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**THE EQUATIONS  $\Delta u = Pu$  ( $P \geq 0$ ) ON RIEMANN SURFACES  
AND ISOMORPHISMS BETWEEN RELATIVE HARDY SPACES**

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# THE EQUATIONS $\Delta u = Pu$ ( $P \geq 0$ ) ON RIEMANN SURFACES AND ISOMORPHISMS BETWEEN RELATIVE HARDY SPACES

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It has been demonstrated by M. Nakai that the Banach spaces  $PB$  (the space of bounded solutions on  $R$  of the equation  $\Delta u = Pu$ ,  $P \geq 0$ ) and  $HB$  (the space of bounded harmonic functions on  $R$ ) are isometrically isomorphic whenever the condition

$$\int_R P(z) G(z, w_0) dx dy < +\infty$$

is valid for some point  $w_0$  in  $R$  ( $z = x + iy$ ). Here,  $G(z, w)$  is the harmonic Green's function on  $R$ . In this paper we shall show, under the preceding condition that the Hardy space  $H^p$ ,  $1 < p \leq +\infty$ , of harmonic functions on a hyperbolic Riemann surface  $R$  is isometrically isomorphic to the relative Hardy space  $PH_w^p$  of quotients of solutions of  $\Delta u = Pu$  by the  $P$ -elliptic measure  $w$  of  $R$ .

1. Introduction. Throughout this paper, let  $R$  be a hyperbolic Riemann surface. We consider a density  $P$  on  $R$ , that is, a non-negative Hölder continuous function on  $R$  which depends on the local parameter  $z = x + iy$  in such a way the partial differential equation

$$(1.1) \quad \Delta u = Pu, \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

is invariantly defined on  $R$ . Let  $P \not\equiv 0$  on  $R$ . A real valued function  $u$  is called a  $P$ -harmonic function (or  $P$ -solution) in an open set  $U$  of  $R$ , if  $u$  has continuous partial derivatives up to the order 2 and satisfies the equation (1.1) on  $U$ . The totality of bounded  $P$ -harmonic functions on  $R$  is denoted by  $PB$ . Then,  $PB$  is a Banach space with the uniform norm

$$(1.2) \quad \|u\| = \sup_{z \in R} |u(z)|.$$

Also,  $HB$  is the Banach space of the totality of bounded harmonic functions on  $R$  with the uniform norm (1.2).

Many works ([5, 6, 11, 12 and 14] among others) were done on the comparison theorem, that is, to compare the spaces  $PB$  for different choices of  $P$ . For example, in 1960 ([11]) it was proved that, if two densities  $P$  and  $Q$  on  $R$  satisfy the condition

$$(1.3) \quad \int_R |P(z) - Q(z)| \{G^P(z, w_0) + G^Q(z, w_1)\} dx dy < +\infty$$

for some points  $w_0$  and  $w_1$  in  $R$ , then Banach spaces  $PB$  and  $QB$  are isometrically isomorphic, where  $G^P(z, w)$  and  $G^Q(z, w)$  are Green's functions of  $R$  with pole  $w$  associated with the equations (1.1) and  $\Delta u = Qu$  ( $Q \geq 0$ ) respectively. Here, in particular we consider the case  $Q \equiv 0$  on  $R$ . In this case we can conclude that under the assumption

$$(1.4) \quad \int_R P(z) G(z, w_0) dx dy < +\infty$$

for some point  $w_0$  in  $R$ , the Banach spaces  $PB$  and  $HB$  ( $=QB$ ,  $Q \equiv 0$ ) are isometrically isomorphic, where  $G(z, w)$  is the harmonic Green's function of  $R$  with pole  $w$  in  $R$ .

Let a pair  $(R, P)$ ,  $P \not\equiv 0$ , be hyperbolic. Then there exists the positive  $P$ -solution  $w$  on  $R$  which takes the constant 1 at the ideal boundary of  $R$ , which we call the  $P$ -elliptic measure of  $R$ . The  $P$ -elliptic measure  $w$  plays a role somewhat analogous to that played by the constant 1. A  $w$ - $P$ -harmonic function is a quotient of a  $P$ -harmonic function by the  $P$ -elliptic measure  $w$ . The relative Hardy class  $PH_w^p$ ,  $1 \leq p \leq +\infty$ , of  $w$ - $P$ -harmonic functions in  $R$  is defined by the way analogous to that of Hardy class  $H^p$  of harmonic functions on  $R$ . We are interested in the comparison problem of Banach space structures of  $PH_w^p$  and  $H^p$ . In this paper, we shall give the theorem: under the assumption (1.4) the Banach spaces  $PH_w^p$  and  $H^p$ ,  $1 < p \leq +\infty$ , isometrically isomorphic.

