Pacific Journal of Mathematics

HOLOMORPHIC MAPPING OF PRODUCTS OF ANNULI IN C^n

Eric Bedford

Vol. 87, No. 2 February 1980

PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 2, 1980

HOLOMORPHIC MAPPING OF PRODUCTS OF ANNULI IN C^n

ERIC BEDFORD

Let $\Omega_1, \Omega_2 \subset C^n$ be bounded pseudoconvex Reinhardt domains with the property that $z_1 \cdots z_n \neq 0$ for all $(z_1, \cdots, z_n) \in \bar{\Omega}_j$. A holomorphic mapping $f: \Omega_1 \to \Omega_2$ is discussed in terms of the induced mapping on homology $f_*: H_1(\Omega_1, R) \to H_1(\Omega_2, R)$. Specifically, there is a norm on $H_1(\Omega_j, R)$ which must decrease under f_*. As a consequence we prove that a domain Ω as above is rigid in the sense of H. Cartan: if $f: \Omega \to \Omega$ is holomorphic and $f_*: H_1(\Omega, R) \to H_1(\Omega, R)$ is nonsingular, then f is an automorphism.

1. Introduction. Let $A(R_j) = \{z \in C: 1/R_j < |z| < R_j\}$ be an annulus in the complex plane. If $f: A(R_1) \to A(R_2)$ is a holomorphic mapping, then the topological behavior of f is restricted in terms of the moduli R_1 and R_2 (see Schiffer [6] and Huber [4]). With the methods of Landau and Osserman [5] it will be possible to generalize this result to certain domains which are (topologically) the products of plane annuli. Domains satisfying (2) are also shown to be rigid; see Theorem 2 and Remark 1. In [1] the homology group H_{2n-1} was used to prove rigidity; here we discuss H_1.

Let $\Omega \subset C^n$ be a complex manifold and let

$$\mathscr{F} = \{u \in C^2(\Omega),\ 0 < u < 1,\ u \text{ pluriharmonic}\}\ .$$

If $\gamma \in H_1(\Omega, R)$ is a homology class, then a seminorm on γ may be defined by

$$N\{\gamma\} = \sup_{u \in \mathscr{F}} \int_\gamma d^c u \tag{1}$$

where $d^c = i(\bar{\partial} - \partial)$, (see Chern, Levine, and Nirenberg [2]). If $F: \Omega_1 \to \Omega_2$ is a holomorphic mapping, then the map on homology $F_*: H_1(\Omega_1, R) \to H_1(\Omega_2, R)$ must decrease this norm.

2. Computation of the intrinsic norm. We will compute this norm for domains $\Omega \subset C^n$ satisfying

(2) Ω is connected, bounded, pseudoconvex, Reinhardt (i.e., $(e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n) \in \Omega$ if $z \in \Omega$ and $\theta_1, \cdots, \theta_n \in R$), and if $z \in \bar{\Omega}$, then $z_1 \cdots z_n \neq 0$.

Let $\omega \subset R^n$ be the logarithmic image of Ω, i.e.,

$$\omega = \{(\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n : (e^{\xi_1}, \cdots, e^{\xi_n}) \in \Omega\} .$$

Since Ω satisfies (2), ω is convex. Choosing a point $\zeta \in \Omega$, we define $\gamma_j \in H_1(\Omega, \boldsymbol{R})$ to be the homology class of the circle $\theta \to (\zeta_1, \cdots, e^{i\theta}\zeta_j, \cdots, \zeta_n)$, $0 \leqq \theta \leqq 2\pi$. Thus $\{\gamma_1, \cdots, \gamma_n\}$ forms a basis for $H_1(\Omega, \boldsymbol{R})$. For $u \in \mathscr{F}$, we set

$$u^0(r_1, \cdots, r_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} u(r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n}) \cdot d\theta_1 \cdots d\theta_n .$$

Since d^c is linear and invariant under complex rotations,

$$\int_{\gamma_j} d^c u = \int_{\gamma_j} d^c u^0$$

for all $u \in \mathscr{F}$. Let $\mathscr{F}^0 = \{u \in \mathscr{F} : u = u(r_1, \cdots, r_n)\}$. We note that every element of $\mathscr{F}^0$ has the form $u = c + c_1 \log r_1 + \cdots + c_n \log r_n$. For $u^0 \in \mathscr{F}^0$, the function $\mathfrak{l}(\xi_1, \cdots, \xi_n) = u^0(e^{\xi_1}, \cdots, e^{\xi_n})$ is affine (linear plus constant). A simple computation gives

$$\int_{\gamma_j} d^c u^0 = \int_{\gamma_j} \frac{\partial u}{\partial r_j} r_j d\theta_j = 2\pi \frac{\partial \mathfrak{l}}{\partial \xi_j} .$$

