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V. KARUNAKARAN AND MICHAEL ROBERT ZIEGLER

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# THE RADIUS OF STARLIKENESS FOR A CLASS OF REGULAR FUNCTIONS DEFINED BY AN INTEGRAL

V. KARUNAKARAN AND M. R. ZIEGLER

**Let $F(z)$, $f(z)$, and $g(z)$ be regular in the unit disc $E=\{z: z<1\}$, be normalized by $F(0)=f(0)=g(0)=0$ and $F'(0)=f'(0)=g'(0)=1$, and satisfy the equation $z^{c-1}(c+1)f(z)=[F(z)g(z)^c]'$, $c\geqq 0$. This paper is concerned with studying relationships between the mapping properties of these functions. The principle result is the determination of the radius of $\beta$-starlikeness of $f(z)$ when $F(z)$ and $g(z)$ are restricted to certain classes of univalent starlike functions. Conversely, a lower bound for the radius of $\beta$-starlikeness of $F(z)$ is obtained when $f(z)$ and $g(z)$ satisfy similar conditions.**

Problems of this nature were first studied by Libera [9], where he showed that if $f(z)$ is a convex, starlike, or close-to-convex univalent function and $F(z)$ is defined by

$$F(z)=\frac{2}{z}\int_0^z f(t)dt\,, \tag{1}$$

then $F(z)$ is also convex, starlike, or close-to-convex, respectively. Livingston then considered the converse of this problem and determined that if $F(z)$ satisfies one of these geometric conditions in $E$ and $f(z)=(F(z)+zF'(z))/2$, then $f(z)$ satisfies the same condition in $\{z: |z|<1/2\}$ [11]. Refinements of Livingston's results can be found in [1], [2], [10], [12], and [13], while results dealing with generalizations of (1) appear in [3], [4], [5], [6], [7], and [8]. Most recently, Lewandowski et al have shown that if $f(z)$ is starlike in $E$ and $F(z)$ is the solution of

$$cF(z)+zF'(z)=(1+c)f(z)\,, \tag{2}$$

then $F(z)$ is starlike whenever $\operatorname{Re} c\geqq 0$ [8].

Before proceeding any further, it will be convenient to introduce the following notation. Let $S^*(\alpha)$ denote the collection of functions $f(z)$ which are regular in $E$, are normalized by $f(0)=0$ and $f'(0)=1$, and satisfy $\operatorname{Re}[zf'(z)/f(z)]\geqq\alpha$ for $z$ in $E$. Such functions are said to be starlike of order $\alpha$. Normally one only considers $\alpha$ in the interval $[0, 1)$, however, in order to relate the results presented here to earlier works, it is advantageous to allow $\alpha=1$, with the understanding that $S^*(1)$ consists only of the function $f(z)=z$.

In this paper we continue the investigation of a generalization of (1) which was introduced by the first author in [7]. Let $\mathscr{F}_1(\alpha, \gamma, c)$ denote the family of functions $F(z)$ which satisfy

$$F(z) = \frac{c+1}{[g(z)]^c}\int_0^z t^{c-1}f(t)dt\,, \tag{3}$$

where $f(z)$ is in $S^*(\alpha)$, $g(z)$ is in $S^*(\gamma)$ and $c \geqq 0$. Let $\mathscr{F}_2(\alpha, \gamma, c)$ denote the family of functions $f(z)$ which satisfy

$$(c+1)f(z) = c[g(z)/z]^{c-1}g'(z)F(z) + [g(z)/z]^c zF'(z) \tag{4}$$

for $F(z)$ in $S^*(\alpha)$, $g(z)$ in $S^*(\gamma)$ and $c \geqq 0$. Theorem 1 provides a lower bound for the radius of $\beta$-starlikeness of $\mathscr{F}_1(\alpha, \gamma, c)$ and Theorem 3 gives the radius of $\beta$-starlikeness of $\mathscr{F}_2(\alpha, \gamma, c)$.

We begin by stating a slight generalization of the result obtained by Lewandowski et al mentioned above. Since our result follows directly from the techniques used in [8], the proof will be omitted.

LEMMA 1. *If $F(z)$ and $f(z)$ satisfy* (2), *$f(z)$ is in $S^*(\alpha)$ and $c \geqq 0$, then $F(z)$ is in $S^*(\alpha)$.*

This lemma now enables us to determine a lower bound for the radius of $\beta$-starlikeness of $\mathscr{F}_1(\alpha, \gamma, c)$.

