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**BOREL SELECTORS FOR SEPARATED QUOTIENTS**

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**We use a nonstandard version of the Kuratowski, Ryll-Nardzewski general selection theorem to establish the existence of Borel measurable selectors for certain equivalence relations on Polish spaces.**

The two main results of this paper establish the existence of Borel measurable selectors of determined complexity for certain equivalence relations on Polish spaces.

**THEOREM A.** *Let  $G$  be a Polish topological group which acts continuously on a Polish space  $Y$  inducing an equivalence  $E$  on  $Y$ . Suppose  $A \subseteq Y$  is an invariant Borel set of ambiguous class  $\alpha \geq 0$  such that quotient Borel space  $A/E$  is countably separated by projections of invariant sets of ambiguous class  $\alpha$ . Then there is an  $\alpha$ -Borel measurable selector function for  $E$  on  $A$ .*

Given  $\gamma \in \omega_1$ , let  $\gamma^* = \sup(\gamma + \beta : \beta < \gamma)$ , so  $\gamma^* = \gamma + \beta$  when  $\gamma = \beta + 1$ ,  $\gamma^* = \gamma \cdot 2$  when  $\gamma$  is a limit ordinal.

**THEOREM B.** *Let  $Y$  be a Polish space and let  $E$  be an equivalence on  $Y$  whose equivalence classes are  $G_\delta$  sets. Suppose that the  $E$ -saturation of each open set in a given basis is of ambiguous class  $\gamma > 0$ . Then there is a  $\gamma^*$ -Borel selector function for  $E$  on  $Y$ .*

This work was stimulated by a recent paper of Burgess [1] in which some techniques from Vaught [11] were used to obtain a Borel selector in the case  $A = Y$  of Theorem A. The selector in [1] is obtained via an application of Suslin's theorem, so its complexity is very hard to estimate. We will use Vaught's method in a different way to provide a more direct (and simpler) construction in which the complexity of the selector is apparent.

Burgess's result extends a theorem of E. Effros [2] on locally compact groups. Theorem A will be proved as a special case of a more general result (3.2) about Borel actions. The existence of a Borel selector for noncontinuous actions appears to be new in all cases.

The existence of a Borel selector (of unknown complexity) under the hypothesis of Theorem B is a recent result of S. M. Srivastava [10]. The special case,  $\gamma = 1$ , was previously established by the present author in [8]. These results extend earlier work by Kallman and Mauldin [3] and Kuratowski and Maitra [5]. Theorem B is a

special case of a slightly stronger result (3.4) with a weaker requirement on the equivalence classes.

Our arguments are based on a "nonstandard" version of the well-known general theorem on selectors of Kuratowski and Ryll-Nardzewski. This result is proved in §1. Section 2 collects some material concerning topologies generated by Borel sets and the main results are proved in §3.

Throughout the paper we reserve boldface notation ( $Y$ , etc.) for topological spaces. Formally, we regard  $Y$  as a pair  $(Y, \mathcal{T})$  where  $Y$  is the underlying set and  $\mathcal{T}$  is the collection of open subsets of  $Y$ . However, we make free use of the common identification of  $Y$  with  $Y$  when no confusion will result (as in the previous sentence).  $X \times Y$  is  $X \times Y$  with the product topology;  $A \subseteq Y$  is  $A$  with the relative topology;  $2^\omega$  is  $2^\omega$  with the product topology (i.e., the Cantor space).

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1. Measurable choice functions. Suppose  $Y = (Y, \mathcal{T})$  is a topological space,  $X$  is an arbitrary set, and  $F$  is a function on  $X$  to the power set  $\mathcal{P}(Y)$ . A *choice function* for  $F$  is a map  $f: X \rightarrow Y$  such that  $f(x) \in F(x)$  for each  $x \in X$ . (We reserve the common term "selector" for its other meaning connected with equivalence relations.) Given  $\mathcal{D} \subseteq \mathcal{P}(X)$ , we say that  $f$  is  $\mathcal{D}$ -measurable provided  $f^{-1}(O) \in \mathcal{D}$  whenever  $O \in \mathcal{T}$ .  $\mathcal{D}_\sigma$  is the countably additive family generated by  $\mathcal{D}$ .

An operator  $\#: \mathcal{T} \rightarrow \mathcal{P}(X)$  is countably additive provided  $\bigcup_{i \in \omega} (O_i^*) = (\bigcup_{i \in \omega} O_i)^*$  whenever  $\{O_i: i \in \omega\} \subseteq \mathcal{T}$ .  $\#$  is a semisaturation operator for  $F$  provided

- (i)  $Y^* = X$
- (ii)  $O^* \subseteq \{x: F(x) \cap O \neq \emptyset\}$  for each  $O \in \mathcal{T}$ .

The simplest example of an additive semisaturation operator is obtained by setting  $O\# = O^+ = \{x: F(x) \cap O \neq \emptyset\}$ . Two further examples will play essential roles in §3. The standard case " $\# = +$ " of Theorem 1.1 is the main result of Kuratowski and Ryll-Nardzewski [6]. Our proof is abstracted from theirs.

