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ARNOLD SAMUEL KAS

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A. KAS

The classical Lefschetz hyperplane theorem in algebraic geometry describes the homology of a projective algebraic manifold M in terms of "simpler" data, namely the homology of a hyperplane section X of M and the vanishing cycles of a Lefschetz pencil containing X. This paper is a first step in proving a diffeomorphism version of the Lefschetz hyperplane theorem, namely a description of the diffeomorphism type of M in terms of "simpler" data.

Let $\tilde{M}$ be the manifold obtained from M by blowing up the axis of a Lefschetz pencil. There is a holomorphic mapping $f: \tilde{M} \to \mathrm{CP}^1$ which is a Lefschetz fibration, i.e., f has only nondegenerate critical points (in the complex sense). Using the Morse function $z \to |f(z)|^2$ on $\tilde{M}-f^{-1}(\infty)$, one obtains a handlebody decomposition of $\tilde{M}-f^{-1}(\infty)$ which may be described as follows: Let $X=f^{-1}(a)$ be a regular fiber of f. Choose a system of smooth arcs $\gamma_1, \cdots, \gamma_\mu$ starting at a and ending at the critical values of f such that the γ's are pairwise disjoint except for their common initial point. The γ's are ordered such that the tangent vectors $\gamma_1'(0), \cdots, \gamma_\mu'(0)$ rotate in a counter clockwise manner. To each γ_j one may associated a "vanishing cycle", i.e., an imbedding ϕ_j: $S^n \to X$ ($\dim X=2n$) defined up to isotopy, together with a bundle isomorphism ϕ_j': $\tau \to \nu$ where τ is the tangent bundle to S^n and ν is the normal bundle of S^n in X corresponding to the imbedding ϕ_j. ϕ_j' together with the well known bundle isomorphism $\tau \oplus \varepsilon \simeq \varepsilon^{n+1}$ determines a trivialization of the normal bundle of $e^{2\pi i j/\mu} \times \phi_j(S^n)$ in $S^1 \times X$. This trivialization allows one to attach a n-handle to $D^2 \times X$ with the core $e^{2\pi i j/\mu} \times \phi_j(S^n)$. If this is done for each j, $j=1, \cdots, \mu$, the resulting manifold is diffeomorphic to $\tilde{M}$-(tubular neighborhood of $f^{-1}(\infty)$).

Using the bundle isomorphism ϕ_j' and the tubular neighborhood theorem one may identify a closed tubular neighborhood T of $\phi_j(S^n)$ in X with the tangent unit disk bundle to S^n. One may then define a diffeomorphism, up to isotopy, $\delta_j: X \to X$ with support in T. δ_j is a generalization of the classical Dehn-Lickorish twist. δ_j is the geometric monodromy corresponding to the jth critical value of f. It follows that the composition $\delta_\mu \circ \cdots \circ \delta_1$ is smoothly isotopic to the identity $1_X: X \to X$. A smooth isotopy is given by a smooth arc λ in $\mathrm{Diff}(X)$ joining the identity to $\delta_\mu \circ \cdots \circ \delta_1$. The choice of λ, up to homotopy, determines the way in which one closes off $\tilde{M}$-(tubular neighborhood of $f^{-1}(\infty)$) to obtain $\tilde{M}$.

Thus the diffeomorphism type of $\tilde{M}$ is determined by the invariants $\phi_1, \phi_1', \cdots, \phi_\mu, \phi_\mu'$ and $\{\lambda\}$, the homotopy class of λ. Conversely, given a compact oriented $2n$ dimensional manifold X, imbeddings $\phi_j: S^n \to X$, $j=1, \cdots, \mu$, and bundle isomorphisms $\phi_j': \tau \sim \nu_j$ such that $\delta_\mu \circ \cdots \circ \delta_1$ is smoothly isotopic to 1_X, and a homotopy class $\{\lambda\}$ of arcs in $\mathrm{Diff}(X)$ with initial point 1_X and end point $\delta_\mu \circ \cdots \circ \delta_1$, one may construct a $2n+2$ dimensional manifold $\tilde{M}$ and a Lefschetz fibration $f: \tilde{M} \to \mathrm{CP}^1$. It is shown that in the case $n=1$, apart from certain exceptions, $\tilde{M}$ is uniquely determined by $\phi_1, \cdots, \phi_\mu$, i.e., the bundle isomorphisms $\phi_1', \cdots, \phi_\mu'$ and the smooth isotopy class $\{\lambda\}$ are superfluous.

0. Introduction. The classical Lefschetz hyperplane theorem in algebraic geometry describes the homology groups of an algebraic manifold M in terms of those of a hyperplane section X and in terms of the "vanishing cycles" of X. This paper was inspired by the Morse theoretic proof of the Lefschetz hyperplane theorem due to Andreotti and Frankel [1]. Their approach is to blow up the base locus of a generic Lefschetz pencil so as to obtain a manifold $\tilde{M}$ and a "Lefschetz fibration" $f: \tilde{M} \to \boldsymbol{P}^1(\boldsymbol{C})$. They then use the Morse function $|f|^2$ to describe $\tilde{M}$, at least up to homotopy type, and finally they show how to relate the homology groups of M to those of $\tilde{M}$. Now according to Smale's handlebody theory [11], it should be possible to use the Morse function $|f|^2$ to determine the full diffeomorphism class of $\tilde{M}$, not just its homotopy type. In order to do this we must describe the framings of the imbedded spheres (vanishing cycles) corresponding to the critical points of $|f|^2$. In general, the framing associated to a critical point of index $n+1$, has an ambiguity measured by $\pi_n(SO(n+1))$. In our situation, we can improve this to $\pi_n(SO(n))$. This completely determines the framing in certain cases, most notably if M is a complex surface.

In this paper, we describe a set of invariants associated to a Lefschetz fibration $f: M \to \boldsymbol{P}^1$, which allows one, in principle, to give a handlebody decompositions of certain complex surfaces.

There is a certain amount of overlap between some of the ideas of this paper and certain papers of B. Moishezon and R. Mandelbaum (cf. e.g., [6], [7], [10]).