THEOREM 1. *If $F(z)$ is in $\mathscr{F}_1(\alpha, \gamma, c)$, then $F(z)$ is $\beta$-starlike for $|z| < \sigma = \sigma(\alpha, \beta, \gamma, c)$, where $\sigma$ is the least positive root of the equation*

$$\begin{aligned} 1 - \beta - r[2(1-\alpha) + 2c(1-\gamma)] \\ - r^2[2\alpha - 1 - \beta + 2c(1-\gamma)] = 0\,. \end{aligned} \tag{5}$$

*Proof.* If $h(z) = [(c+1)/z^c]\int_0^z t^{c-1}f(t)dt$ then $F(z) = [z/g(z)]^c h(z)$ and Lemma 1 implies $h(z)$ is in $S^*(\alpha)$. Differentiating logarithmically and applying the usual inequalities we obtain

$$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geqq \frac{1+(2\alpha-1)r}{1+r} + \frac{2c(1-\gamma)r}{1-r}\,.$$

Thus $\operatorname{Re}\{zF'(z)/F(z)\} \geqq \beta$ whenever $|z| < \sigma$ where $\sigma$ is the least positive root of (5).

Before turning our attention to the principal result of this paper, we state without proof two lemmas which appear in [7] and are fundamental to what follows.

LEMMA 2. *If $\omega(z)$ is analytic and satisfies $|\omega(z)| \leqq |z|$ in $E$ and* *if* $p(z) = (1 + D\omega(z))/(1 + B\omega(z))$, $-1 \leqq D < B \leqq 1$, *then for* $|z| = r < 1$ *we have*

$$\operatorname{Re}\left\{\frac{z\omega'(z)}{(1 + D\omega(z))(1 + B\omega(z))}\right\}$$
$$\leqq \frac{-1}{(B - D)^2}\left[\operatorname{Re}\left\{\frac{D}{p(z)} + Bp(z)\right\} - \frac{r^2|Bp(z) - D|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|}\right] + \frac{B + D}{(B - D)^2} .$$

LEMMA 3. *If* $p(z)$ *and* $\omega(z)$ *satisfy the conditions of Lemma 2, then for any* $K \geqq B$ *we have on* $|z| = r$

$$\operatorname{Re}\left\{Kp(z) + \frac{D}{p(z)}\right\} - \left[\frac{r^2|Bp(z) - D|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}\right] \geqq \begin{cases} P_1(r) & for \quad R_0 \leqq R_1 \\ P_2(r) & for \quad R_0 \geqq R_1 \end{cases}$$

*where*

$$P_1(r) = P_1(r, K, B, D) = K\frac{1 + Dr}{1 + Br} + D\frac{1 + Br}{1 + Dr} ,$$
$$P_2(r) = P_2(r, K, B, D)$$
$$= \frac{2}{(1 - r^2)}[(1 + D)(1 + K - (B^2 + K + D(1 + K))r^2 + D(B^2 + K)r^4)]^{1/2} - \frac{2(1 - BDr^2)}{1 - r^2} ,$$
$$R_0^2 = [(1 + D)(1 - Dr^2)]/[(1 + K) - (K + B^2)r^2] ,$$

*and*

$$R_1 = (1 + Dr)/(1 + Br).$$

*The above estimates are sharp.*

THEOREM 2.

$$\min_{f \in \mathscr{F}_2(\alpha,\gamma,c)} \min_{|z|=r} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \begin{cases} Q_1(r) , & R_0 \leqq R_1 \\ Q_2(r) , & R_0 \geqq R_1 \end{cases}$$

*where*

$$Q_1(r) = \frac{1 + r(1 + 2D - K) + r^2D(2 - K)}{(1 + r)(1 + Dr)} , \tag{6}$$

$$Q_2(r) = \frac{2}{1 - D}\left[\left\{\frac{(1 + D)(1 + K)(1 - Dr^2)}{1 - r^2}\right\}^{1/2} - \frac{1 - Dr^2}{1 - r^2}\right] - \frac{K - 1 + 2D}{1 - D} , \tag{7}$$

$$R_0^2 = [(1 + D)(1 - Dr^2)]/[(1 + K)(1 - r^2)] , \tag{8}$$

$$R_1 = (1 + Dr)/(1 + r) , \tag{9}$$

$\delta = (\alpha + c\gamma)/(1 + c)$, $D = 2\delta - 1$, *and* $K = 1 + (c + 1)(1 - D)$.

