$  is similar to an approximately finite C\*-subalgebra of B(H) if and only if  $\mathscr{A}$  is approximately finite and the following condition holds: there is an admissable selection of matrix units  $\{e_k^{(m)}(i, j): i, j =$  $1, \dots, [k, n]\}, k = 1, \dots, r_n$  for  $\mathscr{A}$  and a constant M > 0 such that:

(i) for each fixed n and for all sets  $\{\delta_i^{(k)}: i = 1, \cdots, [n, k]\}$  of zero's and one's,

$$\left\|\sum_{k=1}^{r_n}\sum_{i=1}^{[n,k]}\delta_i^{(k)}e_k^{(n)}(i,i)
ight\|\leq M$$
 ,

(ii) for each k and n,

$$||e_k^{(n)}(i, j)|| \leq M, i, j = 1, \dots, [n, k].$$

Moreover, if these conditions are met, an invertible operator implementing the similarity can be chosen in the von Neumann algebra generated by  $\mathcal{A}$ .

Proof. ( $\Rightarrow$ ). Suppose  $\mathscr{C} = T \mathscr{A} T^{-1}$  is an approximately finite  $C^*$ -algebra for some invertible T in B(H). Thus,  $\mathscr{C} = (\bigcup_n \mathscr{C}_n)^-$ , where  $\{\mathscr{C}_n\}$  is an ascending sequence of finite dimensional  $C^*$ -subalgebras. By Proposition 1.7 of [1], there is an admissable selection of  $C^*$ -matrix units  $\{f_k^{(n)}(i, j): i, j = 1, \dots, [n, k]\}, k = 1, \dots, r_n$  for  $\mathscr{C}$  relative to  $\{\mathscr{C}_n\}$ . If  $\mathscr{A}_n = T^{-1}\mathscr{C}_n T$ , then  $\{\mathscr{A}_n\}$  is an increasing sequence of finite dimensional, semisimple subalgebras of  $\mathscr{A}$  such that  $\mathscr{A} = (\bigcup_n \mathscr{A}_n)^-$ , so that  $\mathscr{A}$  is approximately finite, and if  $e_k^{(n)}(i, j) = T^{-1}f_k^{(n)}(i, j)T$ , then  $\{e_k^{(n)}(i, j): i, j=1, \dots, [n, k]\}, k=1, \dots, r_n$  is an admissable selection of matrix units for  $\mathscr{A}$ .

For each positive integer n, let diag $(\lambda_1, \dots, \lambda_n)$  denote the  $n \times n$  diagonal matrix with main diagonal  $\{\lambda_1, \dots, \lambda_n\}$ . Let  $\{\delta_i^{(k)}: i = 1, \dots, [n, k]\}, k = 1, \dots, r_n$  be sets of zero's and one's. Let

Then

$$egin{aligned} \|TAT^{-1}\| &= \left\|igoplus_{k=1}^{r_n}igoplus_{i=1}^{[n,k]}\delta_i^{(k)}f_k^{(n)}(i,i)
ight\| \ &= \max_{1\leq k\leq r_n}\|\mathrm{diag}(\delta_1^{(k)},\,\cdots,\,\delta_{[n,k]}^{(k)})\| \ &\leq 1 \;. \end{aligned}$$

It therefore follows that (i) obtains with  $M = ||T|| ||T^{-1}||$ . (ii) follows on noticing that  $||f_k^{(n)}(i, j)|| = 1$  for all i, j, k, and n.

( $\Leftarrow$ ). It will first be shown that there exists an invertible T in the von Neumann algebra generated by  $\mathscr{A}$  such that  $Te_k^{(n)}(i, i)T^{-1}$  is self-adjoint for all i, k, and n. Set

$$F_n = \sum_{k=1}^{r_n} \sum_{i=1}^{[n,k]} e_k^{(n)}(i, i)$$
 .

Then if  $E_n = I - F_n$ ,  $\mathscr{C}_n = \{E_n\} \cup \{e_k^{(n)}(i, i): i = 1, \dots, [n, k], k = 1, \dots, r_n\}$ is a pairwise independent set of idempotents in B(H) with sum I. It follows by (i) that  $\mathscr{C}_n$  satisfies the hypotheses of Lemma 2.7 with constant 2M + 1. By that lemma, an invertible  $T_n$  in the C<sup>\*</sup>-algebra generated by  $\mathscr{A}$  and I may hence be chosen such that  $T_n e_k^{(n)}(i, i) T_n^{-1}$ is self-adjoint for  $i = 1, \dots, [n, k], k = 1, \dots, r_n$ , and such that

(3.1) 
$$||T_n^{\pm 1}|| \leq 2(2M+1)$$
,  $n = 1, 2, \cdots$ .