**THEOREM 1.1.** *Assume that  $Y$  is a Polish space,  $F$  is function with domain  $X$  whose values are closed subsets of  $Y$  and  $\#: \mathcal{T} \rightarrow \mathcal{P}(X)$  is a countably additive, semisaturation operator for  $F$ . Suppose that  $\mathcal{D}$  is a field of subsets of  $X$  such that for every  $O \in \mathcal{T}$ ,  $O^* \in \mathcal{D}_\sigma$ . Then there is a  $\mathcal{D}_\sigma$ -measurable choice function for  $F$ .*

*Proof.* For  $U \subseteq Y$ , let  $\bar{U}$  be its closure. Fix a complete metric

$d$  for  $Y$  which is bounded by 1, and let  $\mathcal{H}$  be a countable basis for  $Y$  with  $Y \in \mathcal{H}$ . We also fix an enumeration of  $\omega \times \omega$  as  $\{(a_i, b_i): i \in \omega\}$ .

We enumerate  $\mathcal{H}$  as  $\{U_n: n \in \omega\}$  such that  $U_0 = Y$  and any singletons are listed infinitely often. For each  $n$ , we choose a sequence  $\{B_{n,k}: k \in \omega\} \subseteq \mathcal{D}$  such that  $U_n^\# = \bigcup_k B_{n,k}$ . We also define

$$I_n = \{m > n: \text{diameter}(\bar{U}_m) < 1/2n \text{ and } \bar{U}_m \subseteq U_n\}.$$

It follows that  $U_n = \bigcup_{m \in I_n} U_m$ .

For each  $x \in X$ , we define, by induction on  $n$ , a sequence  $\{p_n(x): n \in \omega\}$  satisfying:

- (a)  $p_n(x) \geq n$  and  $p_{n+1}(x) \in I_{p_n}(x)$
- (b)  $x \in U_{p_n(x)}^\#$ .

The construction is started by setting  $p_0(x) = 0$ . Now assume that  $p_n(x) = p_n$  is given; we show how to define  $p_{n+1}(x)$ .

By condition (b) and the fact that  $\#$  is countably additive,  $x \in \bigcup_{m \in I_{p_n}} U_m^\# = \bigcup_{m \in I_{p_n}} \bigcup_{k \in \omega} B_{m,k}$ . Let  $i$  be the first natural number such that  $a_i \in I_{p_n}$  and  $x \in B_{a_i, b_i}$ . We set  $p_{n+1}(x) = a_i$ . Clearly (b) holds and since  $a_i \in I_{p_n}$ ,  $a_i > p_n \geq n$  so (a) holds as well.

Now fix  $x$  and write  $p_k = p_k(x)$  for each  $k \in \omega$ . By (a) and the definition of the sets  $I_n$ , we have  $U_{p_0} \supseteq \bar{U}_{p_1} \supseteq U_{p_1} \supseteq \bar{U}_{p_2} \supseteq \dots$  with each  $\text{diam}(\bar{U}_{p_k}) < 1/k$ . It follows that  $\bigcap_{k \in \omega} U_{p_k}$  is a singleton. We define  $f(x)$  to be the unique element of  $\bigcap_k U_{p_k}$ . We must show that  $f$  has the required properties.

By (b), for each  $k$ ,  $x \in U_{p_k}^\# \subseteq U_{p_k}^+$ . Thus, each  $F(x) \cap U_{p_k} \neq \emptyset$ . It follows that  $f(x)$  is the limit of a convergent sequence from  $F(x)$ . Since  $F(x)$  is closed,  $f(x) \in F(x)$ . Thus  $f$  is a choice function for  $F$ .

It remains to show that  $f$  is  $\mathcal{D}_\sigma$ -measurable. First we prove a lemma:

$$(1) \quad \text{For each } n, f^{-1}(U_n) = \bigcup_{m \in I_n} \bigcup_{k \leq m} \{x: m = p_k(x)\}.$$

Inclusion from right to left is obvious since  $m = p_k(x)$  implies  $f(x) \in U_m \subseteq U_n$ . For the reverse inclusion suppose  $f(x) \in U_n$ . Since  $U_n$  is open we may choose  $k > n$  such that  $\{y: d(f(x), y) < 1/k\} \subseteq U_n$ . Since  $\text{diam}(U_{p_k}(x)) < 1/k$  and  $f(x) \in U_{p_k(x)}$ , we have  $U_{p_k(x)} \subseteq U_n$ . Thus  $p_k(x) \in I_n$  and the lemma is established.

Now set  $C_{k,m} = \{x: m = p_k(x)\}$ . By (1) it suffices to show that each  $C_{k,m}$  belongs to  $\mathcal{D}_\sigma$ . This is easily checked by induction on  $k$  using the assumption that  $\mathcal{D}$  is a field of sets, together with the observation:

$$p_{k+1}(x) = m \text{ if and only if}$$

$$(2) \quad (\exists r < m)[m \in I_r \text{ and } p_k(x) = r \text{ and } (\exists i)(m = a_i \text{ and } x \in B_{a_i, b_i} \text{ and } (\forall j < i)(a_j \notin I_r \text{ or } x \notin B_{a_j, b_j}))]. \quad \square$$

REMARK. Note that if we drop the assumption that each  $F(x)$  is closed, the construction still provides a map  $f$  with the property that  $f(x) \in F(x)$  whenever  $F(x)$  closed.

2. **Borel-generated topologies.** The second set of observations that we require deals with the notion of Borel generated topologies on Polish spaces. It is well-known that if  $B$  is a Borel subset of a Polish space  $Y$ , then there is a topology on  $Y$  compatible with the Borel structure on  $Y$  in which  $B$  is both closed and open (cf. [4], p. 448). In this section we examine this remark with special concern for the Borel complexity of the sets generating the new topology on  $Y$ . Our approach is closely akin to the techniques from mathematical logic of closing a set of formulas under subformulas and of adding Skolem predicates to a language.