*Proof.* Let $s(z) = z[F(z)/z]^{1/(c+1)}[g(z)/z]^{c/(c+1)}$ where in each multivalued expression we choose the branch which has value 1 at $z = 0$. Combining this with (4) yields

$$f(z) = [s(z)/z]^c zs'(s) . \tag{10}$$

Since

$$\frac{zs'(z)}{s(z)} = \frac{1}{(1 + c)}\left[\frac{zF'(z)}{F(z)} + c\frac{zg'(z)}{g(z)}\right],$$

$s(z)$ is in $S^*(\delta)$ for $\delta = (\alpha + c\gamma)/(1 + c)$, $p(z) = zs'(z)/s(z)$ is analytic in $E$, $p(0) = 1$ and $\operatorname{Re}[p(z)] \geqq \delta$, $z$ in $E$. Consequently, there exists a function $\omega(z)$ analytic in $E$ and satisfying $|\omega(z)| \leqq |z|$, $z \in E$, such that

$$p(z) = \frac{1 + D\omega(z)}{1 + \omega(z)} , \quad D = 2\delta - 1 . \tag{11}$$

Now differentiating (10) and making use of (11), we have

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= (c + 1)p(z) + \frac{zp'(z)}{p(z)} - c \\ &= (c + 1)p(z) + \frac{z\omega'(z)(D - 1)}{(1 + D\omega(z))(1 + \omega(z))} - c\end{aligned}$$

and Lemma 2 now yields

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1}{(1 - D)}\Bigg[&\operatorname{Re}\left\{Kp(z) + \frac{D}{p(z)}\right\} \\ &- \frac{r^2|p(z) - D|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|}\Bigg] - c - \frac{1 + D}{1 - D} ,\end{aligned}$$

where $B = 1$ and $K = 1 + (c + 1)(1 - D)$. An application of Lemma 3 now completes the proof. Sharpness follows directly from the sharpness of Lemma 3.

In [7] the radius of $\beta$-starlikeness of $\mathscr{F}_2(\alpha, \gamma, c)$ is determined in the case $c = 1$ and $\alpha + \gamma \leqq 1$. The following result extends this to include all permissible values of $\alpha$, $\gamma$ and $c$.

THEOREM 3. *Let* $r_* = r_*(\alpha, \gamma, c, \beta)$ *be the radius of* $\beta$*-starlikeness of* $\mathscr{F}_2(\alpha, \gamma, c)$. *Let* $D = 2\delta - 1$, $\delta = (\alpha + c\gamma)/(1 + c)$, $c \geqq 0$, $0 \leqq \alpha < 1$, *and* $0 \leqq \gamma \leqq 1$. *For each fixed* $c$ *in* $[0, \infty)$, *let* $r(D)$ *be the* *unique solution in* $(0, 1]$ *of the equation*

$$(12)\qquad (2+c)-(4-2D-2Dc+c)r-D(5-D+2c-Dc)r^2 + D(1-D-Dc)r^3=0\,.$$

*If* $Q_1(r)$ *and* $Q_2(r)$ *are defined by* (6) *and* (7) *and* $\mu(D)=Q_1(r(D))$, *then the equation* $\mu(D)=0$ *has a unique solution* $D_0$ *in* $(-1, 1)$. *Furthermore, if* $D$ *satisfies* $D_0 < D < 1$ *and* $0 \leqq \beta \leqq \mu(D)$, *then* $r_*$ *is the unique root in* $(0, 1)$ *of the equation* $Q_2(r)=\beta$. *For all other values of* $D$, $r_*$ *is the unique root in* $(0, 1)$ *of the equation* $Q_1(r)=\beta$.

*Proof.* Let $I(r)=\min_{f\in\mathscr{F}_2(\alpha,\gamma,c)}\min_{|z|=r}\mathrm{Re}\,\{zf'(z)/f(z)\}$ and let $R_0$ and $R_1$ be defined by (8) and (9). A differentiation shows $R_0$ is a decreasing function of $r$ and $R_1$ is an increasing function of $r$, hence the equation $R_0=R_1$ has a unique solution $r(D, c)$ which is the unique root in $(0, 1]$ of (12). Thus

$$I(r)=\begin{cases}Q_1(r) & 0\leqq r<r(D, c)\\ Q_2(r) & r(D, c)\leqq r<1\end{cases}\,,$$

with the understanding that the second inequality holds vacuously when $r(D, c)=1$. An examination of (12) shows this happens only when $D=-1$, in which case $r_*$ is the solution of $Q_1(r)=\beta$. Since $\alpha<1$ implies $D<1$, we can now restrict our attention to $D\in(-1, 1)$.

It follows from the minimum principle and the compactness of $\mathscr{F}_2(\alpha, \gamma, c)$ that $I(r)$ is a continuous, decreasing function of $r$. [In fact one can show $Q_1'(r(D, c))=Q_2'(r(D, c))$ so that $I(r)$ is differentiable and $I'(r)<0$ on $(0, 1)$.] Since $r(D, c)<1$ for $D>-1$, $\lim I(r)$ $(r\to 1^-)=\lim Q_2(r)$ $(r\to 1^-)=-\infty$, and, since $I(0)=1$, the equation $I(r)=\beta$ will always have a unique solution $r_*$ in $(0, 1)$. Clearly $r_*$ is always the solution of either $Q_1(r)=\beta$ or $Q_2(r)=\beta$, depending on the relationship between the roots of these equations and $r(D, c)$, or equivalently, on the relationship between $I(r(D, c))$ and $\beta$. The remainder of this argument is concerned with determining this relationship.

