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A NOTE ON GAP-FREQUENCY PARTITIONS

DAVID BRESSOUD

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D. M. BRESSOUD

George Andrews has introduced gap-frequency partitions in order to interpret the Rogers-Selberg q -series identities related to the modulus seven. In this paper, we give a direct derivation of the generating function for such partitions. Our approach makes it much easier to extend and generalize the notion of gap-frequency partitions.

L. J. Rogers is known today primarily for his discovery of the Rogers-Ramanujan identities:

$$(1) \quad \prod_{\substack{n=1 \\ n \neq 0 \pmod{5}}}^{\infty} (1 - q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m},$$

$$(2) \quad \prod_{\substack{n=1 \\ n \neq 0 \pmod{5}}}^{\infty} (1 - q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m},$$

where $(a)_{\infty} = (a; q)_{\infty} = \prod_{i=0}^{\infty} (1 - aq^i)$,

$$(a)_m = \frac{(a)_{\infty}}{(aq^m)_{\infty}}.$$

These analytic identities came to prominence largely because of P. A. MacMahon's combinatorial interpretation of them:

- (3) For $r = 1$ or 2 , and any positive integer n , the partitions of n into parts not congruent to $0, \pm r \pmod{5}$ are equinumerous with the partitions of n into parts with difference at least two between parts, and in which one appears as a part at most $r - 1$ times.

Statement (3) can be proved from equations (1) and (2) by viewing each side of the equations as a generating function (see [3], § 19.13).

It is less well known that Rogers also discovered similar identities for the modulus 7:

$$(4) \quad \prod_{\substack{n=1 \\ n \neq 0, \pm 1 \pmod{7}}}^{\infty} (1 - q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2+2m}}{(q^2; q^2)_m} (-q^{2m+2})_{\infty}$$

$$(5) \quad \prod_{\substack{n=1 \\ n \neq 0, \pm 2 \pmod{7}}}^{\infty} (1 - q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2+2m}}{(q^2; q^2)_m} (-q^{2m+1})_{\infty}$$

$$(6) \quad \prod_{\substack{n=1 \\ n \neq 0, \pm 3 \pmod{7}}}^{\infty} (1 - q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q^2; q^2)_m} (-q^{2m+1})_{\infty}.$$

Equations (4) and (6) first appeared in [4]. All three are proved by Rogers in [5]. A. Selberg rediscovered them in [6].

There is also a combinatorial theorem for the modulus seven. It is a special case of a combinatorial theorem by B. Gordon, [2], which was stated for all odd moduli greater than or equal to five.

- (7) For $r = 1, 2$ or 3 , and any positive integer n , the partitions of n into parts not congruent to $0, \pm r \pmod{7}$ are equinumerous with the partitions of n in which each part appears at most twice, the difference between nonidentical parts is at least two if either appears twice, and one appears as a part at most $r - 1$ times.

While many proofs of statement (7) exist, until recently there was no proof which showed (7) as a direct consequence of equations (4)–(6). It was to supply such a proof that George Andrews introduced the notion of *gap-frequency partitions* (abbreviated *g-f partitions*) in [1]. The purpose of this paper is to provide a simpler derivation of the generating function for *g-f partitions*. This yields a more direct proof that equations (4)–(6) imply statement (7), and also leads to certain natural generalizations of *g-f partitions*.

The generating function for *g-f partitions*.

DEFINITION. A partition π is said to be a *gap-frequency* (or *g-f*) partition if whenever a summand s appears exactly t times, the next larger part is at least $s + t$, and if it is exactly $s + t$ it can appear at most t times.

EXAMPLE. $1 + 4 + 4 + 4 + 7 + 7 + 7$ is a *g-f partition*. Neither $1 + 3 + 3 + 3 + 6 + 6 + 6 + 6$ nor $2 + 3 + 5 + 5 + 5 + 7 + 7$ is a *g-f partition*.

DEFINITION. For positive integers r, x and n , let $S_{r,x}(n)$ denote the number of *g-f partitions* of n in which no part appears more than x times and one appears at most $r - 1$ times.