Thus we conclude that

$$N\{a_1\gamma_1 + \cdots + a_n\gamma_n\} = 2\pi \sup_{\mathfrak{l} \in \mathscr{L}} \left(a_1 \frac{\partial \mathfrak{l}}{\partial \xi_1} + \cdots + a_n \frac{\partial \mathfrak{l}}{\partial \xi_n} \right)$$

where

$$\mathscr{L}(\omega) = \{\mathfrak{l}(\xi) \text{ affine: } 0 < \mathfrak{l}(\xi) < 1,\ \xi \in \omega\} .$$

We define the norm

$$||\mathfrak{l}|| = \max_{\bar{\omega}} \mathfrak{l} - \min_{\bar{\omega}} \mathfrak{l}$$

so that $\mathscr{L}$ is identified via the map $\mathfrak{l} \to \mathfrak{l} - \mathfrak{l}(0)$ with $\Gamma = \{\mathfrak{l}$ linear: $||\mathfrak{l}|| \leqq 1\}$. Clearly $\Gamma = -\Gamma$ and Γ is convex. Let $\boldsymbol{R}^n_\Gamma$ denote the Banach space $\boldsymbol{R}^n$ with Γ as its unit ball. By (3) the unit ball B of $H_1(\Omega, \boldsymbol{R})$ is

$$B = \left\{\gamma = \sum_{j=1}^n a_j\gamma_j : \left|\sum_{j=1}^n a_j \frac{\partial \mathfrak{l}}{\partial \xi_j}\right| < \frac{1}{2\pi} \text{ for } \mathfrak{l};\ ||\mathfrak{l}|| < 1\right\}$$

which is $1/2\pi$ times the unit ball of $(\boldsymbol{R}^n_\Gamma)'$.

If $\omega = -\omega$, then $(\boldsymbol{R}^n_\omega)' = \boldsymbol{R}^n_{2\Gamma}$, and thus B is naturally identified in $\boldsymbol{R}^n_\omega$ as $B = (1/\pi)\omega$. If ω is any convex set, then the convex set $\tilde{\omega} = \pi B \subset \boldsymbol{R}^n$ satisfies $\tilde{\omega} = -\tilde{\omega}$ and has the same unit ball, B, as ω. For a general convex set ω, we may assume that $0 \in \omega$ and let $\rho(\xi)$ be its support function, i.e., $\rho(\xi)$ is the distance from 0 of the hyperplane which supports ω and has outward normal ξ. It follows that

$$\Gamma = \left\{ \mathfrak{l}(\xi) = \sum c_j \xi_j : (\sum c_j^2)^{1/2} \leqq \frac{1}{(\rho(c) + \rho(-c))} \right\} .$$

In terms of the basis $\{d\theta_1, \cdots, d\theta_n\}$, Γ may be identified as a subset of $H^1(\Omega, \boldsymbol{R})$, and so H^1 inherits the dual norm. Thus, for each $a \in H^1(\Omega, \boldsymbol{R})$ with $a \in \partial\Gamma$, there exists $\gamma \in H_1(\Omega, \boldsymbol{R})$ such that $\gamma \cdot a = N\{\gamma\}$.

For $u \in \mathscr{F}$, $\mathfrak{l} \in \Gamma$, we will use the notation:

$$Lu(\xi) = u^0(e^\xi)$$
$$\tilde{\mathfrak{l}}(z) = \mathfrak{l}(\log |z|) .$$

It is useful to know, given a homology class $\gamma \in H_1(\Omega, \boldsymbol{Z})$, whether there is an imbedded annulus $\varphi: A(R) \to \Omega$ such that $\varphi_*(|z| = 1) = \gamma$ and $N\{|z| = 1\} = N\{\gamma\}$. We do not know this in general, but this happens when $\omega = -\omega$. For integers $m_1, \cdots, m_n$, we define the map $\varphi: A(R) \to \boldsymbol{C}^n$ by $\varphi(\tau) = (\tau^{m_1}, \cdots, \tau^{m_n})$, and thus $\varphi_*(|z| = 1) = \sum m_j \gamma_j$. It is easily seen that $\varphi(A(R)) \subset \Omega$ for $\log R = \mu$ if $(\mu m_1, \cdots, \mu m_n) \in \omega$. By the identification $B = (1/\pi)\omega$, we have

$$N(\sigma) = \frac{\pi}{\mu} = N\{\varphi_*(\sigma)\} = N\{\sum m_j \gamma_j\} \tag{4}$$

for $\mu = \log R$ and $\mu(m_1, \cdots, m_n) \in \partial\omega$.