Throughout this section,  $Y = (Y, \mathcal{T})$  is a fixed Polish space.  $\mathcal{H}$  is a fixed countable basis for  $Y$  and  $\mathcal{B}$  is the collection of Borel subsets of  $Y$ .  $S_\alpha$ ,  $M_\alpha$ , and  $A_\alpha$  respectively denote the  $\alpha$ th additive, multiplicative and ambiguous classes in the Borel hierarchy on  $Y$  (so  $S_0 = \mathcal{T}$ ,  $S_1 = F_\sigma$ ,  $A_1 = F_\sigma \cap G_\delta$ , etc.). A function  $s: Y \rightarrow Z$  is a  $\alpha$ -Borel provided  $S^{-1}(0) \in S_\alpha$  whenever  $0$  is open in  $Z$ .

For any Borel set  $B$  define  $r(B)$  to be the least ordinal  $\alpha$  such that  $B \in A_\alpha$ . We say that a collection  $\mathcal{C}$  of Borel sets is *supported* provided  $\mathcal{C}$  is closed under complementation and for all  $B \in \mathcal{C}$ , if  $r(B) = \alpha > 1$  then there exist collections  $\{B_{ij}: i, j \in \omega\}$  and  $\{C_{ij}: i, j \in \omega\}$  included in  $\mathcal{C} \cap \bigcup_{\beta < \alpha} A_\beta$  such that

$$B = \bigcap_i \bigcup_j B_{ij} = \bigcup_i \bigcap_j C_{ij}.$$

LEMMA 2.1. *Suppose  $\alpha \geq 1$ . Let  $\mathcal{S}$  be a countable collection of  $A_\alpha$  sets which is closed under complementation. Then there exists a countable collection  $\mathcal{C} \subseteq \bigcup_{\beta < \alpha} A_\beta$  such that  $\mathcal{C} \cup \mathcal{S}$  is supported.*

*Proof.* We proceed by induction on  $\alpha$ . The case  $\alpha = 1$  is treated trivially by setting  $\mathcal{C} = \emptyset$ .

Let  $\alpha$  and  $\mathcal{S}$  be given with  $\alpha > 1$  and  $\mathcal{S} = \{R_n: n \in \omega\}$ . Assume that the lemma holds for all  $\beta < \alpha$ . For each  $n \in \omega$  choose sets  $B_{ij}^n, C_{ij}^n \in \bigcup_{\beta < \alpha} A_\beta (i, j \in \omega)$  such that

$$R_n = \bigcap_i \bigcup_j B_{ij}^n = \bigcup_i \bigcap_j C_{ij}^n.$$

Applying the induction hypothesis, choose for each triple  $n, i, j$  a supported collection  $\mathcal{C}_{ij}^n \subseteq \bigcup_{\beta < \alpha} A_\beta$  such that  $B_{ij}^n, C_{ij}^n \in \mathcal{C}_{ij}^n$ . It is evident that  $\mathcal{C} = \bigcup_n \bigcup_i \bigcup_j \mathcal{C}_{ij}^n$  has the required property.  $\square$

LEMMA 2.2. *Suppose  $\mathcal{S} \subseteq \mathcal{B}$  is countable and supported. Let*

$\mathcal{T}'$  be the topology generated by  $\mathcal{S} \cup \mathcal{T}$ . Then  $Y' = (Y, \mathcal{T}')$  is Polish.

*Proof.* Enumerate  $\mathcal{S}$  as  $\{B_i: i \in \omega\}$ . Given  $y \in Y$  define  $\xi_y \in 2^\omega$  by the equation  $\xi_y(i) = 1 \Leftrightarrow y \in B_i$ . Define  $G = \{(y, \xi_y): y \in Y\} \subseteq Y \times 2^\omega$ . The map  $f: Y' \rightarrow G$  defined by  $f(y) = (y, \xi_y)$  is apparently one-to-one and onto. Since for each  $i$ ,

$$f(B_i) = G \cap \{(y, \xi): \xi(i) = 1\}$$

and for each  $O \in \mathcal{T}$ ,  $f(O) = G \cap \{(y, \xi): y \in O\}$ ,  $f$  is open. Since  $f^{-1}(O \times \{(y, \xi): \xi(i) = k\})$  is either  $O \cap B_i$  or  $O \cap (\sim B_i)$  (as  $k = 0$  or  $1$ ) and  $\mathcal{S}$  is closed under complementation,  $f$  is continuous. Thus,  $Y'$  is homeomorphic to  $G$ . Since any  $G_\delta$  subspace of a Polish space is Polish, it suffices to show that  $G$  is  $G_\delta$  in  $Y \times 2^\omega$ .