THEOREM. For positive integers r, x and n and for $|q| < 1$, let $M(m_1, \dots, m_x) = M = \sum_{j=1}^x j^2 \binom{m_j}{2} + \sum_{1 \leq i < j \leq x} i j m_i m_j$. Then

$$(8) \quad \sum_{m_1, \dots, m_x \geq 0} \frac{q^{M + m_1 + 2m_2 + \dots + (r-1)m_{r-1} + 2(rm_r + (r+1)m_{r+1} + \dots + xm_x)}}{(q; q)_{m_1} (q^2; q^2)_{m_2} \dots (q^x; q^x)_{m_x}} = \sum_{n=0}^{\infty} S_{r,x}(n) q^n.$$

For fixed values of m_2, \dots, m_x , the left side of (8) can be summed using Euler's formula:

$$(9) \quad \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}} a^m}{(q; q)_m} = (-a; q)_{\infty}.$$

It is then a straightforward exercise to verify that when $x = 2$ the right-hand sides of equations (4)–(6) are obtained.

Note that $S_{r,2}(n)$ counts those partitions described in the second part of (7). The theorem is sufficient to prove that equations (4)–(6) imply statement (7).

Proof of the theorem.

DEFINITION. A *partition with attributes* is a partition in which parts of equal value may be distinguished by some attribute or characteristic. For example, parts may be colored red, blue, green, etc. A *partition with x attributes* is a partition in which at most x attributes or characteristics are used. In a partition with x attributes, each part will be denoted by an ordered pair, (d_i, a_i) , where d_i is the value of the part and a_i is its attribute, $1 \leq a_i \leq x$.

The generating function for partitions into exactly m parts, each part greater than or equal to b is given by

$$q^{bm}(q; q)_m^{-1}.$$

It follows that

$$q^{b_1 m_1 + b_2 m_2 + \cdots + b_x m_x} (q; q)_{m_1}^{-1} (q; q)_{m_2}^{-1} \cdots (q; q)_{m_x}^{-1}$$

is the generating function for partitions with x attributes such that for $1 \leq i \leq x$, there are exactly m_i parts with attribute i , and each such part is greater than or equal to b_i .

DEFINITION. Let $R_r(m_1, \dots, m_x; n)$ denote the number of partitions of n with x attributes such that for $1 \leq i \leq x$ there are exactly m_i parts with attribute i , each part with attribute i is divisible by i , and all parts with attribute $i \geq r$ are greater than or equal to $2i$.

LEMMA 1.

$$\frac{q^{m_1 + 2m_2 + \cdots + (r-1)m_{r-1} + 2(rm_r + (r+1)m_{r+1} + \cdots + xm_x)}}{(q; q)_{m_1} (q^2; q^2)_{m_2} \cdots (q^x; q^x)_{m_x}} = \sum_{n=0}^{\infty} R_r(m_1, \dots, m_x; n) q^n.$$

Proof. This lemma follows from the discussion given above and the definition of $R_r(m_1, \dots, m_x; n)$.

DEFINITION. Let $S_r(m_1, \dots, m_x; n)$ denote the number of g-f

partitions of n in which no part appears more than x times, one appears at most $r - 1$ times, and for $1 \leq i \leq x$, exactly m_i different integers appear i times.

LEMMA 2. Let $M(m_1, \dots, m_x) = M = \sum_{j=1}^x j^2 \binom{m_j}{2} + \sum_{1 \leq i < j \leq x} i j m_i m_j$. Then $R_r(m_1, \dots, m_x; n - M) = S_r(m_1, \dots, m_x; n)$.

Before proving Lemma 2, we note that it and Lemma 1 imply the theorem, since

$$\begin{aligned} \sum_{n=0}^{\infty} S_{r,x}(n) q^n &= \sum_{m_1 \dots m_x \geq 0} \sum_{n=M}^{\infty} S_r(m_1, \dots, m_x; n) q^n \\ &= \sum_{m_1 \dots m_x \geq 0} \sum_{n=M}^{\infty} R_r(m_1, \dots, m_x; n - M) q^n \\ &= \sum_{m_1 \dots m_x \geq 0} q^M \sum_{n=0}^{\infty} R_r(m_1, \dots, m_x; n) q^n \\ &= \sum_{m_1, \dots, m_x \geq 0} \frac{q^{M+m_1+\dots+(r-1)m_{r-1}+2(rm_r+\dots+xm_x)}}{(q; q)_{m_1} \dots (q; q)_{m_x}}. \end{aligned}$$

Proof of Lemma 2. We shall prove this lemma by establishing a one-to-one correspondence between partitions counted by $R_r(m_1, \dots, m_x; n - M)$ and those counted by $S_r(m_1, \dots, m_x; n)$.