3. Extremal functions. To study holomorphic mappings we will need to know that the function achieving the supremum in (1) is unique.

PROPOSITION 1. *If γ is the homology class of $\{|z| = 1\}$ in the annulus $A(R)$, then*

$$u = \frac{\log R|z|}{2 \log R}$$

is the unique function in $\mathscr{F}$ satisfying

$$N\{\gamma\} = \int_\gamma d^c u . \tag{5}$$

If $v \in \mathscr{F}$ satisfies

$$cN\{\gamma\} = \int_\gamma d^c v$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} |v(re^{i\theta}) - u(r)| d\theta \leqq 4(1 - c)$$

for $1/R < r < R$.

Proof. The first assertion is well known. The idea of the proof is that if $v \in \mathscr{F}$, and if $\{u > v\}$ is nonempty, then the homology class of $\gamma' = \partial\{u > v\}$ is homologous to γ. Thus if v satisfies (5), then

$$\int_\gamma d^c(u - v) = \int_{\gamma'} d^c(u - v) = 0 .$$

Thus $\nabla(u - v) = 0$ on γ', and by unique continuation, $u = v$ on $A(R)$. For details, see Landau and Osserman [5], or [1].

For the second assertion, we consider the Laurent expansion

$$v(z) = cu(z) + c_0 + \operatorname{Re} g(z)$$

where $g(z) = \sum_{j \neq 0} c_j z^j$. Since $\operatorname{Re} g(z)$ is a bounded harmonic function on $A(R)$, it has nontangential boundary limits a.e. on $|z| = R$ and $|z| = 1/R$. It follows that

$$\int_0^{2\pi} \operatorname{Re} g(re^{i\theta}) d\theta = 0$$

for $1/R \leqq r \leqq R$. Since $v \in \mathscr{F}$, it follows that $c_0 + c \leqq 1$ and $\operatorname{Re} g(z) \leqq 1 - c - c_0$ for $|z| = R$; and $c_0 \geqq 0$, $\operatorname{Re} g(z) \geqq -c_0$ for $|z| = 1/R$. Therefore

$$\frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} g(re^{i\theta})| d\theta \leqq 2(1 - c)$$

for $r = R$ and $r = 1/R$. Since $\operatorname{Re} g$ is harmonic on $A(R)$, this bound holds for $1/R \leqq r \leqq R$. Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |u(r) - v(re^{i\theta})| d\theta \leqq 1 - c + c_0 + \frac{1}{2\pi} \int_0^{2\pi} |\operatorname{Re} g(e^{i\theta})| d\theta$$

which gives the desired estimate.

PROPOSITION 2. *Let Ω satisfy* (2), *and let $\gamma \in H_1(\Omega, \boldsymbol{R})$ be given. If u satisfies* (5), *then $u(z) = u^0(z)$ for all $z \in \Omega$ such that* $\log|z|$ *belongs to the convex hull of $\{\xi \in \partial\omega : Lu(\xi) = 0$ or $1\}$. In particular, if*

(6) *there exist*
p_0 $p_1 \in \bar{\omega}$, $Lu(p_1) = 1$, $Lu(p_0) = 0$
$c = (c_1, \cdots, c_n) = p_1 - p_0$ *and the*
set $\{c_1, \cdots, c_n\}$ *is rationally*
independent

then $u(z) = u^0(z)$ *for all* $z \in \Omega$.

Proof. Let us begin by recalling that $\int_\gamma d^c(u^0 - u) = 0$ for all $\gamma \in H_1(\Omega, \boldsymbol{R})$. Thus there is a holomorphic function $f \in \mathcal{O}(\Omega)$ such that $u = u^0 + \operatorname{Re} f$. If the first part of the proposition is proved, then it follows that $\operatorname{Re} f(z) = 0$ on $S = \{z \in \Omega: \log|z| = \lambda c, \lambda \in \boldsymbol{R}\}$, if $p_0 = 0$. If (6) holds there is a one-dimensional complex manifold $M = \{(\tau^{c_1}, \cdots, \tau^{c_n}): \tau \in \boldsymbol{C}\} \cap \Omega$ which is dense in S. Since M is complex, it follows that $f = 0$ on M. Thus $f = 0$ on S, and so $f = 0$ on Ω.