Let  $\Delta_1$  and  $\Delta_{P_1}$  are the sets of minimal boundary points of Martin and  $P$ -Martin compactifications, respectively. And, let  $\chi$  and  $\chi_P$  be the harmonic measure on  $\Delta_1$  and the  $P$ -elliptic measure on  $\Delta_{P_1}$ , respectively. Since L. L. Naim [9] proved that  $H^p$  and  $PH_w^p$ ,  $1 < p \leq +\infty$ , are isometrically isomorphic to Banach spaces  $L^p(\Delta_1, \chi)$  and  $L^p(\Delta_{P_1}, \chi_P)$  respectively, by constructing a measurable transformation defined almost everywhere on  $\Delta_{P_1}$  into  $\Delta_1$  we shall investigate a relation between  $\chi$  and  $\chi_P$  under the assumption (1.4), and so, we can find an isomorphism from  $PH_w^p$  onto  $H^p$ ,  $1 < p \leq +\infty$ .

**2. Preliminaries.** In 1941 Martin [7] introduced a compactification in the investigation of nonnegative harmonic functions. Nakai [10] extended the Martin theory to the setting of  $P$ -harmonic functions. The results of these theories were established by Hervé [4] in the setting of Brelot's axiomatic potential theory. We shall use extensively the Martin compactification  $R^*$  and the Nakai's  $P$ -Martin compactification  $R_P^*$  of  $R$ . We denote by  $\Delta_{P_1}$  (resp.  $\Delta_1$ ) the

set of minimal points of  $R^* - R$  (resp.  $R^* - R$ ) and by  $K_a^P$  (resp.  $K_b$ ) the associated Martin kernel of  $a \in \Delta_{P_1}$  (resp.  $b \in \Delta_1$ ) with pole  $z_0$ . And, let

$$K^P(z, a) = K_a^P(z), \quad z \in R \quad \text{and} \quad a \in \Delta_{P_1},$$

and

$$K(z, b) = K_b(z), \quad z \in R \quad \text{and} \quad b \in \Delta_1.$$

Let  $P$  be a density on  $R$  which is not constantly zero on  $R$ . Almost every theorem in this paper will be proved under Nakai's integral condition:

$$(2.1) \quad \int_R P(z) G(z, w_1) dx dy < +\infty$$

at some point  $w_1$  in  $R$ . If this condition holds at some point  $w_1$  of  $R$ , then it does at all points of  $R$  by Harnack's inequality.

We state the definition of  $P$ -elliptic measure from the work of H. Royden [14]. By a compact region we mean a connected open set whose closure is compact and whose boundary is composed of finite number of analytic curves. Let  $\{R_n\}$  be an exhaustion of  $R$ , i.e., a sequence of compact regions such that  $\bar{R}_n \subset R_{n+1}$  and  $R = \bigcup_{n=1}^{\infty} R_n$ . We define the function  $w_n$  to be the  $P$ -solution in  $R_n$  which is identically one on  $\partial R_n$ . For  $P \not\equiv 0$  we have  $0 < w_n < 1$ . Since the maximum principle implies that the functions  $w_n$  form a monotone decreasing sequence of positive  $P$ -solutions, this sequence converges uniformly on each compact set in  $R$  to a nonnegative  $P$ -solution  $w$ , which is called the  $P$ -elliptic measure of  $R$ .

The  $P$ -elliptic measure  $w$  is either identically zero or else everywhere positive. In the second case we say that the pair  $(R, P)$  is hyperbolic provided  $P \not\equiv 0$ .

The  $P$ -elliptic measure  $w$  may be characterized as the largest  $P$ -solution which is bounded by 1.

For the  $P$ -elliptic measure  $w$  of  $R$ , there exists a unique measure  $\chi_P$  supported by  $\Delta_{P_1}$  such that

$$(2.2) \quad w(z) = \int_{\Delta_{P_1}} K^P(z, a) d\chi_P(a), \quad z \in R,$$

which is called the  $P$ -elliptic measure on the  $P$ -Martin boundary.

And, the harmonic measure is denoted by  $\chi$ , that is, the measure which represents the constant function 1 and is supported by  $\Delta_1$ :

$$1 = \int_{\Delta_1} K(z, b) d\chi(b), \quad z \in R.$$

DEFINITION 2.1. We introduce the set  $\Delta_{P_0}$  of point  $a$  in  $\Delta_{P_1}$  such

that

$$(2.3) \quad \int_R P(z)G(z, w_1)K^P(z, a)dxdy < +\infty$$

for some point  $w_1$  in  $R$ , and hence for every point in  $R$ .