1. The framings associated with a Lefschetz fibration. Let M be a smooth manifold of dimension $\geqq 2$ and let $f: M \to S^2$ be a smooth mapping. A point $p \in M$ will be called a critical point of f if the differential $df_p: T(M)_p \to T(S^2)_{f(p)}$ is not surjective. Now let M be a closed compact oriented smooth manifold of even dimension, say $\dim M = 2n + 2$, $n \geqq 0$, and assume that $f\colon M \to S^2$ is a surjective mapping with a finite number of critical points. We will identify S^2 with the extended complex plane $C \cup \infty$. If $z \in S^2$ is a regular value of f, then $f^{-1}(z)$ is called a regular fiber of f. It is clear that up to diffeomorphism, the regular fibers of f are independent of the regular value z.

DEFINITION 1.1. The smooth mapping $f\colon M \to S^2$ will be called a Lefschetz fibration if each critical point p of f admits a coordinate neighborhood with complex valued coordinates $(w_1, \cdots, w_{n+1})$, consistent with the given orientation of M, and if $f(p)$ has a coordinate neighborhood with a complex coordinate z, consistent with the orientation of S^2, such that locally, f has the form:

$$f(w) = z_0 + w_1^2 + \cdots + w_{n+1}^2 .$$

DEFINITION 1.2. Two Lefschetz fibrations

$$f_1\colon M \longrightarrow S^2 , \quad \text{and} \quad f_2\colon M \longrightarrow S^2$$

are said to be equivalent, if $f_2 = g \circ f_1$ where g is an orientation preserving diffeomorphism of S^2.

Let X denote any regular fiber of the Lefschetz fibration $f\colon M \to S^2$. Notice that up to diffeomorphism, X only depends on the equivalence class of the Lefschetz fibration $f\colon M \to S^2$. Notice also, that X has a unique orientation consistent with the orientations of M and S^2.

We wish to describe a handlebody decomposition of M associated to the Lefschetz fibration $f\colon M \to S^2$. We will assume that a handlebody decomposition of X is already known.

We first recall some standard facts about handles and handlebodies. Let N be a manifold with boundary and let $n = \dim N$. Let

$$\Phi\colon S^{k-1} \times D^{n-k} \longrightarrow \partial N$$

be a smooth imbedding. Form the union $N_1 = N \cup_\Phi D^k \times D^{n-k}$ where we identify each point of $S^{k-1} \times D^{n-k} \subset \partial(D^k \times D^{n-k})$ with its image under Φ. N_1 is a manifold with boundary and corner points. Let V denote the unique manifold (possibly with boundary) obtained from N_1 by "straightening the corners" of N_1 (cf. [2]).

DEFINITION 1.3. V is called: the manifold obtained from N by attaching a k-handle along Φ.

It is easy to see that, up to diffeomorphism, V depends only on the smooth isotopy class of Φ. Let Φ_0 denote the restriction of

Φ to $S^{k-1}\times 0$. It follows easily from the tubular neighborhood theorem [9], that Φ is determined, up to smooth isotopy, by a bundle isomorphism:

$$\Phi'\colon \varepsilon^{n-k} \longrightarrow \nu$$

where ε^{n-k} is the trivial $n-k$ bundle on S^{k-1}, and ν is the normal bundle of S^{k-1} in ∂N under the imbedding Φ_0. Thus the isotopy class of the imbedding Φ is determined by:

(i) A smooth isotopy class of imbeddings

$$\Phi_0\colon S^{k-1} \longrightarrow \partial N\,;$$

(ii) For each Φ_0 in (i), a smooth isotopy class of bundle isomorphisms:

$$\Phi'\colon \varepsilon^{n-k} \longrightarrow \nu\ .$$

Notice that (i) is a "knot invariant". As for (ii), the distinct bundle isomorphisms, up to smooth isotopy, are classified by the group $\pi_{k-1}(SO(n-k))$. Φ' is called a framing of Φ_0.

Let $F\colon M\to \boldsymbol{R}$ be a Morse function, and let $c\in\boldsymbol{R}$ be a critical value such that $F^{-1}(c)$ contains a single critical point p, where F has index λ at p. For each real number $a\in\boldsymbol{R}$, let $M_a=\{x\in M\,|\,F(x)\leqq a\}$. Then for $\varepsilon>0$ sufficiently small, $M_{c+\varepsilon}$ is diffeomorphic to the manifold obtained from $M_{c-\varepsilon}$ by attaching a λ-handle along $\Phi\colon S^{\lambda-1}\times D^{n-\lambda}\to\partial M_{c-\varepsilon}$. To construct Φ explicitely, one may choose (by Morse's lemma) a system of coordinates $x_1,\cdots,x_n$ in M centered at p, such that $F(x_1,\cdots,x_n)=c-x_1^2-\cdots-x_\lambda^2+x_{\lambda+1}^2+\cdots+x_n^2$ (cf. e.g., [8]). Then if $\xi=(\xi_1,\cdots,\xi_\lambda)\in S^{\lambda-1}$, $\eta=(\eta_{\lambda+1},\cdots,\eta_n)\in D^{n-\lambda}$, $\Phi\colon S^{\lambda-1}\times D^{n-\lambda}\to F^{-1}(c-\varepsilon)$ is defined by setting

$$\Phi(\xi,\eta)=(x_1,\cdots,x_n)$$

where

$$(x_1,\cdots,x_\lambda)=\sqrt{\varepsilon+|\eta|^2}\,\xi$$
$$(x_{\lambda+1},\cdots,x_n)=\eta\ .$$

Similarly, if $F^{-1}(c)$ contains several nondegenerate critical points $p_1,\cdots,p_\mu$ of indices $\lambda_1,\cdots,\lambda_\mu$, then $M_{c+\varepsilon}$ is diffeomorphic to the manifold obtained from $M_{c-\varepsilon}$ by attaching μ handles, where the jth handle is a λ_j-handle attached along $\Phi_j\colon S^{\lambda_j-1}\times D^{n-\lambda_j}\to F^1(c-\varepsilon)$, $j=1,\cdots,\mu$, and where the images of the Φ_j are disjoint.

DEFINITION 1.4. A handlebody decomposition of a manifold M is given by:

(i) a sequence $M=M_N\supset\cdots\supset M_0$, where each M_j is a submanifold of M;

(ii) diffeomorphisms $\Phi_j\colon S^{\lambda_j-1}\times D^{n-\lambda_j}\to\partial M_k$, and

$$\begin{aligned}&\Psi_{j+1}\colon M_{j+1}\longrightarrow M_j\cup_{\Phi_j}D^{\lambda_j}\times D^{n-\lambda_j}\,,\\&\Psi_0\colon M_0\longrightarrow D^n\,.\end{aligned}$$

A Morse function $F\colon M\to \boldsymbol{R}$ together with a gradient-like vector field for F, determine a handlebody decomposition of M (cf. [6]).