Now we establish the first part of the proposition. Let $p_0, p_1 \in \partial\omega$ be such that $Lu(p_0) = 0$ and $Lu(p_1) = 1$. Without loss of generality we may assume that $p_1 = -p_0$. We first consider the case where the ratios c_j/c_k are all rational. Thus there are integers $(m_1, \cdots, m_n)$ such that $c_j = \mu m_j$ for some $\mu \in \boldsymbol{R}$. The mapping $\varphi_m(\tau) = (\tau^{m_1}, \cdots, \tau^{m_n})$ maps the annulus $A(e^\mu)$ into Ω, and the logarithmic image of $\varphi(A(e^\mu))$ is the segment (p_0, p_1). It follows that $u(\varphi)$ and $u^0(\varphi)$ both satisfy (5), and thus by Proposition 1 $u(\varphi) = u^0(\varphi)$ on A. Since this argument applies to all mappings $\varphi(\tau) = (e^{i\theta_1}\tau^{m_1}, \cdots, e^{i\theta_n}\tau^{m_n})$, we conclude that $u(z) = u^0(z)$ for all z such that $\log|z| \in (p_0, p_1)$.

For general c, we may take a sequence $\{c^s\}$, $c^s = \mu_s(m_1^s, \cdots, m_n^s)$, $\mu_s \in \boldsymbol{R}$, $m_i^s \in \boldsymbol{Z}$ such that $\pm c^s \in \bar{\omega}$ and c^s converges to p_1. As before we set $\varphi_{m^s} = \varphi_s: A(e^{\mu_s}) \to \Omega$. Thus

$$u^0(\varphi_s(z)) = \frac{\log e^{\mu_s}|z|}{2 \log e^{\mu_s}} + \varepsilon(s)$$

where $\varepsilon(s)$ is a function on $A(e^{\mu_s})$ such that

$$\lim_{s\to\infty} \|\varepsilon(s)\| = 0 \text{ (here } \|\varepsilon(s)\| = \sup_{A(e^{\mu_s})} |\varepsilon(s)|) .$$

If σ is the class of $\{|z| = 1\}$ in $A(e^{\mu_s})$ then

$$\int_\sigma d^c u^0(\varphi_s) \geqq (1 - \|\varepsilon(s)\|)N\{\sigma\} .$$

Since

$$\int_{(\varphi_s)_*\sigma} d^c u = \int_{(\varphi_s)_*\sigma} d^c u^0 ,$$

we have

$$\int_\sigma d^c u(\varphi_s) \geqq (1 - \|\varepsilon(s)\|)N\{\sigma\} .$$

By Proposition 1, then,

$$\frac{1}{2\pi}\int_0^{2\pi}|u(\varphi_s(re^{i\theta})) - u^0(\varphi_s(r))|\,d\theta \leqq 4\|\varepsilon(s)\| .$$

Clearly the same holds if φ_s is replaced by $\varphi(\gamma) = (e^{i\theta_1}\tau^{m_1}, \cdots, e^{i\theta_n}\tau^{m_n})$ with $\theta_1, \cdots, \theta_n \in \boldsymbol{R}$.

Finally we will show that $u(r) = u^0(r)$ for $r = \lambda c$, $0 < \lambda < 1$. If this does not hold, then there exists $\delta > 0$ such that $|u(z) - u^0(|z|)| > \delta$ for all z such that $|z - r| < \delta$. Now we may cover the set $T = \{z \in \Omega: |z_j| = r_j\}$ with K balls (K large) of radius δ and centers $q_1, \cdots, q_K \in T$. At least one of these balls has the property that

$$\frac{2\pi}{K} \leqq \text{measure}\,\{0 < \theta < 2\pi: |\varphi_s(\rho e^{i\theta}) - q_j| < \delta\} ,$$

where $\varphi_s(\rho) = r$. Denote $\operatorname{Arg}(q_j)$ by $(\psi_1, \cdots, \psi_n)$. It follows that

$$\int_0^{2\pi}|u(\tilde{\varphi}_s(\rho e^{i\theta})) - u^0(r)|\,d\theta$$

$$\geqq \delta \text{ measure } \{0 < \theta < 2\pi: |\tilde{\varphi}_s(\rho e^{i\theta}) - r| < \delta\} \geqq \frac{2\pi\delta}{K}$$

where $\tilde{\varphi}_s = (e^{-i\psi_1}\tau^{m_1}, \cdots, e^{-i\psi_n}\tau^{m_n})$. Since this contradicts our previous estimate, we conclude that $u(z) = u^0(z)$ if $|z| = r$, which was what we wanted to prove.