For each  $i \in \omega$  we define a  $G_\delta$  set  $G_i$  as follows: If  $r(B_i) > 1$  we use the fact that  $\mathcal{S}$  is supported to choose functions  $p_i, q_i$  such that

$$B_i = \bigcap_k \bigcup_1 B_{p_i(k,1)} = \sim \bigcap_k \bigcup_1 B_{q_i(k,1)}$$

with each  $r(B_{p_i(k,1)}), r(B_{q_i(k,1)}) < r(B_i)$ . In this case we define

$$G_i = \{(y, \xi): \xi(i) = 1 \text{ and } (\forall k)(\exists l)(\xi(p_i(k, l) = l))\} \\ \cup \{(y, \xi): \xi(i) = 0 \text{ and } (\forall k)(\exists l)(\xi(q_i(k, l) = l))\}.$$

If  $r(B_i) \leq 1$  we choose open sets  $U_k, V_k$  such that

$$B_i = \bigcap_k U_k = \sim \bigcap_k V_k,$$

then define

$$G_i = \{(y, \xi): \xi(i) = 1 \text{ and } (\forall k)y \in U_k\} \\ \cup \{(y, \xi): \xi(i) = 0 \text{ and } (\forall k)y \in V_k\}.$$

A straightforward induction on  $r(B_i)$  shows

- (a) For all  $y \in Y$ ,  $(y, \xi_y) \in G_i$  and
- (b) For all  $(y, \xi) \in \cap G_i$ ,  $\xi(i) = \xi_y(i)$ .

Thus  $G = \bigcap_i G_i$  and the lemma is proved.  $\square$

Combining 2.1 and 2.2 we obtain

**THEOREM 2.3.** *Suppose  $\mathcal{S} \subseteq A_\alpha$  is countable and closed under complementation with  $\alpha \geq 1$ . Then there is a countable collection  $\mathcal{C} \subseteq \bigcup_{\beta < \alpha} A_\beta$ , such that  $\mathcal{H} \cup \mathcal{S} \cup \mathcal{C}$  generates a Polish topology on  $Y$ .*

**3. Selectors.** Suppose  $E$  is an equivalence relation on a space  $A$ . We write  $A/E$  for the set of  $E$ -equivalence classes and

denote the associated projection map  $\pi: y \mapsto [y]$ . A *selector* for  $E$  on  $A$  is a map  $s: A \rightarrow A$  satisfying (i)  $(\forall y)(s(y) \in [y])$  and (ii)  $(\forall y_1 y_2)(y_2 \in [y_1] \Rightarrow s(y_1) = s(y_2))$ . A *transversal* for  $E$  is a set which intersects each equivalence class in a singleton. Note that if  $s$  is a selector for  $E$ , then the fixed points of  $s$  form a transversal  $T_s$  for  $E$ . If  $s$  is  $\alpha$ -Borel, then  $T_s$  is an  $M_\alpha$  subset of  $A$ . Given  $B \subseteq A$ , we say that a collection  $\mathcal{S}$  of  $E$ -invariant sets *separates*  $B/E$  provided  $(\forall y \in B)([y] = \bigcap \{S \in \mathcal{S} : [y] \subseteq S\})$ .

As promised, we will use the constructions of §1 and §2 to obtain two results on the existence of Borel selectors for certain equivalence relations on Polish spaces. In each case we introduce a Borel-generated topology on the relevant space  $A$  in which the equivalence classes are closed. We then consider the identity map  $F: A/E \rightarrow \mathcal{P}(A)$  and introduce a semisaturation operator derived from the constructions of Vaught [11]. Theorem 1.1 provides a choice function  $f$  for  $I$ ,  $f \circ \pi$  is the required selector.  $\square$

**A. Borel actions.** Throughout this subsection,  $G$  is a nonmeager topological group with a countable basis. In the most notable case,  $G$  is Polish.  $Y$  is a Polish space and  $J: G \times Y \rightarrow Y$  is a Borel measurable function. For  $g \in G$ ,  $J^g$  is the function  $y \mapsto J(g, y) = gy$ . We assume that  $J$  defines an *action* of  $G$  on  $Y$ , i.e., that the map  $g \mapsto J^g$  is a homomorphism from  $G$  to the group of permutations of  $Y$ . The action induces the equivalence relation  $E_J = \{y, gy\} : y \in Y, g \in G\}$  on  $Y$ .

Following Vaught [11] we write, for  $B \subseteq Y$ ,  $y \in Y$   $B^y = \{g : gy \in B\}$ ,  $B^+ = \{y : B^y \neq \emptyset\}$ ,  $B^d = \{y : B^y \text{ is meager}\}$ .

It follows from the definitions and the fact that  $G$  is a Baire space that for  $A, B, B_i \subseteq Y$ .

- (2)  $B^d$  is  $E_J$ -invariant and  $B^d \subseteq B^+$ .
- (3)  $(\bigcup_{i \in \omega} B_i)^d = \bigcup_{i \in \omega} B_i^d$ .
- (4) If  $A$  is invariant then  $A^d = A$  and  $(A \cap B)^d = A \cap B^d$ .

The following lemma is proved in [11] for the case  $\gamma = 0$  and in [7] for the general case. (In [7] it is asserted only that  $B^d$  is Borel, but the argument establishes the stronger result 3.1.)

**LEMMA 3.1.** *Suppose  $J$  is  $\gamma$ -Borel and  $B \in S_\beta$ . Then  $B^d \in S_{\gamma+\beta}$ .*

**THEOREM 3.2.** *Let  $\alpha \geq 1, \gamma \geq 0$ . Assume that  $J$  is a  $\gamma$ -Borel action on  $Y$ . Suppose  $A \subseteq Y$  is an invariant  $A_\alpha$  set and that  $\mathcal{S}$  is a countable collection of invariant  $A_\alpha$  sets which separates  $A/E_J$ . Then there exists a  $\gamma + \alpha$ -Borel selector for  $E$  on  $A$ .*

*Proof.* We assume without loss of generality that  $A \in \mathcal{S}$  and

that  $\mathcal{S}$  is closed under complementation and finite intersection. By 2.3 we can choose a countable set  $\mathcal{C}' \subseteq \bigcup_{\beta < \alpha} A_\beta$  such that  $\mathcal{S} \cup \mathcal{C}' \cup \mathcal{H}$  generates a Polish topology  $\mathcal{S}'$  on  $Y$ . Write  $Y' = (Y, \mathcal{S}')$ . Let  $\mathfrak{B}$  be the closure of  $\mathcal{C} \cup \mathcal{H}$  under finite intersections. Then  $Y'$  has the basis  $\mathcal{K}' = \{B \cap S : B \in \mathfrak{B}, S \in \mathcal{S}\}$ .