Now let $f\colon M\to S^2$ be a Lefschetz fibration with regular fiber X, where $\dim X=2n$, $\dim M=2n+2$. We will assume that 0 and ∞ are regular values of f. Define $F\colon M\to \boldsymbol{R}\cup\infty$ by $F(p)=|f(p)|^2$. Then it is easy to verify that outside of $f^{-1}(0)\cup f^{-1}(\infty)$, F has only nondegenerate critical points, each of index $n+1$ (cf. [1]).

DEFINITION 1.5. The Lefschetz fibration $f\colon M\to S^2$ is said to be normalized if the critical values of f are precisely the μ roots of unity.

Every Lefschetz fibration is equivalent to a normalized Lefschetz fibration. Explicitely, if $a_1,\cdots,a_\mu\in S^2$ are the critical values of f, and if $a\in S^2$ is a regular value of f, choose arcs γ_j from a to a_j, $j=1,\cdots,\mu$, which do not intersect except at a. Construct an orientation preserving diffeomorphism between a regular neighborhood of $\bigcup_{j=1}^{\mu}\gamma_j$ and the disk of radius $1+\varepsilon$ such that a_j is mapped to $e^{2\pi ij/\mu}$. Let $g\colon S^2\to S^2$ be an extension of this diffeomorphism. Then $g\circ f\colon M\to S^2$ is a normalized Lefschetz fibration.

Now assume that $f\colon M\to S^2$ is a normalized Lefschetz fibration, and let $D_t\subset S^2$ denote the disk of radius t centered at the origin. Then $f^{-1}(D_t)\cong X\times D^2$ for $t<1$, while for $t>1$, $f^{-1}(D_t)$ is diffeomorphic to the manifold obtained from $X\times D^2$ by attaching μ $(n+1)$-handles by means of imbedding $\Phi_j\colon S^n\times D^{n+1}\to\partial(X\times D^2)=X\times S^1$, $j=1,\cdots,\mu$, where the Φ_j have disjoint images.

LEMMA 1.6. *The imbeddings Φ_j may be chosen so that $\Phi_j(S^n\times 0)\subset f^{-1}(z_j)$ for some z_j, $j=1,\cdots,\mu$.*

The proof of Lemma 1.6 will be given together with the proof of Theorem 1.7.

Let $\phi_j\colon S^n\to X$ be the imbedding defined by Φ_j restricted to $S^n\times 0$, and the identification $f^{-1}(D_t)\cong X\times D^2$, for $t<1$. Let ν_1 denote the normal bundle of S^n in X corresponding to the imbedding ϕ_j, and let ν denote the normal bundle of S^n in $F^{-1}(1-\varepsilon)$. Clearly we have $\nu\cong\nu_1\oplus\varepsilon$. Let τ denote the tangent bundle of S^n.

THEOREM 1.7. *There exists a bundle isomorphism $\phi_j'\colon\tau\to\nu_1$. The framing $\phi_j'\colon\varepsilon^{n+1}\to\nu$ coincides with the isomorphism.*

$$\varepsilon^{n+1} \xrightarrow{\sim} \tau \oplus \varepsilon \xrightarrow{\phi_j' \oplus 1} \nu_1 \otimes \varepsilon \xrightarrow{\sim} \nu .$$

Moreover ϕ_j' is determined, up to smooth isotopy, by the local behavior of the function f at the critical point p.

The proof of Lemma 1.6 and Theorem 1.7 are based on the following easy lemma.

LEMMA 1.8. *Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ be a smooth function, and let $a = (a_1, \cdots, a_n)$ be a regular point of f. Let $f(x) = f(a) + L(x-a) + R(x-a)$ where $L(x)$ is a linear function and $|R(x-a)| \leqq k|x-a|^2$ for $|x-a|$ sufficiently small. Then for $\varepsilon > 0$ sufficiently small, there exists a smooth function $F: \boldsymbol{R}^n \times I \to \boldsymbol{R}$ satisfying:*

(i) $F(x, 0) = f(x)$;

(ii) $F(x, t) = f(x)$ *for* $|x-a| > \varepsilon$;

(iii) $F(x, 1) = f(a) + L(x-a)$ *for* $|x-a| < \varepsilon/2$;

(iv) $x \to F(x, t)$ *has no critical points in* $\{|x-a| < \varepsilon\}$.

Proof. We may assume that $x = 0$, and $f(a) = 0$. Let $\rho_\varepsilon(s)$ be a smooth function such that

$$\rho_\varepsilon(s) = \begin{cases} 1, & s < \varepsilon/2 \\ 0, & s > \varepsilon \end{cases} .$$

Let $F(x, t) = (tL(x) + (1-t)f(x))\rho_\varepsilon(|x|) + f(x)(1 - \rho_\varepsilon(|x|))$. It is obvious that F satisfies conditions (i), (ii), and (iii). To check condition (iv), we restrict our choice of $\rho_\varepsilon(s)$. Let $\rho(s)$ be a smooth function such that $\rho(s) = \begin{cases} 1, & s < 1/2 \\ 0, & s > 1 \end{cases}$. Then set $\rho_\varepsilon(s) = \rho(s/\varepsilon)$. Let M be a constant such that $|\rho'(s)| \leqq M$. Then $|\rho_\varepsilon'(s)| \leqq M/\varepsilon$. Letting $\vec{\nabla} f = (\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$, and defining $F_t(x) = F(x, t)$ we have:

$$\vec{\nabla} F_t(x) = -t\rho_\varepsilon(|x|)\vec{\nabla} R(x) - tR(x)\rho_\varepsilon'(|x|)\frac{\vec{x}}{|x|} + \vec{\nabla} f$$

where $\vec{x} = (x_1, \cdots, x_n)$. The first term on the right is small for small values of ε since $\vec{\nabla} R(x) \to 0$ as $x \to 0$. As for the second term, we have $|R(x)| < k\varepsilon^2$ and $|\rho_\varepsilon'(x)| \leqq M/\varepsilon$ for $|x| < \varepsilon$. Finally, $\vec{\nabla} f(0) \neq 0$ since 0 is a regular point. Therefore for ε sufficiently small $\vec{\nabla} F_t(x) \neq 0$ for $|x| < \varepsilon$.