PROPOSITION 3. *Let $\omega \subset \boldsymbol{R}^n$ be a bounded convex set. Given $c \in \boldsymbol{R}^n$, $c \neq 0$, there exists $u \in \mathscr{F}$, $p_0, p_1 \in \partial\omega$ such that $p_1 - p_0 = \lambda c$, $\lambda \in \boldsymbol{R}$, and $Lu(p_j) = j$ for $j = 0, 1$. Furthermore, there exist $u_1, \cdots, u_n \in \mathscr{F}$ satisfying* (6) *and such that $Lu_1, \cdots, Lu_n$ are linearly independent.*

Proof. Let us first suppose that $\partial\omega$ is smooth and strictly convex. Let $\alpha: S^{n-1} \to \partial\omega$ be the Gauss map, i.e., the outward normal to $\partial\omega$ at $\alpha(\xi)$ is ξ. Consider the map $\beta: S^{n-1} \to S^{n-1}$ given by

$$\beta(\xi) = \frac{\alpha(\xi) - \alpha(-\xi)}{|\alpha(\xi) - \alpha(-\xi)|} .$$

Clearly $\beta(\xi)\cdot\xi > 0$, and thus β has degree 1, so that β is onto. Let ξ_0 be a vector such that $\beta(\xi_0) = c/|c|$. Then we take $p_1 = \alpha(\xi_0)$, $p_0 = \alpha(-\xi_0)$, and grad $Lu = \beta(\xi)$.

For general ω, we take an increasing sequence $\{\omega_j\}$ of smoothly bounded strictly convex sets. If u^j, p_0^j, p_1^j have the desired properties on ω_j, we pass to a convergent subsequence to obtain u, p_0, p_1.

Now we show that we can obtain the family $\{u_1, \cdots, u_n\}$. Let us suppose that we have found $\{u_1, \cdots, u_j\}$ with $\{Lu_1, \cdots, Lu_j\}$,

$1 \leqq j < n$, linearly independent and satisfying (6). Pick $c \in \bigcap_{k \leqq j} \operatorname{Ker} Lu_k$, $c \neq 0$. It follows that if u_{j+1} satisfies the conclusion of the first part of the proposition, then $\{Lu_1, \cdots, Lu_{j+1}\}$ are linearly independent. Now we perturb c slightly so that (6) is satisfied and the set is still independent.

4. Application to holomorphic mappings. Let $F: \Omega_1 \to \Omega_2$ be a holomorphic mapping of domains satisfying (2). Then by the integer matrix T_F we will denote the map on integral homology classes $F_* = T_F: \boldsymbol{Z}^n \to \boldsymbol{Z}^n$ in terms of basis $\{\gamma_1, \cdots, \gamma_n\}$. It follows that $T_F(B_1) \subset B_2$ and $T'_F(\Gamma_2) \subset \Gamma_1$, where T'_F is the transpose of T_F, and T'_F gives the action of F^* on H^1. If $\mathrm{l}(\xi) = \sum c_j \xi_j$, then $F^* d^c \tilde{\mathrm{l}}$ represents the same cohomology class as $T'_F(c)$. Writing $u(z) = \tilde{\mathrm{l}}(F(z))$ we have $Lu(\xi) = T'_F(c) \cdot \xi$.

THEOREM 1. *Let Ω_1, Ω_2 satisfy (2), and assume that $\omega_1 = -\omega_1$, $\omega_2 = -\omega_2$. Let T be an $n \times n$ matrix with integer entries. There exists a holomorphic mapping $F: \Omega_1 \to \Omega_2$ with $T_F = T$ if and only if $T(\omega_1) \subset \omega_2$. Furthermore $T(\omega_1) = \omega_2$* (i.e., *F_* is an isometry*) *if and only if F is a proper covering map, and in this case F has the form*

$$F(z) = (e^{i\theta_1} z^{t_1}, e^{i\theta_n} z^{t_n})$$

where $\theta_1, \cdots, \theta_n \in \boldsymbol{R}$ and $t_1, \cdots, t_n$ are the rows of T.