Let  $A'$  be  $A$  with the relative topology from  $Y'$ .  $A'$  has the basis  $\mathcal{K} = \{K \in \mathcal{K}' : K \subseteq A\}$ . Since  $A$  is closed in  $Y'$ ,  $A'$  is Polish.

Set  $X = A/E$  and let  $F$  be the identity function from  $X$  to  $\mathcal{P}(A')$ . For  $O \subseteq A'$  define  $O^\# = \pi(O^\Delta)$ . It follows from (2) and (4) that  $\#$  is a semisaturation operator for  $F$  and from (3) that  $\#$  is countably additive.

Let  $\mathcal{D}$  be the field generated by  $\{K^\Delta : K \in \mathcal{K}\}$  and let  $\mathcal{D}' = \{\pi^{-1}(D) : D \in \mathcal{D}\}$ . Since each  $K^\Delta$  is invariant,  $\mathcal{D}'$  is just the field generated by  $\{K^\Delta : K \in \mathcal{K}\}$ . Each  $K \in \mathcal{K}$  has the form  $B \cap S$  with  $S$ -invariant  $A_\alpha$  and  $B \in \bigcup_{\beta < \alpha} S_\beta$ ; by (4) and 3.1,  $K = B \cap S \in A_{\gamma+\alpha}$ .

It follows that

$$(5) \quad \mathcal{D}' \subseteq A_{\gamma+\alpha}.$$

Since  $\mathcal{S}$  separates  $A/E_j$ , each  $[y]$  is closed in  $A'$ . Thus, we may apply 1.1 to obtain a  $\mathcal{D}_o$ -measurable choice function  $f: A/E_j \rightarrow A'$ . Define  $s = f \circ \pi$ . Clearly,  $s$  is a selector for  $E$  on  $A$ . We claim that  $s$  is  $\gamma + \alpha$ -Borel with respect to  $A$ . Since  $A'$  refines  $A$  it suffices to establish:

$$(6) \quad s \text{ is a } \gamma + \alpha\text{-Borel map from } A \text{ to } A'.$$

Consider an arbitrary open set  $O \subseteq A'$ . Since  $f$  is  $\mathcal{D}_o$ -measurable we may choose  $D_i \in \mathcal{D}$  such that  $f^{-1}(O) = \bigcup_{i \in \omega} D_i$ . Then  $s^{-1}(O) = \pi^{-1}(f^{-1}(O)) = \bigcup_{i \in \omega} \pi^{-1}(D_i)$ . By (5) each  $\pi^{-1}(D_i) \in A_{\gamma+\alpha}$  so  $s^{-1}(O) \in S_{\gamma+\alpha}$  as required.  $\square$

**B. Equivalence relations with relatively large equivalence classes.** Assume throughout this subsection that  $E$  is an equivalence relation on a Polish space  $Y$ , such that each equivalence class is Borel in  $Y$  and a Baire space in its relative topology. In the most notable special case, each equivalence is  $G_\delta$  in  $Y$ . For  $B \subseteq Y$  we continue to write  $B^+ = \{y : [y] \cap B \neq \emptyset\}$ .

$\mathcal{H}$  is a fixed countable basis for  $Y$  with  $Y \in \mathcal{H}$ .  $U, V$  will always denote elements of  $\mathcal{H}$ . We write

$$\mathcal{S} = \{U^+ : U \in \mathcal{H}\} \cup \{\sim U^+ : U \in \mathcal{H}\}.$$

LEMMA 3.3 (Compare [8] Lemma 2.)  $\mathcal{S}$  separates  $Y/E$ .

*Proof.* Given  $y \in Y$  let



$$\mathcal{S}_y = \{U^+ : U \cap [y] \neq \emptyset\} \cup \{\sim(U^+) : U \cap [y] = \emptyset\}.$$

We must show  $[y] = \cap \mathcal{S}_y$ .

Inclusion from left to right is immediate from the definition of  $\mathcal{S}_y$ . For the reverse inclusion suppose  $x \notin [y]$ .

There are two cases to consider.

*Case 1.*  $[x] \not\subseteq \overline{[y]}$  (the closure of  $[y]$ ).

Let  $z \in [x]$ ,  $z \notin \overline{[y]}$ . Then for some  $U \in \mathcal{H}$ ,  $z \in U$  and  $U \cap [y] = \emptyset$ . Then  $\sim(U^+) \in \mathcal{S}_y$  and  $x \notin \sim(U^+)$  so  $x \notin \cap \mathcal{S}_y$ .

*Case 2.*  $[x] \subseteq \overline{[y]}$ .