Proof of Lemma 1.6 *and Theorem* 1.7. Assume, for simplicity, that $j = 1$. Since the Morse function F is the composition of the Lefschetz fibration $w \to f(w)$, and the function $z \to |z|^2$, we can find, by Lemma 1.8, an arc of Morse functions $F_t: M \to \boldsymbol{R}$ such that

(i) F_t coincides with F on $\{w \in M \,|\, |f(w) - 1| > \varepsilon\}$

(ii) $F_0 = F$

(iii) $F_1(w) = 2\mathrm{Re}\, f(w) - 1$ on $\{w \in M \mid |f(w) - 1| \leqq \varepsilon/2\}$.

Since F_1 can be connected to F_0 by an arc of Morse functions, the handlebody decomposition associated with F_1 differs from that associated with F_0 by a smooth isotopy. Now if we use local complex coordinates $(w_1, \cdots, w_{n+1})$ in a neighborhood of the critical point p_1 such that: $f(w_1, \cdots, w_{n+1}) = 1 + w_1^2 + \cdots + w_{n+1}^2$, then $F_1(w) = 2\,\mathrm{Re} f(w) - 1 = 1 + 2|u|^2 - 2|v|^2$ where $w = u + iv$, $u, v \in \boldsymbol{R}^{n+1}$. Then $\Phi_1(S^n \times 0) = \{(u, v) \mid u = 0, 1 - 2|v|^2 = 1 - \varepsilon\} \subset f^{-1}(\sqrt{1-\varepsilon})$, which proves Lemma 1.6.

To prove Theorem 1.7, consider the function: $g\colon S^n \times \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ defined by $g(\xi, \eta) = \xi \cdot \eta$. $g^{-1}(0)$ is the total space of the tangent bundle to S^n; thus if ν_1 is the normal bundle of $S^n \times 0$ in $g^{-1}(0)$, then ν_1 is naturally isomorphic to $\tau = \tau_{S^n}$. Since g is a fibration, the normal bundle to $g^{-1}(0)$ in $S^n \times \boldsymbol{R}^{n+1}$ is trivial. Restricting these bundles to $S^n \times 0$, and using the standard metric on $S^n \times \boldsymbol{R}^{n+1}$, we get the splitting of bundles on S^n, $\varepsilon^{n+1} \cong \tau \oplus \varepsilon$. An easy computation shows that this coincides with the splitting induced by the standard imbedding of S^n in $\boldsymbol{R}^{n+1}$. Now consider the diagram:

$$\begin{array}{ccc}
S^n \times D^{n+1} & \xrightarrow{\Phi} & F_1^{-1}(1-\varepsilon) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Im\, f} \\
\boldsymbol{R} & \xrightarrow{\sqrt{2\varepsilon}} & \boldsymbol{R}
\end{array}$$

The diagram does not commute, but it does "commute to first order" on $S^n \times 0$. More precisely, this diagram induces a commutative diagram of bundle maps which proves Theorem 1.7.

DEFINITION 1.8. $\phi_j(S^n) \subset X$ is called a vanishing cycle for the Lefschetz fibration $f\colon M \to S^2$. The bundle isomorphism $\phi_j'\colon \tau \to \nu$ will be called a normalization of ϕ_j. The pair $\tilde{\phi}_j = (\phi_j, \phi_j')$ will be called a normalized vanishing cycle. The sequence of normalized vanishing cycles, $(\tilde{\phi}_1, \cdots, \tilde{\phi}_\mu)$ will be called an admissible sequence of normalized vanishing cycles.

Note that the bundle isomorphism $\phi'\colon \tau \to \nu$ preserves or reverses orientations according to the factor $(-1)^{n(n-1)/2}$. Thus, e.g., if X is a real oriented surface, then a normalized vanishing cycle is completely determined by an imbedded circle. If X is an oriented 4 manifold, the "core" of a normalized vanishing cycle is an imbedded S^2 with self intersection -2.

Consider the case where $\dim M = 4$, and therefore X is an oriented surface. A vanishing cycle is then an imbedded circle in X, and the normalization of each vanishing cycle is unique. If

$f: M \to S^2$ is a Lefschetz fibration, then for a sufficiently large disk $G \subset S^2$, $f^{-1}(G)$ is diffeomorphic to the manifold obtain from $X \times D^2$ by attaching a finite number of 2-handles using attaching maps $\Phi_j: S^1 \times D^2 \to X \times S^1$, $j = 1, \cdots, \mu$, where $\Phi_j(S^1 \times 0)$ is an imbedded circle γ_j, in a fiber $X \times c_j$ of $X \times S^1$. Now there is a natural way to trivialize the normal bundle of γ_j in $X \times S^1$ corresponding to the fact that the normal bundle of γ_j in X is trivial, and the normal bundle of X in $X \times S^1$ is trivial. However, our Theorem 1.7 states that the framing $\Phi'_j: \varepsilon^2 \to \nu$ is obtained by identifying the normal bundle of γ_j in X with the tangent bundle τ of γ_j and then using the isomorphism

$$\varepsilon^2 \cong \tau \oplus \varepsilon .$$

If one also pays attention to orientations, it is not hard to see that, relative to the "natural" framing, $\Phi'_j: \varepsilon^2 \to \nu$ has framing -1. In other words, if we identify ν with ε^2 by the "natural" framing, then $\Phi'_j: \varepsilon^2 \to \nu$ is given by the mapping $S^1 \to SO(2) \cong S^1$ of degree -1. This explains the well known fact (??) in algebraic geometry that relative vanishing cycles have self intersection number -1.

2. **Invariants of a Lefschetz fibration.** Let $TS^n(1)$ denote the closed tangent unit disk bundle to S^n. We wish to describe a diffeomorphism

$$\delta: TS^n(1) \longrightarrow TS^n(1)$$

such that:

(i) δ is the identity on the boundary of $TS^n(1)$; and

(ii) δ is the antipodal map on $S^n \subset TS^n(1)$.

Let $(p, v) \in TS^n(1)$, where $p \in S^n$ and $v \in TS^n_p$ is a tangent vector of length ≤ 1. Form the geodesic arc on S^n with initial point p, initial velocity v, and length $\pi|v|$, where $|v|$ denotes the length of v. Let q be the end point, and w the terminal velocity of this arc. Then we put $\delta(p, v) = (q^*, w^*)$ where (q^*, w^*) is the image of (q, w) under the antipodal map.