DEFINITION 2.2. We introduce the set  $\Delta'_{0P}$  of points  $b$  in  $\Delta_1$  such that

$$\int_R P(z)G(z, w_1)K(z, b)dxdy < +\infty$$

for some point  $w_1$  in  $R$ .

LEMMA 2.1. Let  $u$  be a positive  $P$ -solution on  $R$  such that

$$(2.4) \quad \int_R P(z)G(z, w_1)u(z)dxdy < +\infty$$

for some point  $w_1$  in  $R$ , and let  $\mu$  be the canonical measure on  $\Delta_{P1}$  which represents  $u$ :

$$u(z) = \int_{\Delta_{P1}} K^P(z, a)d\mu(a), \quad z \in R.$$

Then,  $\Delta_{P1} - \Delta_{P0}$  is a measurable set of  $\mu$ -measure zero.

*Proof.* For each positive integer  $n$ , let  $E_n$  be a set of points  $a$  in  $\Delta_{P1}$  such that

$$\int_R P(z)G(z, w_1)K^P(z, a)dxdy \geq n,$$

where  $w_1$  is a fixed point in  $R$ . Since  $E_n$  is measurable and, by Fubini's theorem,

$$\begin{aligned} n\mu(E_n) &\leq \int_{\Delta_{P1}} \left\{ \int_R P(z)G(z, w_1)K^P(z, a)dxdy \right\} d\mu(a) \\ &= \int_R P(z)G(z, w_1) \left\{ \int_{\Delta_{P1}} K^P(z, a)d\mu(a) \right\} dxdy \\ &= \int_R P(z)G(z, w_1)u(z)dxdy, \end{aligned}$$

we have

$$\begin{aligned} \mu(\Delta_{P1} - \Delta_{P0}) &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &\leq \mu(E_n) \\ &\leq \frac{1}{n} \int_R P(z)G(z, w_1)u(z)dxdy \end{aligned}$$

for every positive integer  $n$ . Hence, it follows that  $\mu(\Delta_{P_1} - \Delta_{P_0}) = 0$ .

**LEMMA 2.2.** *Let  $v$  be a positive harmonic function on  $R$  such that*

$$\int_R P(z)G(z, w_1)v(z)dxdy < +\infty$$

*for some  $w_1$  in  $R$  and  $\nu$  be the canonical measure on  $\Delta_1$  which represents  $v$ :*

$$v(z) = \int_{\Delta_1} K(z, b)d\nu(b), \quad z \in R.$$

*Then,  $\Delta_1 - \Delta'_{0P}$  is a measurable set of  $\nu$ -measure zero.*

*Proof.* This can be shown by the same proof as that of Lemma 2.1.

**THEOREM 2.3.** *Let  $P$  be a density on  $R$  which satisfies Nakai's integral condition (2.1). Then, the  $P$ -elliptic measure of the set  $\Delta_{P_1} - \Delta_{P_0}$  is zero:*

$$\chi_P(\Delta_{P_1} - \Delta_{P_0}) = 0.$$

*Proof.* Since  $w < 1$  on  $R$ , from (2.1) it follows that

$$\begin{aligned} \int_R P(z)G(z, w_1)w(z)dxdy \\ \leq \int_R P(z)G(z, w_1)dxdy < +\infty. \end{aligned}$$

Therefore, by the fact that  $w$  is represented as the integral (2.2) by the measure  $\chi_P$ , Lemma 2.1 gives this theorem.

Lemma 2.2 gives the following:

**THEOREM 2.4.** *Under the same assumption as that in Theorem 2.3, the harmonic measure of the set  $\Delta_1 - \Delta'_{0P}$  is zero.*

*Proof.* The constant function 1 and the harmonic measure  $\chi$  play the roles of  $v$  and  $\nu$  in Lemma 2.2.

**3. Relations between minimal  $P$ -solutions and minimal harmonic functions.** To give an isomorphism between  $PB$  and  $QB$ , Nakai [11] has defined the transform  $T^{PQ}f$  for a function  $f$  as follows:

$$T^{P^0}f(z) = f(z) + \frac{1}{2\pi} \int_R (P(w) - Q(w))G^Q(w, z)f(w)dudv,$$

where  $w = u + iv$ . And, Lahtinen [5], Nakai, and Satō [15] showed some properties of the transformation  $T^{P^0}$ . In this paper we consider only the case in which  $Q$  is identically zero on  $R$ . Some usefull properties of  $T^{P^0}$  will be shown in this section.