We claim that  $[x]$  is not dense in  $[y]$ . This suffices since we then obtain  $U \in \mathcal{H}$  with  $U \cap [y] \neq \emptyset$ ,  $U \cap [x] = \emptyset$  so  $U^+ \in \mathcal{S}_y$  and  $x \notin U^+$ . To verify the claim suppose *arguendo* that  $[x]$  is dense in  $\overline{[y]}$ , i.e.,  $\overline{[y]} = \overline{[x]}$ . Let  $C = \overline{[y]}$ . Since  $[x]$  is a Baire space,  $[x]$  is not meager in itself, and hence  $[x]$  is not meager in  $C$ . Since  $[x]$  is Borel,  $[x]$  is almost open in  $C$  so  $[x] \cap U \cap C$  is comeager in  $U \cap C$  for some  $U$ . Since  $[y]$  is dense in  $C$ ,  $[y] \cap U \cap C \neq \emptyset$ . Since  $[y]$  is a Baire space,  $[y] \cap U \cap C$  is not meager in  $C$ . But  $[y] \cap U \cap C \subseteq U \cap C \sim [x]$  which is meager in  $C$ . This contradiction establishes the claim and, thereby, the lemma.  $\square$

When  $\mathcal{S} \subseteq A_\gamma$  we say that  $E$  is a  $\gamma$ -decomposition for  $(Y, \mathcal{H})$ . We assume that this is the case,  $\gamma \geq 0$ . Thus we are in a situation similar to subsection A.

We need an operator analogous to Vaught's  $\Delta$ . Our solution is a kind of local version of the transform, cf. the remarks in [8] and [9] where such a version was described. For  $B \subseteq Y$ ,  $U \in \mathcal{H}$ , we define

$$B^{4U} = \{y : U \cap B \cap [y] \text{ is not meager in } [y]\}, \quad B^d = B^{4Y}.$$

We have for  $B, B_i$  Borel in  $Y$  (and, by convention,  $U, V \in \mathcal{H}$ )

$$(7) \quad B^{4U} \subseteq (B \cap U)^+$$

$$(8) \quad V^{4U} = (U \cap V)^+$$

$$(9) \quad \left( \bigcup_{i \in \omega} B_i \right)^{4U} = \bigcup_{i \in \omega} B_i^{4U}$$

$$(10) \quad (\sim B)^{4U} = \bigcup_{V \subseteq U} (V^+ \sim B^{4V})$$

$$(11) \quad B^d \text{ is invariant and if } S \text{ is invariant then } (S \cap B)^d = S \cap B^d.$$

The proofs of (7)-(11) are essentially similar to corresponding arguments in [11]. For completeness we indicate them here.

For (7): The empty set is meager in any space so a nonmeager subset of  $[y]$  is nonempty. Thus, for any  $y$ ,  $y \in B^U \Rightarrow U^d \cap B \cap [y] \neq \emptyset$ .

For (8): By (7) it suffices to check inclusion from right to left. Since  $[y]$  is a Baire space, any nonempty open subset of  $[y]$  is nonmeager. Thus,  $U \cap B \cap [y] \neq \emptyset \Rightarrow y \in B^{dU}$ .

For (9): Using the fact that a countable union of meager sets is meager, note the equivalence  $y \in (\bigcup_i B_i)^{dU} \Leftrightarrow U \cap \bigcup_i B_i \cap [y]$  is nonmeager  $\Leftrightarrow (\exists i)(U \cap B_i \cap [y]$  is nonmeager)  $\Leftrightarrow y \in \bigcup_i (B_i^{dU})$ .

For (10): Since  $B$  is a Borel set,  $[y] \cap B$  is almost open in  $[y]$  for each  $y$ ; say  $B \cap [y]$  is congruent to  $C_y$  modulo a  $[y]$ -meager set, where  $C_y$  is closed in  $[y]$ . Let  $[y] \sim C_y = O_y \cap [y]$  where  $O_y$  is open in  $Y$ . Then making repeated use of the fact that  $[y]$  is a Baire space and  $\mathcal{H}$  is a basis, we compute

$$\begin{aligned}
 y \in (\sim B)^{dU} &\iff (U \sim B) \cap [y] \text{ is nonmeager in } [y] \\
 &\iff (U \cap O_y) \cap [y] \text{ is nonmeager in } [y] \\
 &\iff (U \cap O_y \cap [y]) \text{ is nonempty} \\
 &\iff (\exists V \subseteq U)(V \cap [y] \neq \emptyset \text{ and } V \subseteq O_y) \\
 &\iff (\exists V \subseteq U)(V \cap [y] \neq \emptyset \text{ and } V \cap C_y = \emptyset) \\
 &\iff (\exists V \subseteq U)(V \cap [y] \neq \emptyset \text{ and } V \cap C_y \text{ is meager in } [y]) \\
 &= (\exists V \subseteq U)(V \cap [y] \neq \emptyset \text{ and } V \cap B \text{ is meager in } [y]) \\
 &= y \in \bigcup_{V \subseteq U} (V^+ \sim (B^V)) .
 \end{aligned}$$

For (11): The invariance of  $B^d$  (and of any  $B^{dU}$ ) is apparent from the definition. If  $S$  is invariant, then for any  $y$ , either  $[y] \cap S = \emptyset$  or  $[y] \subseteq S$ . Thus,

$$\begin{aligned}
 y \in (S \cap B)^d &\iff S \cap B \cap [y] \text{ is nonmeager in } [y] \\
 &\iff [y] \subseteq S \text{ and } B \cap [y] \text{ is nonmeager in } [y] \\
 &\iff y \in S \text{ and } y \in B^d .
 \end{aligned}$$

Corresponding to 3.1 we have the somewhat stronger:

**LEMMA 3.3.** *For all  $\alpha$ , if  $B \in S_\alpha$  then  $B^{dU}$  is a countable union of invariant  $A_{\gamma+\alpha}$  sets.*

*Proof.* We proceed by induction on  $\alpha$ .