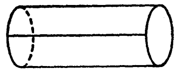
Now let X be a closed manifold of dimension $2n$; let $\phi: S^n \to X$ be an imbedding, and let $\phi': \tau \to \nu$ be a bundle isomorphism where τ is the tangent bundle to S^n, and ν is the normal bundle of S^n in X. By the tubular neighborhood theorem, ϕ' induces a diffeomorphism between $TS^n(1)$ and a tubular neighborhood of $\phi(S^n)$ in X. Thus we can apply δ to a tubular neighborhood of $\phi(S^n)$ in X, and after smoothing near the boundary, we can extend δ by the identity to a diffeomorphism of X which we denote

$$\delta_{\phi,\phi'}: X \longrightarrow X .$$

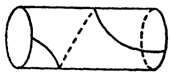
Note that up to smooth isotopy $\delta_{\phi,\phi'}$ only depends on the smooth isotopy class of the imbedding ϕ and the bundle isomorphism ϕ'.

DEFINITION 2.1. $\delta_{\phi,\phi'}$ is called the Dehn twist of X with center (ϕ, ϕ').

Notice that if $\dim X = 2$, then $\delta_{\phi,\phi'}$ is the classical right handed Dehn twist, i.e., $\delta_{\phi,\phi'}$ maps the cylinder



to



Now let $f\colon M \to S^2$ be a Lefschetz fibration with regular fiber X, and let $p \in M$ be a critical point of f. The part of M which lies over the boundary of a small disk about $f(p)$ is then a fiber bundle over S^1 with typical fiber X. It is well known that such a fiber bundle is diffeomorphic to $X \times I/(x, 1) \sim (h(x), 0)$ where $h\colon X \to X$ is a diffeomorphism which is uniquely defined up to smooth isotopy. h is the geometric monodromy of the fibering $f\colon M \to S^2$, associated with the critical value $f(p)$. Assume now that p is the only critical point with critical value $f(p)$. Then as in § 1, we may construct an imbedding $\phi\colon S^n \to X$, and normalization $\phi'\colon \tau \to \nu$.

THEOREM 2.1 (cf. [3], page 148). *The Dehn twist* $\delta_{\phi,\phi'}\colon X \to X$ *is the geometric monodromy of* M *about* $f(p)$.

Proof. This theorem is proved in [3], but we will give another proof which is more consistent with our point of view. Let $I = [-1, 1]$ and fix an imbedding $\iota\colon I \to S^1$, e.g., $t \mapsto e^{i\varepsilon t}$. Now given a Lefschetz fibration $f\colon M \to S^2$ with critical value $f(p)$, choose a small circle γ about $f(p)$; then $f^{-1}(\gamma) = Y$ is the manifold obtained from $X \times S^1$ by surgery. The surgery is performed as follows: look at $\phi' \times \iota\colon TS^n(1) \times I \to X \times S^1$. $TS^n(1) \times I$ may be identified with $S^n \times D^n$ in a standard way (this involves straightening corners). The surgery consists of removing $S^n \times D^n = \phi' \times \iota(TS^n(1) \times I)$ and sewing it back in by the automorphism of the boundary $S^n \times S^n$ given by $(\xi, \eta) \to (\eta, \xi)$. Y is given a differentiable structure by identifying Y with $(X \times S^1 - \phi(S^n) \times \{1\}) \cup D^{n+1} \times S^n$ where $(\xi', \eta') \in (D^{n+1} - 0) \times S^n$ is identified with a point of $X \times S^1 - \phi(S^n) \times \{1\}$ by the following map: first send (ξ', η') to $(\xi, \eta) = (\xi'/|\xi'|, |\xi'|\eta') \in S^n \times D^{n+1}$; then send (ξ, η) to $(u, v) \times t \in TS^n(1) \times I$, and finally send this point to $\phi'(u, v) \times$

$\iota(t)$. We may write

$$Y = (X \times S^1 - \phi(S^n) \times \{1\}) \bigcup_h TS^n(1) \times I$$

where $h\colon TS^n(1) \times I - S^n \times \{0\} \to X \times S^1 - \phi(S^n) \times \{1\}$ is given as follows: if $(u, v) \times t \in TS^n(1) \times I - S^n \times \{0\}$, then $h((u, v), t) = \phi'(u', v') \times \iota(t)$ where (u', v') are given by:

$$u' = \frac{t}{\sqrt{t^2 + |v|^2}}\ u + \frac{1}{\sqrt{t^2 + |v|^2}} v$$
$$v' = \frac{|v|^2}{\sqrt{t^2 + |v|^2}}\ u - \frac{t}{\sqrt{t^2 + |v|^2}} v\,.$$

Thus the vector $\begin{pmatrix} u' \\ v' \end{pmatrix}$ is the product of the vector $\begin{pmatrix} u \\ v \end{pmatrix}$ by the matrix:

$$\begin{pmatrix} \frac{t}{\sqrt{t^2 + |v|^2}} & \frac{1}{\sqrt{t^2 + |v|^2}} \\ \frac{|v|^2}{\sqrt{t^2 + |v|^2}} & \frac{-t}{\sqrt{t^2 + |v|^2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & |v| \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 \\ 0 & |v|^{-1} \end{pmatrix}$$

where $\theta = \text{arc } \sin t/\sqrt{t^2 + |v|^2}$. These formulae show how the projection $X \times S^1 - \phi(S^n) \times \{1\} \to S^1$ extends to a fibration $f\colon Y \to S^1$ where f is defined on $TS^n(1) \times I$ by $f((u, v), t) = \iota(t)$.

The mapping $h\colon TS^n(1) \times I - S^n \times 0 \to TS^n(1) \times I - S^n \times 0$ given by $(u, v, t) \mapsto (u', v', t)$ has the following geometric description: Given $u \in S^n$ and $v \neq 0$, v tangent to S^n at u, define (u', v') so that u' lies on the great circle through u in the direction v and the angle between u and u' is $\theta + \pi/2$. v' is tangent to the great circle, has the same length as v and points backwards towards u. For fixed t, as v approaches 0, $\theta = \text{arc } \sin t/\sqrt{t^2 + |v|^2}$ approaches $\pi/2$ for $t > 0$ and $-\pi/2$ for $t < 0$. Thus $h|_{S^n \times \{t\}}$ is the identity for $t < 0$ and the antipodal map for $t > 0$. For $|v| = 1$, $\theta = \theta(t)$ goes from $-\pi/4$ to $\pi/4$ as t goes from -1 to 1.