Since the selection of matrix units is admissable, for each fixed *i*, *k*, and *n*,  $e_k^{(m)}(i, i)$  is a sum of a subfamily of idempotents  $e_k^{(n+1)}(j, j)$ . It follows that

(3.2) 
$$T_m e_k^{(n)}(i, i) T_m^{-1}$$
 is self-adjoint, for all  $m \ge n$ .

Since closed balls are compact in the weak operator topology on B(H), (3.1) implies the existence of a subsequence  $\{n_k\}$  and an invertible T in the von Neumann algebra generated by  $\mathscr{A}$  for which

$$(3.3) T^{2}_{n_{k}} \longrightarrow T^{2}(WOT) , \quad k \longrightarrow \infty .$$

Since each  $T_n$  is positive, we may assume T is positive.

Now fix i, k, and n. By (3.2), for x, y in H,

$$(3.4) (T_m e_k^{(n)}(i, i)x, T_m y) = (T_m x, T_m e_k^{(n)}(i, i)y), m \ge n.$$

The self-adjointness of T and each  $T_n$  together with (3.3) and (3.4) hence yield

$$(Te_{k}^{(n)}(i, i)x, Ty) = (Tx, Te_{k}^{(n)}(i, i)y)$$

i.e.,  $Te_k^{(n)}(i, i)T^{-1}$  is self-adjoint. There is therefore no loss of generality in assuming that  $e_k^{(n)}(i, i)$  is a projection for each i, k, and n.

We have  $\mathscr{A} = (\bigcup_n \mathscr{A}_n)^-$ , where

$$\mathscr{A}_n = \bigoplus \{ \mathscr{A}_k^{(n)} \colon k = 1, \dots, r_n \},$$
  
 $\mathscr{A}_k^{(n)} = \bigvee \{ e_k^{(n)}(i, j) \colon i, j = 1, \dots, [n, k] \}.$ 

Set

$$egin{aligned} P_k^{(n)} &= igoplus \{ e_k^{(n)}(i,\,i) \colon i=1,\,\cdots,\,[n,\,k] \} \ , \ &P_n = igoplus \{ P_k^{(n)} \colon k=1,\,\cdots,\,r_n \} \ , \ &M_k^{(n)} = ext{range of } P_k^{(n)} \ . \end{aligned}$$

For each k and n,  $\{e_k^{(n)}(i, j): i, j = 1, \dots, [n, k]\}$  can be considered as a system of matrix units in  $B(M_k^{(n)})$ . By (ii),  $\{e_k^{(n)}(i, j)\}$  satisfies the hypotheses of Lemma 2.6, so if

$$T_k^{(n)} = rac{1}{[k,\,n]^{1/2}} \Bigl( \sum_{1 \leq i,\,j \leq [n,\,k]} e_k^{(n)}(i,\,j)^* e_k^{(n)}(i,\,j) \Bigr)^{1/2} \Bigr) \,,$$

then by that lemma,

(3.5) 
$$\frac{||x||^2}{M^2} \leq ||T_k^{(n)}x||^2 \leq M^2 ||x||^2, \quad x \in M_k^{(n)}$$

Now  $H = (\text{range of } P_n)^{\perp} \bigoplus (\bigoplus \{M_k^{(n)}: k = 1, \dots, r_n\})$ , and therefore if

$$T_n = (I - P_n) \bigoplus (\bigoplus \{T_k^{(n)} \colon k = 1, \cdots, r_n\}),$$

then by (3.5)

(3.6) 
$$\frac{||x||^2}{M^2} \leq ||T_n x||^2 \leq M^2 ||x||^2, x \in H, n = 1, 2, \cdots$$

The proof of Lemma 2.4 shows that  $T_n$  orthogonalizes  $\{e_k^{(n)}(i, j):$  $i, j = 1, \dots, [n, k], k = 1, \dots, r_n\}$ . Since the selection of matrix units is admissable, each  $e_k^{(n)}(i, j)$  is a sum of the form (2.1) of a subfamily of matrix units of  $\mathscr{N}_{n+1}$ . (3.6) hence allows one to use the previous compactness argument to find a invertible operator T in the von Neumann algebra generated by  $\mathscr{N}$  which orthogonalizes  $e_k^{(n)}(i, j)$  for all i, j, k, and n. It follows that  $T \mathscr{N} T^{-1}$  is an approximately finite  $C^*$ -subalgebra of B(H).

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