Proof. Let $F: \Omega_1 \to \Omega_2$ be given. Since F_* must be norm-decreasing, and since $1/\pi\omega_j = B_j$, it follows that $T(\omega_1) \subset \omega_2$. Conversely, if $T(\omega_1) \subset \omega_2$, we set $F(z_1, \cdots, z_n) = (z^{t_1}, \cdots, z^{t_n})$. Exponentiating the inclusion $T(\omega_1) \subset \omega_2$, we obtain $F(\Omega_1) \subset \Omega_2$.

Now we assume that T_F is an isometry, and let $\{u_1, \cdots, u_n\} \subset \mathscr{F}^0(\Omega_1)$ be the set constructed in Proposition 3. We may assume that $d^c u_j \in \partial \Gamma$, so there exists $\{\gamma_1, \cdots, \gamma_n\} \subset H_1(\Omega_1, \boldsymbol{R})$ such that $N\{\gamma_j\} = \int_{\gamma_j} d^c u_j$. Now we pick $u'_1, \cdots, u'_n \in \mathscr{F}^0(\Omega_2)$ such that the cohomology class of $d^c u_j$ is the same as $F^*(d^c u'_j)$. Thus

$$\int_{\gamma_j} d^c u_j = N\{\gamma_j\} = N\{F_* \gamma_j\} = \int_{\gamma_j} F^*(d^c u_j) .$$

Since F is holomorphic,

$$\int_{\gamma_j} F^*(d^c u_j) = \int_{\gamma_j} d^c(u'_j(F)) .$$

Since u_j satisfies (6), we conclude by Proposition 2, that $u_j = u'_j(F)$. This gives n independent equations which have the form

$$\sum_{i=1}^{n} c_{ij} \log |z_i| = \sum_{i=1}^{n} c'_{ij} \log |F_i(z)|$$

for $j = 1, \cdots, n$. Thus $\log |F_i(z)| = \sum a_{ij} \log |z_j|$, $i = 1, \cdots, n$. Since $T_F = T$, it follows that $a_{ij} = t_{ij}$, and so F has the desired form. Thus

$$\frac{\partial F_i}{\partial z_j} = \frac{t_{ij}}{z_j} F_i$$

so that $\det(\partial F_i/\partial z_j) = (\prod_{k=1}^{n} F_k/z_k) \det T \neq 0$. Since $T(\omega_1) = \omega_2$ it follows that F is in fact a covering map and is proper.

Conversely, we shows that if F is a covering, then F_* is an isometry. We consider first the one-dimensional case $f: A(R_1) \to A(R_2)$, where f is a d-to-1 covering. If $\varphi: A(R_2^{1/d}) \to A(R_2)$ is given by $\varphi(z) = z^d$, then taking a suitable branch of $\varphi^{-1}(f)$ we obtain a biholomorphism between $A(R_1)$ and $A(R_2^{1/d})$. Since $R_1 = R_2^{1/d}$, f_* is an isometry.

For the general case, we consider integral homology classes $\gamma' = \sum m_j \gamma'_j \in H_1(\Omega_2, \boldsymbol{Z})$. Let $\varphi: A' \to \Omega_2$ be an imbedding of an annulus so that $\varphi_*(\sigma) = \gamma'$ and (4) holds. If we set $A = F^{-1}(\varphi A')$, then $F_{|A}: A \to \varphi A'$ is a covering. F is proper, so $F^{-1}\gamma'$ is a closed curve in Ω_1; thus A is a 1-dimensional annulus and so $(F_{|A})_*$ is an isometry. We let σ be the generator of $H_1(A, \boldsymbol{Z})$, and we let $\gamma = \gamma_\sigma$ be the induced element of $H_1(\Omega_1, \boldsymbol{Z})$. Thus $F_*(\gamma) = \gamma'$, and so $N\{\gamma\} \geqq N\{\gamma'\}$. On the other hand, since $A \subset \Omega_1$,

$$N\{\gamma'\} = N\{\sigma'\} = N\{\sigma\} \geqq N\{\gamma\} ,$$

and so $N\{\gamma\} = N\{F_*(\gamma)\}$. Since this holds for all integral classes in $H_1(\Omega_2, \boldsymbol{R})$, it follows that F_* is an isometry.