Now to calculate the geometric monodromy of $f\colon Y \to S^1$, we construct a flow on Y. This is accomplished by constructing a smooth map $g\colon X \times [\varepsilon, 2\pi + \varepsilon] \to Y$ such that $g(x, s) \in f^{-1}(e^{is})$ and $g_s\colon X \to f^{-1}(e^{is})$ is a diffeomorphism (the flow is $g_*(\partial/\partial s)$). The geometric monodromy is given by $g_{2\pi+\varepsilon} \circ g_\varepsilon^{-1}$.

Identifying Y with

$$X \times S^1 - \phi(S^n) \times \{1\} \bigcup_h TS^n(1) \times I$$

we first define g on $X \times [\varepsilon, 2\pi + \varepsilon] - \phi'(TS^n(1/2)) \times [2\pi - \varepsilon/2, 2\pi + \varepsilon]$ by setting $g(x, s) = (x, e^{is})$. We extend g by a mapping $\tilde{g}\colon TS^n(1) \times I \to TS^n(1) \times I$ such that on $TS^n(1) \times I - TS^n(1/2) \times [-1/2, 1] h \circ g = \mathrm{id}$. To construct $\tilde{g}$, choose a smooth function $\zeta(|v|, t)$ such that: (i) $\zeta(|v|, t) = \theta(|v|, t)$ on $\{t \leqq -1/2\} \cup \{|v| \geqq 1/2\}$ and (ii) for fixed t, $\zeta(|v|, t) \to -\pi/2$ as $|v| \to 0$. Now define $\tilde{g}$ just as we defined h, but use ζ instead of θ. Specifically,

$$\tilde{g}((u, v), t) = \begin{cases} ((u, v), t) & \text{if} \quad v = 0 \\ ((u', v'), t) & \text{if} \quad v \neq 0 \end{cases}$$

where u' lies on the great circle through u in the direction v, the angle between u and u' is $\pi/2 + \zeta$, and v' points backwards towards u along this geodesic and has the same length as v.

To calculate the monodromy $g_{2\pi+\varepsilon} \circ g_\varepsilon^{-1}$, it is clear that the monodromy is concentrated on $TS^n(1) \subset X$, i.e., it is the identity outside of $TS^n(1)$. Notice that g_ε is defined by the following diagram:

$$\begin{array}{ccc} X & = X - S^n \underset{\mathrm{id}}{\bigcup} & TS^n(1) \\ g_\varepsilon \big\downarrow & \mathrm{id} \big\downarrow & \big\downarrow h_1^{-1} \\ f^{-1}(e^{i\varepsilon}) & = X - S^n \underset{h_1}{\bigcup} & TS^n(1) \end{array}$$

where $h_t = h|_{TS^n(1) \times t} \cdot g_{2\pi+\varepsilon}$ is defined by:

$$\begin{array}{ccc} X & = X - S^n \underset{\mathrm{id}}{\bigcup} & TS^n(1) \\ g_{2\pi+\varepsilon} \big\downarrow & \big\downarrow \mathrm{id} & \big\downarrow \tilde{g}_1 \\ f^{-1}(e^{i\varepsilon}) & = X - S^n \underset{h_1}{\bigcup} & TS^n(1) \,. \end{array}$$

Thus the geometric monodromy $g_{2\pi+\varepsilon} \circ g_\varepsilon^{-1}$ is concentrated on $TS^n(1) \subset X$ and there it is equal to $\tilde{g}_1^{-1} \circ h_1$. This mapping sends a point $(u, v) \in TS^n(1)$ to (u', v') where u' lies on the great circle through u tangent to v, $|v'| = |v|$, and the angle between u and u' is $\phi(|v|, 1) - \zeta(|v|, 1)$. This angle is a decreasing function of $|v|$, equal to π for $|v| = 0$ and equal to 0 for $|v| = 1$. This map is therefore isotopic to the Dehn twist $\delta_{\phi, \phi'}$.

Let $f\colon M \to S^2$ be a normalized Lefschetz fibration with fiber X, and critical values $e^{2\pi i j/\mu}$, $j = 0, 1, \cdots, \mu - 1$. Let E denote the subset of the disk of radius $1 + \varepsilon$ obtained by removing the sets E_j, $j = 1, \cdots, \mu$ where E_j is described in polar coordinates (r, θ) by:

$$E_j = \left\{(r, \theta) | 1 - \varepsilon < r \leqq 1 + \varepsilon, \frac{2\pi j}{\mu} - \varepsilon < \theta < \frac{2\pi j}{\mu} + \varepsilon\right\} .$$

Identifying $f^{-1}(E)$ with $X \times E$, one may construct a vector field σ on $f^{-1}\{|z| = 1 + \varepsilon\}$ such that $\sigma = (0, \partial/\partial\theta)$ outside of the E_j, while on E_j the flow along σ maps the fiber over $\theta = 2\pi j/\mu - \varepsilon$ to the fiber over $\theta = 2\pi j/\mu + \varepsilon$ by a Dehn twist δ_j, $j = 1, \cdots, \mu$. In particular, if we let X_θ denote the fiber over θ, then the flow along σ defines diffeomorphisms,

$$h_\theta: X_0 \longrightarrow X_\theta \quad \text{such that ,}$$
$$h_{2\pi} = \delta_\mu \circ \cdots \circ \delta_1. \quad \text{Since } f \text{ has no critical}$$

values outside the unit circle, the fibration defined by f may be trivialized over the complement of E. In particular each fiber X_θ is then identified with $X = X_0$, and $\theta \to h_\theta$ may be regarded as an arc in $\mathrm{Diff}(X)$ joining the identity to $\delta_\mu \circ \cdots \circ \delta_1$. Any two such trivializations of the fibering f are related by a diffeomorphism

$$D^2 \times X \longrightarrow D^2 \times X$$

of the form: $(z, x) \to (z, g_z(x))$. The function $D^2 \to \mathrm{Diff}(X)$ given by $z \to g_z$ gives a homotopy between the arcs $\theta \to h_\theta$, corresponding to the two trivializations. Thus the homotopy class of the arc $\theta \to h_\theta$ is an invariant of the normalized Lefschetz fibration $f: M \to S^2$.