**DEFINITION 3.1.** Let  $P$  be a density on  $R$  and  $f$  be a continuous function on  $R$  for which

$$(3.1) \quad \int_R P(w)G(w, z_0)|f(w)|dudv < +\infty, \quad w = u + iv,$$

is true at some point  $z_0$  in  $R$  (then it holds at all points  $z$  in  $R$ ). Then, the linear transformation  $T^{P^0}f$  of  $f$  is well defined by

$$(3.2) \quad T^{P^0}f(z) = f(z) + \frac{1}{2\pi} \int_R P(w)G(w, z)f(w)dudv.$$

By changing the role of  $P$  and  $0$  we define also the transformation  $T^{0P}$ . For a continuous function  $g$  on  $R$  such that

$$(3.3) \quad \int_R P(w)G^P(w, z_0)|g(w)|dudv < +\infty$$

for some point  $z_0$  in  $R$ ,  $T^{0P}g$  is defined by

$$(3.4) \quad T^{0P}g(z) = g(z) - \frac{1}{2\pi} \int_R P(w)G^P(w, z)g(w)dudv.$$

To derive properties of  $T^{P^0}$  we consider an auxiliary sequence of transformations  $T_n^{P^0}$ ,  $n = 1, 2, \dots$ , of a real valued continuous function  $f$  defined on the closure  $\bar{R}_n$  of  $R_n$  as follows:

$$T_n^{P^0}f(z) = f(z) + \frac{1}{2\pi} \int_{R_n} P(w)G(R_n, w, z)f(w)dudv,$$

where  $G(R_n, w, z)$  is the harmonic Green function on  $R_n$ . It is evident that, if  $f$  is a  $P$ -solution on  $R_n$ , then  $T_n^{P^0}f$  is a continuous function on  $\bar{R}_n$  which is harmonic on  $R_n$  and satisfies

$$T_n^{P^0}f|_{\partial R_n} = f|_{\partial R_n}$$

(see, for example, Nakai [11] or Lahtinen [5]).

The following lemma is a special case of Lahtinen's lemma in [5] in which  $P$  is acceptable in the sense of his definition.

**LEMMA 3.1 (Lahtinen).** Let  $f$  be a  $P$ -solution on  $R$  and  $\{f_n\}$  a

sequence of  $P$ -solutions each defined on  $\bar{R}_n$  such that  $\lim_{n \rightarrow +\infty} f_n = f$ . If there exists a function  $u$  continuous on  $R$  such that  $|f_n| \leq u$  for each positive integer  $n$  and  $u$  fulfils the inequality obtained by replacing  $f$  by  $u$  in (3.1) at some point in  $R$ , then  $T^{P_0}f$  is well defined and has the following properties: (1)  $\lim_{n \rightarrow +\infty} T_n^{P_0}f_n = T^{P_0}f$ , (2)  $T^{P_0}f$  is harmonic on  $R$ .

By changing the roles of  $P$  and 0, the transformation  $T_n^{0P}$  is defined and we can state the following:

**LEMMA 3.1'.** *Let  $g$  be a harmonic function on  $R$  and  $\{g_n\}$  be a sequence of harmonic functions each defined on  $R_n$  such that  $\lim_{n \rightarrow +\infty} g_n = g$ . If there exists a function  $v$  continuous on  $R$  such that  $|g_n| \leq v$  for each positive integer  $n$  and  $v$  fulfils the inequality (3.3) at some point in  $R$ , then  $T^{0P}g$  is well defined and has the following properties: (1)  $\lim_{n \rightarrow +\infty} T_n^{0P}g_n = T^{0P}g$ , (2)  $T^{0P}g$  is  $P$ -harmonic on  $R$ .*

**LEMMA 3.2.** *Let  $P$  be a density on  $R$ . If  $P$  satisfies Nakai's condition*

$$(3.5) \quad \int_R P(z)G(z, w_0)dx dy < +\infty$$

for some point  $w_0$  in  $R$ , then we have

$$(3.6) \quad \begin{aligned} G(z, w) &= G^P(z, w) + \frac{1}{2\pi} \int_R P(\zeta)G(\zeta, z)G^P(\zeta, w)d\xi d\eta \\ &= G^P(w, z) + \frac{1}{2\pi} \int_R P(\zeta)G(\zeta, w)G^P(\zeta, z)d\xi d\eta, \end{aligned}$$

for each point  $(z, w)$  in  $R \times R$  with  $w \neq z$ , where  $\zeta = \xi + i\eta$ .