THEOREM 2. *Let Ω_1, Ω_2 satisfy (2). If $F: \Omega_1 \to \Omega_2$ is a holomorphic mapping such that $F_*: H_1(\Omega_1, \boldsymbol{R}) \to H_1(\Omega_2, \boldsymbol{R})$ is an isometry, then F is a covering map of the form*

$$F(z) = (c_1 z^{t_1}, \cdots, c_n z^{t_n})$$

where $c_1, \cdots, c_n \in \boldsymbol{C}$ and $t_1, \cdots, t_n$ are the rows of T_F. In particular, if $\Omega_1 = \Omega_2$ and F_* is nonsingular, then F is a biholomorphism.

Proof. We repeat the appropriate portion of the proof of Theorem 1 and conclude that if F_* is an isometry, then

$$c_{0j} + \sum_{i=1}^{n} c_{ij} \log |z_i| = c'_{0j} + \sum_{i=1}^{n} c'_{ij} \log |F_i(z)|$$

for $j = 1, \cdots, n$. Thus $|F_j(z)| = b_j |z_1|^{b_{1j}} \cdots |z_n|^{b_{nj}}$, and so F has the desired form since $F_* = T_F$. As before, $\det(\partial F_i/\partial z_j) \neq 0$. To show that F is a covering, we show that F is proper. We have already shown that $F(z) = (c_1 z^{t_1}, \cdots, c_n z^{t_n})$ and so for $\mathfrak{l}' \in \Gamma_2$, $L\tilde{\mathfrak{l}}'(F) \in \Gamma_1$. We set $U_j(z) = \sup_{\mathfrak{l} \in \partial \Gamma_j} \tilde{\mathfrak{l}}(z)$. By the convexity of ω_j, U_j is an exhaustion for Ω_j: $\partial\Omega_j = \{z \in \bar{\Omega}_j : U_j(z) = 1\}$. As was noted above,

$$T'_F \mathfrak{l}' = \tilde{\mathfrak{l}}'(\log |F|)$$

for $\mathfrak{l}' \in \Gamma_2$. Since F_* is an isometry, $F^*\Gamma_2 = \Gamma_1$, and so

$$U_1(z) = U_2(F(z)) .$$

Thus F is proper.

In case $\Omega_1 = \Omega_2$, then $F_* B_1 \subset B_1$. Since T_F has integer coefficients and is invertible, $\det T_F = \pm 1$. Thus T_F preserves volume, and so $T_F B_1 = B_1$. The inverse mapping is easily constructed as $G(z) = (\zeta^{s_1}, \cdots, \zeta^{s_n})$ where $\zeta_j = z_j/c_j$ and s_j is the jth row of the inverse $S = T^{-1}$.

REMARK 1. It follows that domains satisfying (2) are rigid in the sense of H. Cartan [2]: if $f: \Omega \to \Omega$ is holomorphic and induces a nonsingular mapping on $H_1(\Omega, \boldsymbol{R})$, then f is an automorphism. By topological considerations, it follows that if f_* is nonzero on the generator of $H_n(\Omega, \boldsymbol{R})$, then f_* is nonsingular on $H_1(\Omega, \boldsymbol{R})$ and is thus an automorphism. If T is a complex 1-dimensional torus and if $D \subset \boldsymbol{C}$ is a disk, then $T \times D$ is a complex manifold homeomorphic to $A(R) \times A(R)$ but is not rigid. We would expect, however, that a bounded domain in $\boldsymbol{C}^n$, homeomorphic to $A(R) \times \cdots \times A(R)$, would be rigid.

REMARK 2. The problem of finding nontrivial automorphisms (i.e., other than $z \to (e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n)$) of domain satisfying (2) is thus reduced to finding $T \in GL(n, \boldsymbol{Z})$ such that $TB = B$. For instance, if $1 \leqq p < \infty$, this argument shows that the automorphisms of the domain

$$\Omega = \left\{ z \in \boldsymbol{C}^n : \sum_{j=1}^{n} \left(\log \frac{|z_j|}{R_j} \right)^p < 1 \right\}$$

are generated by the nontrivial automorphisms $z \to (z_1, \cdots, z_j^{-1}, \cdots, z_n)$ and $z_j \to z_k$ if $R_j = R_k$. Since a "generic" norm on $\boldsymbol{R}^n$ does not have any nontrivial isometries, a "generic" domain satisfying (2) has only trivial automorphisms.