THEOREM 2.3. *The normalized Lefschetz fibration $f: M \to S^2$ is uniquely determined by:*

(1) *a sequence of normalized vanishing cycles* $(\tilde{\phi}_1, \cdots, \tilde{\phi}_\mu)$; *and*

(2) *a homotopy class of arcs $\theta \to h_\theta$ in* $\mathrm{Diff}(X)$, *where* $h_0 = \mathrm{id}_X$ *and* $h_{2\pi} = \delta_\mu \circ \cdots \circ \delta_1$, *where* $\delta_j = \delta_{\tilde{\phi}_j}$ *is the Dehn twist of X with center* $\tilde{\phi}_j$.

Proof. This theorem is essentially a direct consequence of Theorem 1.7, and our above discussion. More precisely, the manifold M and the fibration f are constructed as follows: starting with $D^2 \times X$, one attaches μ $(n + 1)$-handles along the centers $\Phi_{j0}: S^n \to \varepsilon_j \times X$, $\varepsilon_j = e^{2\pi i j/\mu}$, where Φ_{j0} coincides with $\phi_j: S^n \to X$, and the framing of Φ_{j0} is given by the prescription of Theorem 1.7. The boundary of the resulting manifold fibers over a circle, with fiber X, and by Theorem 2.1, one may construct a vector field on this boundary whose flow gives rise to the diffeomorphism $\delta_\mu \circ \cdots \circ \delta_1$. The smooth arc $\theta \to h_\theta$ in $\mathrm{Diff}(X)$ joining id_X to $\delta_\mu \circ \cdots \circ \delta_1$ is then used to identify the boundary with $S^1 \times X$, and we finally use this identification to close up the manifold by attaching a copy of $D^2 \times X$.

The normalized vanishing cycle $\tilde{\phi}$ was defined (Def. 1.8) as a pair (ϕ, ϕ') where $\phi: S^n \to X$ is an imbedding, and $\phi': \tau_{S^n} \to \nu$ is a bundle isomorphism. ϕ and ϕ' are defined up to smooth isotopy. The distinct bundle isomorphisms $\tau_{S^n} \to \nu$ are classified, up to isotopy, by the group $\pi_n(SO(n))$. Thus for those values of n for which $\pi_n(SO(n)) = 0$, ϕ' is uniquely determined. This is true, in particular, for $n = 1$ and 2.

Given an admissible sequence $(\tilde{\phi}_1, \cdots, \tilde{\phi}_\mu)$, let M_1 be the manifold with boundary obtained by attaching μ $(n+1)$-handles to $D^2 \times X$ as in Theorem 1.7. The distinct homotopy classes of arcs $\theta \to h_\theta$ in $\mathrm{Diff}(X)$ joining id_X to $\delta_\mu \circ \cdots \circ \delta_1$ are classified by $\pi_1(\mathrm{Diff}(X))$. Now a closed loop in $\mathrm{Diff}(X)$ gives rise to a fiber preserving diffeomorphism of $S^1 \times X$, and a homotopy class corresponds to an isotopy class of such fiber preserving diffeomorphisms. Let $\{\theta \to h_\theta\}$, $\{\theta \to h'_\theta\}$ be two homotopy classes of arcs in $\mathrm{Diff}(X)$ joining id_X to $\delta_\mu \circ \cdots \circ \delta_1$ and $g: S^1 \times X \to S^1 \times X$ be a diffeomorphism representing their difference. If g can be extended to a fiber preserving diffeomorphism of M_1, then the Lefschetz fibering, $f: M \to S^2$ associated to h_θ is equivalent to the Lefschetz fibration $f': M' \to S^2$ associated to h'_θ.

THEOREM 2.4. *If X is an oriented surface of genus > 1, then every Lefschetz fibration $f: M \to S^2$ with fiber X is uniquely determined, up to equivalence, by an admissible sequence of vanishing cycles. In case X has genus 1, the theorem is still true provided that $f: M \to S^2$ has at least one vanishing cycle $\phi: S^1 \to X$ which is not homologous to zero.*

Proof. It is clear that there is a unique normalization for any vanishing cycle in a surface (cf. the remarks on orientation, following Def. 1.8). The theorem in the case of genus $(X) > 1$ follows from the fact that $\mathrm{Diff}(X)$ is contractible (cf. [4[). The case, genus $(X) = 1$, follows from results of Moishezon [10] (Lemma 8 on page 179 and Lemma 7 on page 164).

It follows from Theorem 2.4, that to each equation of the form:

$$\delta_\mu \circ \cdots \circ \delta_1 = 1$$

in the mapping class group of the oriented surface X, where δ_j is the right-handed Dehn twist of X, centered at some circle γ_j in X, one may associate a 4-manifold M and a normalized Lefschetz fibration $f: M \to S^2$ whose critical values are the μ roots of unity and whose vanishing cycles are $\gamma_1, \gamma_2, \cdots, \gamma_\mu$. Let $g: S^2 \to S^2$ be an orientation preserving diffeomorphism which leaves invariant the μ roots of unity. Then $f' = g \circ f: M \to S^2$ is a normalized Lefschetz fibration with vanishing cycles $\gamma'_1, \cdots, \gamma'_\mu$. Following [10], page 180, we will say that the admissible sequence $(\gamma'_1, \cdots, \gamma'_\mu)$ is equivalent to $(\gamma_1, \cdots, \gamma_\mu)$. This relation is generated by the elementary transformations:

$$T_j\colon (\gamma_1, \cdots, \gamma_\mu) \longrightarrow (\gamma'_1, \cdots, \gamma'_\mu)$$

where:

$$\begin{aligned} \gamma'_j &= \delta_j^{-1}(\gamma_{j+1}) \\ \gamma'_{j+1} &= \gamma_j \\ \gamma'_k &= \gamma_k \quad \text{if} \quad k \neq j,\ j+1 \end{aligned}$$

(here the subscript j is computed modulo μ). The corresponding Dehn twists δ'_k are given by:

$$\begin{aligned} \delta'_j &= \delta_j \circ \delta_{j+1} \circ \delta_j^{-1} \\ \delta'_{j+1} &= \delta_j \\ \delta'_k &= \delta_k \quad \text{if} \quad k \neq j \quad \text{or} \quad j+1\,. \end{aligned}$$

EXAMPLE 2.5. Let $X = T^2$ be a surface of genus 1, i.e., a 2-dimensional torus. Let m denote a standard meridian circle, and let l denote a standard longitude on X. Let $(\gamma_1, \cdots, \gamma_\mu)$ be an admissible sequence of imbedded circles on X, such that no γ_j is homologous to zero. Then Moishezon has proved (cf. [10]) that $\mu \equiv 0$ (12) and $(\gamma_1, \cdots, \gamma_\mu)$ is equivalent to the sequence $(m, l, \cdots, m, l)$ (μ terms).