For  $\alpha = 0$  note that any open set  $O$  can be written  $O = \bigcup_{i < \omega} V_i$  with each  $V_i \in \mathcal{H}$ . Then by (9) and (8),  $O^{dU} = (\bigcup_i V_i)^{dU} = \bigcup_i (V_i^{dU}) = \bigcup_i (V_i \cap U)^+$ . Since  $\mathcal{H}$  is a basis,  $V_i \cap U \in \mathcal{H}$ . Since  $E$  is a  $\gamma$ -decomposition for  $(Y, \mathcal{H})$ , each  $(V_i \cap U)^+ \in A_\gamma$ . For  $\alpha \geq 1$ , write  $B = \bigcup_{i \in \omega} \sim B_i$  with each  $B_i \in \bigcup_{\beta < \alpha} S_\beta$  and note that

$$B^{4U} = \bigcup_i (\sim B_i)^{4U} = \bigcup_i \bigcup_{V \subseteq U} (V^+ \cap B_i^{4V}). \quad \square$$

As in the introduction, we write  $\gamma^* = \sup\{\gamma + \beta : \beta < \gamma\}$ .

**THEOREM 3.5.** *Suppose  $E$  is a  $\gamma$ -decomposition for  $(Y, \mathcal{H})$ ,  $\gamma \geq 0$ . Then there is a  $\gamma^*$ -Borel selector for  $E$  on  $Y$ .*

*Proof.* First consider the case  $\gamma = 0$ . In this case Theorem 1.1 (or rather its classical antecedent) can be applied directly. By 3.3 each  $[y]$  is an intersection of clopen sets and, hence, is closed. Set  $X = Y/E$ ,  $\mathcal{D} = \{\pi(C) : C \text{ is an } E\text{-invariant clopen set}\}$ , and for each open  $O \subseteq Y$  set

$$\begin{aligned} O^* &= \pi(O^+) = \{[y] : [y] \cap O \neq \emptyset\} \\ &= \bigcup \{\pi(U^+) : U \in \mathcal{H} \text{ and } U \subseteq O\}. \end{aligned}$$

Theorem 1.1 then applies to provide a choice function  $f: Y/E \rightarrow Y$  which is  $\mathcal{D}_o$ -measurable. Define a selector  $s$  by setting  $s = f \circ \pi$ . Then for any open  $O$ ,  $s^{-1}(O)$  is of the form  $\pi^{-1}(\bigcup_i \pi(C_i)) = \bigcup_i \pi^{-1}\pi(C_i)$  where each  $C_i$  is invariant clopen. Since  $\pi^{-1}\pi(C) = C$  for any invariant set  $C$ , each  $s^{-1}(O)$  is open and  $s$  is continuous as required.

Now suppose  $\gamma \geq 1$ . As before, we imitate the above argument by introducing a new topology and substituting  $\Delta$  for  $+$ . By 2.3 we can choose a countable collection  $\mathcal{C} \subseteq \bigcup_{\beta < \gamma} A_\beta$  such that  $\mathcal{S} \cup \mathcal{C} \cup \mathcal{H}$  generates a Polish topology  $\mathcal{T}'$  on  $Y$ . For  $O \subseteq Y' = (Y, \mathcal{T}')$  we define  $O^* = \pi(O^\Delta)$  using our new  $\Delta$  operator and, using (7) and (9), again observe that  $\#$  is a countably additive semisaturation operator for the identity map  $Y/E \rightarrow \mathcal{P}(Y')$ .

Set  $\mathcal{D} = \{\pi(A) : A \text{ is an invariant } A_\alpha \text{ set}\}$ . We claim

(12) If  $O$  is open in  $Y'$ , then  $O^* \in \mathcal{D}_o$ .

Let  $\mathcal{H}'$  be the collection of sets of the form  $B \cap S$  where  $B$  is a finite intersection of elements of  $\mathcal{C} \cup \mathcal{H}$  and  $S$  is a finite intersection of elements of  $\mathcal{S}$ . Since  $\mathcal{H}'$  is a basis for  $Y'$  and  $\#$  is countably additive, it suffices to check that (12) holds for  $O \in \mathcal{H}'$ . Given  $O = B \cap S$  with  $B, S$  as above we have  $O^* = \pi(B \cap S)^\Delta = \pi(B^\Delta \cap S)$ . Noting that  $B \in S_\beta$  for some  $\beta < \gamma$ , we may apply 3.3 to conclude that  $B^\Delta$  is a countable union of invariant  $A_{\gamma+\beta}$  sets. Since  $S$  is invariant  $A_\gamma$ ,  $B^\Delta \cap S$  is a countable union of invariant  $A_{\gamma+\beta}$  sets. Since  $\gamma + \beta \leq \gamma^*$  and projection commutes with union,  $\pi(B^\Delta \cap S) \in \mathcal{D}_o$  as required, establishing (12).