REMARK 3. Let us consider domains satisfying (7) for some fixed j:

(7) Ω is connected, bounded, pseudoconvex, Reinhardt, if $z \in \bar{\Omega}$, then $z_1, \cdots, z_j \neq 0$, and there are points $P_{j+1}, \cdots, P_n \in \bar{\Omega}$ such that the kth coordinate of P_k is 0 .

Let $p: \boldsymbol{C}^n \to \boldsymbol{C}^j$ be projection onto the first j variables, and set $\Omega_0 = p(\Omega)$. Looking at the logarithmic image of Ω, which is convex, one may deduce that $\Omega_0 \times \{0\} \subseteqq \Omega$. By the norm-decreasing property of inclusion $i: \Omega_0 \to \Omega$ and projection $p: \Omega \to \Omega_0$, it follows that i_* and p_* are isometries of H_1. Thus the norm of a domain satisfying (7) may be computed in terms of Ω_0, which satisfies (2).

REMARK 4. The following observation extends Proposition 2.

PROPOSITION 4. *Let Ω satisfy* (2), *and assume that for each $p \in \partial\omega$ there is a unique supporting hyperplane at p. Then for each homology class $\gamma \in H_1(\Omega, \boldsymbol{R})$ there is a unique function $u \in \mathscr{F}^0$ such that $N\{\gamma\} = \int_\gamma d^c u$.*

Proof. We show that the $\ell \in \mathscr{L}$ which achieves the supremum in (3) is unique. Suppose, to the contrary, that $\ell_1, \ell_2 \in \mathscr{L}$ have this property. Then so does $\ell = (\ell_1 + \ell_2)/2$. Since ℓ is extremal, there must be points $p', p'' \in \partial\omega$ such that $\ell(p') = 0$ and $\ell(p'') = 1$. Thus we must have $\ell_1(p'') = \ell_2(p'') = 1$, and so the half spaces $\{\xi: \ell_1(\xi) \leqq 1\}$ and $\{\xi: \ell_1(\xi) \leqq 1\}$ both support ω at p''. By assumption, then, ℓ_1 is a multiple of ℓ_2. Since $\ell_1(p') = \ell_2(p'') = 0$, it follows that $\ell_1 = \ell_2$, which completes the proof.

EXAMPLE. If $\Omega = A(R) \times A(R)$, then the homology class $\gamma = \gamma_1 + \gamma_2$ has norm $\pi/\log R$. For $0 \leqq \lambda \leqq 1$, the function

$$u_\lambda = \frac{1}{\log R}(\lambda \log |z_1| + (1 - \lambda) \log |z_2|)$$

belongs to $\mathscr{F}^0$ and satisfies (5), and so the extremal function is not unique.

A slight modification of the proof of Proposition 4 shows that uniqueness holds if $\gamma = \sum a_j \gamma_j$ does not have the property:

(8) if $t_0 > 0$ is such that $t_0 a \in \partial\Gamma$, then there is a segment $I \subset \partial\Gamma$ containing $t_0 a$ with $I \perp a$.

Clearly there is a dense subset of H_1 where (8) does not hold.

REFERENCES

1. E. Bedford and D. Burns, *Holomorphic mapping of annuli in* C^n *and the associated extremal function*, Ann. Scuola Norm. Sup. Pisa, **3** (1979), 381–414.

2. H. Cartan, *Sur les fonctions de plusieurs variables complexes*, Math. Z., **35** (1932), 760–773.

3. S.-S. Chern, H. Levine, and L. Nirenberg, *Intrinsic norms on a complex manifold*, Global Analysis. Papers in Honor of K. Kodaira. Princeton Univ. Press, 1969, 119–139.
4. H. Huber, *Ueber analytische Abbildungen von Ringgebieten in Ringgebiete*, Compos. Math., **9** (1951), 161–168.
5. H. Landau and R. Osserman, *On analytic mappings of Riemann surfaces*, J. Analyse Math., **7** (1959–60), 249–279.
6. M. Schiffer, *On the modules of doubly connected domains*, Quart. J. Math., **17** (1946), 197–213.

Received April 24, 1978.

PRINCETON UNIVERSITY
PRINCETON, NJ 08544

../../../FrontMatter/paper.pdf

Pacific Journal of Mathematics

Vol. 87, No. 2 February, 1980