Consider a partition counted by $R_r(m_1, \dots, m_x; n - M)$ with parts given by $(d_1, a_1), (d_2, a_2), \dots$ and with the parts ordered from left to right such that if $d_\lambda/a_\lambda < d_{\mu}/a_{\mu}$, then (d_λ, a_λ) precedes (d_μ, a_μ) and if $d_\lambda/a_\lambda = d_{\mu}/a_{\mu}$ and $a_\lambda > a_\mu$, then (d_λ, a_λ) precedes (d_μ, a_μ) . Clearly there is a unique such ordering of the ordered pairs.

This partition of $n - M$ with x attributes is transformed into a partition of n with x attributes if each ordered pair (d_λ, a_λ) is replaced by the pair (e_λ, a_λ) where

$$e_\lambda = d_\lambda + \sum_{k=1}^{\lambda-1} a_\lambda a_k.$$

We claim that our new partition is a partition of n . The total amount which has been added to our partition is

$$\sum_{\lambda=1}^{m_1+\dots+m_x} \sum_{k=1}^{\lambda-1} a_\lambda a_k,$$

which is the second elementary symmetric function of the numbers

$$\overbrace{1 \dots 1}^{m_1} \quad \overbrace{2 \dots 2}^{m_2} \quad \dots \quad \overbrace{x \dots x}^{m_x},$$

which is equal to

$$\sum_{j=1}^x j^2 \binom{m_j}{2} + \sum_{1 \leq i < j \leq x} i j m_i m_j = M.$$

This proves our claim.

Observe that a_λ divides e_λ , and for each λ

$$\frac{e_\lambda}{a_\lambda} - \frac{e_{\lambda-1}}{a_{\lambda-1}} = \frac{d_\lambda}{a_\lambda} - \frac{d_{\lambda-1}}{a_{\lambda-1}} + a_{\lambda-1} \geq a_{\lambda-1}.$$

Equality occurs when $d_\lambda/a_\lambda = d_{\lambda-1}/a_{\lambda-1}$, which implies that $a_{\lambda-1} \geq a_\lambda$. Also, if $a_1 \geq r$, then $e_1/a_1 = d_1/a_1 \geq 2$. Thus if each part (e_λ, a_λ) is replaced by a_λ equal parts of value e_λ/a_λ , we have a partition counted by $S_r(m_1, \dots, m_x; n)$. This procedure is uniquely reversible (equal parts are added together and the resulting part is given the attribute equal to the number of parts which were added, $\sum_{k=1}^{\lambda-1} a_\lambda a_k$ is then subtracted from the value of the λ 'th part), and so the one-to-one correspondence is established.

This concludes the proof of Lemma 2, and so also the proof of the theorem.

A generalization.

DEFINITION. A partition π is said to be a k -fold g-f partition if whenever a summand s appears exactly t times, then the next larger part is at least $s + kt$, and if it is exactly $s + kt$, it can appear at most t times.

DEFINITION. For positive integers r, x, k and n , let $S_{r,x,k}(n)$ denote the number of k -fold g-f partitions of n in which no part appears more than x times and one appears at most $r - 1$ times.

By the method used above to find the generating function for $S_{r,x}(n) = S_{r,x,1}(n)$, it can be readily verified that

$$\sum_{n=0}^{\infty} S_{r,x,k}(n) q^n = \sum_{m_1, \dots, m_x \geq 0} \frac{q^{kU + m_1 + \dots + (r-1)m_{r-1} + 2(rm_r + \dots + xm_x)}}{(q; q)_{m_1} \cdots (q^x; q^x)_{m_x}}.$$

Since $S_{r,1,k}(n)$ counts the number of partitions of n into parts with minimal difference k , including at most $r - 1$ ones, we see that the right sides of the Rogers-Ramanujan identities are special cases of the generating function for $S_{r,x,k}(n)$.

REFERENCES

1. G. E. Andrews, *Gap-frequency partitions and the Rogers identities*, (to appear).
2. B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math., **38** (1961), 393-399.

3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, Oxford Univ. Press, London, 1960.
4. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc., **25** (1894), 318-343.
5. L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (2), **16** (1917), 315-336.
6. A. Selberg, *Über einige arithmetische Identitäten*, Avhl. Norske Vid., **8** (1936).

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