3. Some open problems. In this section we will discuss several problems on Lefschetz fibrations of 4-manifolds. Such fibrations arise naturally from Lefschetz pencils. Thus let M be an algebraic surface imbedded algebraically in a complex projective space $\boldsymbol{P}^N$. Let H^{N-2} be a generic linear space of codimension 2 in $\boldsymbol{P}^N$, and let $\tilde{M}$ be the manifold obtained from M by blowing up the points of $M \cap H^{N-2}$. Then there is a Lefschetz fibration $f\colon \tilde{M} \to \boldsymbol{P}^1$ such that $f^{-1}(t) \cong M \cap L_t$, where $L_t \subset \boldsymbol{P}^N$ is a hyperplane containing H^{N-2} (such hyperplanes are naturally parametrized by $\boldsymbol{P}^1$). For the details of this construction cf [1]. By Theorem 2.4, the differentiable structure of $\tilde{M}$ together with the Lefschetz fibration is completely determined by an admissible sequence of vanishing cycles. M is obtained from $\tilde{M}$ by blowing down certain exceptional curves. These exceptional curves are holomorphic sections of the fibration $f\colon \tilde{M} \to \boldsymbol{P}^1$ with self intersection -1. Moreover if we assume that M is a minimal surface and that M is neither rational nor ruled, then M is obtained from $\tilde{M}$ by blowing down all holomorphic sections of $f\colon \tilde{M} \to \boldsymbol{P}^1$ of self intersection -1. We do not know whether every continuous section with self intersection -1 is homotopic to a holomorphic section. This leads to the following problem:

Problem 3.1. Is the differentiable structure of an algebraic surface M (minimal, not rational or ruled) determined by an admissible sequence of vanishing cycles arising from a Lefschetz pencil?

Let $f_1: M_1 \to S^2$ and $f_2: M_2 \to S^2$ be two Lefschetz fibrations whose regular fibers are oriented surfaces of genus g. Choose regular fibers $f_1^{-1}(a) \in M_1$ and $f_2^{-1}(b) \in M_2$, and a diffeomorphism $\alpha: f_1^{-1}(a) \to f_2^{-1}(b)$. We may construct a new Lefschetz fibration $f_1 \#_\alpha f_2: M_1 \#_\alpha M_2 \to S^2$ as follows: Let $T_1 \cong f_1^{-1}(a) \times D^2$ be a tubular neighborhood of $f_1^{-1}(a)$ in M_1, and let $T_2 \cong f_2^{-1}(b) \times D^2$ be a tubular neighborhood of $f_2^{-1}(b)$ in M_2. Then $M_1 \#_\alpha M_2$ is the union of M_1-int(T_1) with M_2-int (T_2) where we identify the boundaries by the mapping $f_1^{-1}(a) \times S^1 \to f_2^{-1}(b) \times S^1$ which sends (x, z) to $(\alpha(x), \bar{z})$. The mapping $f_1 \# f_2$ is defined in an obvious way. $f_1 \# f_2: M_1 \#_\alpha M_2 \to S^2$ will be called a fiber connected sum of $f_1: M_1 \to S^2$ and $f_2: M_2 \to S^2$. In general $M_1 \#_\alpha M_2$ will depend on the diffeomorphism α. It is not hard to see that if $\alpha, \beta: f_1^{-1}(a) \to f_2^{-1}(b)$ are two diffeomorphisms, then $M_1 \#_\alpha M_2$ is equivalent (in the sense of Lefschetz fibrations) to $M_1 \#_\beta M_2$ if either $\alpha^{-1} \circ \beta: f_1^{-1}(a) \to f_1^{-1}(a)$ is in the image of the (geometric) monodromy group of M_1, or if $\alpha \circ \beta^{-1}: f_2^{-1}(b) \to f_2^{-1}(b)$ is in the image of the (geometric) monodromy group of M_2.

DEFINITION 3.2. We will say that a Lefschetz fibration $f: M \to S^2$ is irreducible if it is not equivalent to a fiber connected sum $f_1 \# f_2: M_1 \#_\alpha M_2 \to S^2$ where both f_1 and f_2 have at least one critical point. Algebraically, an irreducible Lefschetz fibration with regular fiber X is determined by an equation: $\delta_\mu \cdots \delta_1 = 1$ in the mapping class group of X such that $\delta_\mu \cdots \delta_1$ is not equivalent (in the sense given at the end of § 2) to $\delta'_\mu \cdots \delta'_{\nu+1}\delta'_\nu \cdots \delta'_1$ where $\delta'_\nu \circ \cdots \circ \delta'_1 = 1$ and $\delta'_\mu \circ \cdots \circ \delta'_{\nu+1} = 1$.

Problem 3.3. For a given surface X and integer $\mu > 0$, are there only finitely many (up to equivalence) irreducible Lefschetz fibrations $f: M \to S^2$ with μ vanishing cycles.

REMARK 3.4. The result of Moishezon (Example 2.5) states that if genus $X = 1$, then there is only one irreducible Lefschetz fibration $f: M \to S^2$ where the number μ of critical points, is 12. In this case, every element of the mapping class group of X is realized by a geometric monodromy, and therefore every Lefschetz fibration of genus 1 is equivalent to $M \# \cdots \# M$.

If the answer to problem 3.1 is "yes", we would like to use Lefschetz pencils to study the diffeomorphism types of algebraic surfaces. Specifically, let M be an algebraic surface, minimal, not rational or ruled, and let $f: \tilde{M} \to P^1$ be a Lefschetz fibration arising from a Lefschetz pencil on M. We have given a procedure for obtaining a handlebody decomposition of $\tilde{M}$. We would like to have such a decomposition for M. This could be done if it were possible to rearrange our handlebody decomposition, by sliding handles, so that the sections of $f: \tilde{M} \to P^1$ of self intersection -1 would occur as the cores of 2-handles.

Problem 3.4. Can the handlebody decomposition of $f: \tilde{M} \to P^1$ be altered so as to obtain a handlebody decomposition of M?

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Oregon State University
Corvallis, OR 97331

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