Pacific Journal of Mathematics

$|C, 1|$ SUMMABILITY OF SERIES ASSOCIATED WITH FOURIER SERIES

H. P. DIKSHIT AND S. N. DUBEY

Vol. 91, No. 2 December 1980

PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 2, 1980

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The purpose of this paper is to prove the following theorem. Suppose that for $u \geq n_0$, $g(u)$ and $d(u)$ are positive functions such that $ud(u)$ is nondecreasing and (i) $\sum n^{-1}g(n)d(n) < \infty$. Then the series $\sum d(n)A_n(x)$ is summable $|C, 1|$, if the following hold:

(1.1) $$\Phi(t) = \int_0^t |\varphi(u)|\,du = O(tg(t^{-1}))\ , \quad t \longrightarrow +0\ ;$$

(1.2) $$\sum n^{-1}d(n)I(n^{-1}) = \sum n^{-1}d(n)\int_{n^{-1}}^{\pi} t^{-1}|\varphi(t)|\,dt < \infty\ .$$

1. **The main result.** Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its partial sums. The nth $(C, 1)$ mean of $\{s_n\}$ is given by

$$t_n = \sum_{k=0}^{n} s_k/n + 1$$

and the series $\sum_{n=0}^{\infty} a_n$ is said to be $|C, 1|$ summable, if $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$. It is known that (see [2])

(1.3) $$n(t_n - t_{n-1}) = T_n$$

where T_n is the nth $(C, 1)$ mean of the sequence $\{na_n\}$.

Let $f(t)$ be a Lebesgue integrable periodic function with period 2π and $\sum_{n=0}^{\infty} A_n(t)$ denotes its Fourier series. Then for $k \geqq 1$,

(1.4) $$\pi A_k(x) = \int_0^{\pi} \varphi(t)\cos kt\,dt$$

where $\varphi(t) = f(x + t) + f(x - t) - 2f(x)$. For some positive integer n_0, we write $\sum$ for $\sum_{n=n_0}^{\infty}$. We now turn to the proof of the theorem stated in the first paragraph.

REMARKS. If we take $d(u) = u^{-a}$ and $g(u) = (\log u)^b$, for any $a, b > 0$, then clearly $u\,d(u)$ is nondecreasing for $u \geqq 1$ and (i) holds if $a \leqq 1$. Further, assuming (1.1), we have by integration by parts

(1.5) $$I(n^{-1}) = [t^{-1}\Phi(t)]_{1/n}^{\pi} + \int_{1/n}^{\pi} t^{-2}\Phi(t)\,dt$$

so that $I(n^{-1}) = O[(\log n)^{1+b}]$ and (1.2) follows. Thus, we have

COROLLARY. *Suppose that for any positive b, however, large,*

$$\Phi(t) = \int_0^t |\varphi(u)|\, du = O[t(\log(1/t))^b]\,, \quad t \longrightarrow +0\,. \tag{1.1'}$$

Then the series $\sum n^{-a} A_n(x)$ *with* $n_0 = 1$, *is summable* $|C, 1|$ *for* $0 < a \leqq 1$ *and is absolutely convergent for* $a > 1$.

In order to see the later part of the corollary we notice that

$$\pi A_n(x) = O(\Phi(n^{-1}) + I(n^{-1})) = O[(\log n)^{1+b}]$$

and the consequence follows. Since $\Phi(t) = O(t)$ implies (1.1′) (but not conversely) the result of the corollary is an improvement over that contained in [1, Theorem 1].

Taking $g(u) = [\prod_{r=1}^{k} \log^r u]^{-1}$ and $d(u) = g(u)\{\log^k(u)\}^{-b}$ for any $b > 0$, where $\log^1(u) = \log u$; $\log^2(u) = \log\log u$; $\cdots$; we see that the hypothesis (i) holds. Now assuming (1.1) we have from (1.5), $I(n^{-1}) = O(\log^{k+1} n)$, so that (1.2) holds and we deduce the result of Theorem 2 in [1].

2. Proof of the theorem. In view of (1.3) and (1.4), it follows that in order to prove the theorem it is sufficient to show that

$$J \equiv \sum \{n(n+1)\}^{-1} \left| \int_0^\pi h(n, t)\varphi(t)\, dt \right| < \infty\,, \tag{2.1}$$

where $h(n, t) = \sum_{k=n_0}^{n} (k - n_0)\, d(k) \cos kt$. Since $k\, d(k)$ is nonnegative, nondecreasing and

$$(k - n_0)\, d(k) = k\, d(k)(1 - (n_0/k))$$

therefore, $(k - n_0)\, d(k)$ is nonnegative, nondecreasing for $k \geqq n_0$. Thus, we have by Abel's lemma

$$h(n, t) = O\left(n\, d(n) \max_{n_0 < r \leqq n} \left| \sum_{k=r}^{n} \cot kt \right| \right) = O(sn\, d(n)) \tag{2.2}$$

where $s = n$ or t^{-1}. Thus,

$$\begin{aligned} J_1 &= \sum \{n(n+1)\}^{-1} \left| \int_0^{1/n} h(n, t)\varphi(t)\, dt \right| \\ &= O(\sum d(n)\Phi(n^{-1})) = O(1)\,, \end{aligned} \tag{2.3}$$

by virtue of the hypotheses (i) and (1.1). Using (2.2) with $s = t^{-1}$, we have

$$\begin{aligned} J_2 &= \sum \{n(n+1)\}^{-1} \left| \int_{1/n}^{\pi} h(n, t)\varphi(t)\, dt \right| \\ &= O(\sum n^{-1} d(n) I(n^{-1})) = O(1)\,, \end{aligned} \tag{2.4}$$

by virtue of (1.2). Combining (2.3) and (2.4) with (2.1), we complete the proof of the theorem.

References

1. F. C. Hsiang, *On $|C, 1|$ summability factors of Fourier series,* Pacific J. Math., **33** (1970), 139-147.
2. E. Kogbetliantz, *Sur la séries absolument sommable par la méthode des moyennes arithmétiques,* Bull. Sci. Math., **49** (1925), 234-256.

Received June 5, 1979 and in revised form September 18, 1979.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$84.00 a year (6 Vols., 12 issues). Special rate: \$42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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