Let $c\in[0, \infty)$ be fixed, let $r(D)=r(D, c)$ and let $\mu(D)=Q_1(r(D))=Q_2(r(D))$. We will show $\mu(D)$ is a strictly increasing function of $D$ mapping $(-1, 1)$ onto $(-\infty, 1)$. Now

$$\mu'(D)=\frac{d}{dD}Q_1(r(D))>0$$

if and only if

$$(13)\qquad r'(D)<\frac{r(D)(1+r(D))}{1-D}\cdot\frac{(1+c)(1+Dr(D))^2+1+r(D)}{(1+c)(1+Dr(D))^2+1-Dr(D)^2}\,.$$

Since the second factor in the right hand side of (13) is clearly greater than 1, it is sufficient to show

$$r'(D) < r(D)(1 + r(D))/(1 - D) . \tag{14}$$

Differentiating (12) implicitly yields

$$\begin{aligned} r'(D) = {} & [2(1 + c)r(D) + (2D - 5 - 2c + 2Dc)r(D)^2 \\ & + (1 - 2D - 2Dc)r(D)^3]/[(4 - 2D - 2Dc - c) \\ & + 2D(5 - D + 2c - Dc)r(D) - 3D(1 - D - Dc)r(D)^2] , \end{aligned} \tag{15}$$

and, before substituting (15) in (14), we must determine the sign of the denominator in (15). Let

$$\begin{aligned} p(r) &= (1 + K)(1 - r)(1 + r)^2(R_0^2 - R_1^2) \\ &= (2 + c) - (4 - 2D - 2Dc + c)r \\ &\quad - D(5 - D - 2c - Dc)r^2 + D(1 - D - Dc)r^3 \end{aligned}$$

so that $p(r(D)) = 0$ and the denominator in (15) is $-p'(r(D))$. Since $R_0$ is decreasing and $R_1$ is increasing, $p(r)$ changes sign at $r(D)$ and must have a zero of order 1 or 3 at $r(D)$. If $r(D)$ is a root of order 3 then $p''(r(D)) = 0$ which implies

$$r(D) = (5 + 2c - D - Dc)/(3(1 - D - Dc)) .$$

However this last expression is not in $(0, 1)$ for $D \in (-1, 1)$ and $c \in [0, \infty)$, hence $r(D)$ is a root of order 1 and, since $P(r)$ is decreasing at $r(D)$, $p'(r(D)) < 0$. Thus the denominator in (15) is positive and substituting (15) in (14) then shows that (14) is equivalent to

$$\begin{aligned} & (2 - c) + (9 + D + 3c - 2Dc)r(D) \\ & \quad + (6Dc - D^2c + 10D - 1 - D^2)r(D)^2 \\ & \quad - 3D(1 - D - Dc)r(D)^3 > 0 . \end{aligned} \tag{16}$$

Using the fact that $r(D)$ satisfies (12) to elminate $r(D)^3$ in (16), we find that (16) is equivalent to

$$\begin{aligned} & r(D)(7 - 5r(D))(D + 1) + (8 - 10r(D) + 4r(D)^2) \\ & \qquad + 2D^2r(D)^2 + 2c(1 + Dr(D))^2 > 0 , \end{aligned}$$

which is obviously valid for $r(D)$ in $(0, 1)$, $D$ in $(-1, 1)$ and $c \geqq 0$. Thus $\mu(D)$ is increasing on $(-1, 1)$.

An examination of (12) shows $r(D) \to 1$ when $D \to 1$ or $D \to -1$, hence $\mu(D) \to -\infty$ as $D \to -1$, $\mu(D) \to 1$ as $D \to 1$, and the equation $\mu(D) = 0$ has a unique solution $D_0$ in $(-1, 1)$. If $-1 < D \leqq D_0$, then $\mu(D) = Q_1(r(D)) \leqq 0$ and $r_*$ is the root of $Q_1(r) = \beta$. If $D_0 < D < 1$, then $r_*$ is the root of $Q_1(r) = \beta$ when $\mu(D) \leqq \beta$ and $r_*$ is the root of $Q_2(r) = \beta$ when $\beta \leqq \mu(D)$. This completes the proof.

If we take $c = \gamma = 1$ and $\alpha = \beta = 0$, then we obtain as a special case Livingston's result [11]. If we let $\gamma = 1$ and $\alpha = \beta = 0$, then we obtain Theorem 1 in [4]. Letting $c = \gamma = 1$ yields results found in [1], [2], [10], [13] and, as we have already noted, the case $c = 1$ and $\alpha + \gamma \leqq 1$ appears in [7].

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National College
Tirchirapalli 620001, India
and
Marquette University
Milwaukee, WI 53233

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