As in 3.2 we note that each  $[y]$  is closed in  $Y'$ . Thus, we are again in a position to apply 1.1 to obtain a  $\mathcal{D}_o$ -measurable choice function  $f: Y/E \rightarrow Y'$ . As before, the map  $s = f \circ \pi$  is a selector. In this case, however, it follows directly from our definition of  $\mathcal{D}$  that

$s$  is an  $\alpha$ -Borel map from  $Y$  to  $Y'$ , *a fortiori* an  $\alpha$ -Borel map from  $Y$  to  $Y$ . The theorem is proved.  $\square$

**COROLLARY 3.6.** *Suppose  $A$  is an  $A_\gamma$  subspace of  $Y$  with basis  $\mathcal{H}' = \{U \cap A : U \in \mathcal{H}\}$ . Suppose  $E' \subseteq A \times A$  is a  $\gamma$ -decomposition for  $(A, \mathcal{H}')$ . Then there is an  $\gamma^*$ -Borel selector for  $E'$  on  $A$ .*

*Proof.* When  $\gamma \leq 1$ , we can apply 3.5 directly to the Polish space  $A$ . For  $\gamma > 1$ , set  $E = E' \cup \{(y, y) : y \in Y\}$ . Then for any  $U \in \mathcal{H}$ ,  $U^{+E} = (U \sim A) \cup (U \cap A)^{+\mu'} \in A_\gamma$ . It follows easily that  $E$  is a  $\gamma$ -decomposition for  $(Y, \mathcal{H})$ . 3.5 provides an  $\gamma^*$ -Borel selector for  $E$  whose restriction to  $A$  is an  $\gamma^*$ -Borel selector for  $E'$ .  $\square$

As an application of 3.7 we can derive a multifunction version.<sup>1</sup>

**COROLLARY 3.7.** *Let  $\gamma \geq 1$  and let  $Z$  be Polish.*

*Suppose  $G \subseteq Z \times Y$  is an  $A_\gamma$  set such that*

(i) *For each  $z \in Z$ ,  $G_z = \{y : (z, y) \in G\}$  is  $G_\delta$*

(ii) *For every  $U \in \mathcal{H}$ ,  $\{z : G_z \cap U \neq \emptyset\}$  is  $A_\gamma$ . Then there is an  $M_\gamma$ -set which uniformizes  $G$ .*

*Proof.* Let  $\mathcal{H}'' = \{O \times U : O \text{ is open in } Z \text{ and } U \in \mathcal{H}\}$ ,  $\mathcal{H}' = \{B \cap G : B \in \mathcal{H}''\}$ . Set  $E = \{(z, y), (z', y') \in G \times G : Z = Z'\}$ . For  $W = (O \times U) \cap G \in \mathcal{H}'$  we have  $W^{+E} = \{(z, y) \in G : z \in O \text{ and } G_y \cap U \neq \emptyset\} \in A_\gamma$  so  $E$  is a  $\gamma$ -decomposition for  $(G, \mathcal{H}')$ . Let  $s$  be the  $\gamma^*$ -Borel selector for  $E$  on  $G$  provided by 3.6. Then  $T_s = \{(z, y) \in G : (z, y) = s((z, y))\}$  is an  $M_{\gamma^*}$  uniformization for  $G$ .  $\square$

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Harvey Leslie Abbott, <i>Extremal problems on nonaveraging and nondividing sets</i> .....	1
Marine Bruce Abrahamse and Stephen D. Fisher, <i>Mapping intervals to intervals</i> .....	13
William Wells Adams, <i>The best two-dimensional Diophantine approximation constant for cubic irrationals</i> .....	29
Marilyn Breen, <i>A quantitative version of Krasnosel'skiĭ's theorem in <math>R^2</math></i> .....	31
Stephen LaVern Campbell, <i>Linear operators for which <math>T^*T</math> and <math>TT^*</math> commute. III</i> .....	39
Zvonko Cerin, <i>On cellular decompositions of Hilbert cube manifolds</i> .....	47
J. R. Choike, Ignacy I. Kotlarski and V. M. Smith, <i>On a characterization using random sums</i> .....	71
Karl-Theodor Eisele, <i>Direct factorizations of measures</i> .....	79
Douglas Harris, <i>Every space is a path component space</i> .....	95
John P. Holmes and Arthur Argyle Sagle, <i>Analytic <math>H</math>-spaces, Campbell-Hausdorff formula, and alternative algebras</i> .....	105
Richard Howard Hudson and Kenneth S. Williams, <i>Some new residuacity criteria</i> .....	135
V. Karunakaran and Michael Robert Ziegler, <i>The radius of starlikeness for a class of regular functions defined by an integral</i> .....	145
Ka-Sing Lau, <i>On the Banach spaces of functions with bounded upper means</i> .....	153
Daniel Paul Maki, <i>On determining regular behavior from the recurrence formula for orthogonal polynomials</i> .....	173
Stephen Joseph McAdam, <i>Asymptotic prime divisors and going down</i> .....	179
Douglas Edward Miller, <i>Borel selectors for separated quotients</i> .....	187
Kent Morrison, <i>The scheme of finite-dimensional representations of an algebra</i> .....	199
Donald P. Story, <i>A characterization of the local Radon-Nikodĕm property by tensor products</i> .....	219
Arne Stray, <i>Two applications of the Schur-Nevanlinna algorithm</i> .....	223
N. B. Tinberg, <i>The Levi decomposition of a split <math>(B, N)</math>-pair</i> .....	233
Charles Irvin Vinsonhaler and William Jennings Wickless, <i>A theorem on quasi-pure-projective torsion free abelian groups of finite rank</i> .....	239
Yitzhak Weit, <i>Spectral analysis in spaces of vector valued functions</i> .....	243