*Proof.* Green's formula implies that, for  $(z, w)$  in  $R_n \times R_n$  with  $z \neq w$ ,

$$(3.7) \quad \begin{aligned} G(R_n, z, w) &= G^P(R_n, z, w) \\ &+ \frac{1}{2\pi} \int_{R_n} P(\zeta)G(R_n, \zeta, w)G^P(R_n, \zeta, z)d\xi d\eta, \end{aligned}$$

where  $G^P(R_n, z, w)$  is Green's function of  $R_n$  related to the differential equation (1.1) and  $G(R_n, z)$  is the harmonic Green function of  $R_n$ .

In order to apply Lebesgue's dominated convergence theorem, let

$$F(z, w, \zeta) = P(\zeta)G(\zeta, w)G^P(\zeta, z).$$



And, let  $U$  and  $V$  be small discs with centers  $z$  and  $w$  respectively such that  $U \cap V = \emptyset$ . Then, by the minimum principle, Nakai's condition (3.5) gives that

$$\begin{aligned} \int_V F(z, w, \zeta) d\xi d\eta &\leq \sup_{\zeta \in \partial U} G^P(\zeta, z) \\ &\times \int_R P(\zeta) G(\zeta, w) d\xi d\eta < +\infty \end{aligned}$$

and, by  $G^P(\zeta, z) < G(\zeta, z)$  (which follows from the definition of the Green function),

$$\begin{aligned} \int_{R-V} F(z, w, \zeta) d\xi d\eta &\leq \sup_{\zeta \in \partial V} G(\zeta, w) \\ &\times \int_R P(\zeta) G^P(\zeta, z) d\xi d\eta \\ &< \sup_{\zeta \in \partial V} G(\zeta, w) \\ &\times \int_R P(\zeta) G(\zeta, z) d\xi d\eta < +\infty, \end{aligned}$$

from which it follows that

$$\begin{aligned} (3.8) \quad \int_R F(z, w, \zeta) d\xi d\eta &= \int_V F(z, w, \zeta) d\xi d\eta \\ &+ \int_{R-V} F(z, w, \zeta) d\xi d\eta < +\infty. \end{aligned}$$

Therefore, since

$$P(\zeta) G(R_n, \zeta, w) G^P(R_n, \zeta, z) \leq F(z, w, \zeta)$$

for each positive integer  $n$  and

$$\lim_{n \rightarrow +\infty} P(\zeta) G(R_n, \zeta, w) G^P(R_n, \zeta, z) = P(\zeta) G(\zeta, w) G^P(\zeta, z),$$

Lebesgue's dominated convergence theorem shows (3.6) as  $n$  tends to  $+\infty$  in (3.7).

**LEMMA 3.3.** *Let  $f$  be a continuous function on  $R$  such that*

$$(3.9) \quad \int_R P(z) G(z, w_1) |f(z)| dx dy < +\infty$$

*for some point  $w_1$  in  $R$ . Then, it holds that*

$$(3.10) \quad \int_R P(z) G^P(z, w_0) |T^{p_0} f(z)| dx dy < +\infty$$

*for all points  $w_0$  in  $R$ .*

*Proof.* By the definition of  $T^{P_0}f$ , we have

$$\begin{aligned}
 (3.11) \quad & \int_R P(z)G^P(z, w_0) |T^{P_0}f(z)| dx dy \\
 & \leq \int_R P(z)G^P(z, w_0) |f(z)| dx dy \\
 & + \int_R P(z)G^P(z, w_0) \left\{ \frac{1}{2\pi} \int_R P(w)G(w, z) |f(w)| du dv \right\} dx dy .
 \end{aligned}$$

Here, the first term of the right side of this inequality is finite by the inequality  $G^P(z, w_0) < G(z, w_0)$  on  $R$ .

To apply Fubini's theorem to the second term we define a function  $F(z, w)$  by

$$F(z, w) = \frac{1}{2\pi} P(z)P(w)G^P(z, w_0)G(w, z) |f(w)| .$$

Since, by Lemma 3.2,

$$\begin{aligned}
 (3.12) \quad & \int_R \left\{ \int_R F(z, w) dx dy \right\} du dv \\
 & = \int_R \left\{ \frac{1}{2\pi} \int_R P(z)G^P(z, w_0)G(z, w) dx dy \right\} P(w) |f(w)| du dv \\
 & = \int_R P(w) \left\{ G(w, w_0) - G^P(w, w_0) \right\} |f(w)| du dv \\
 & < \int_R P(w)G(w, w_0) |f(w)| du dv \\
 & < +\infty .
 \end{aligned}$$

Fubini's theorem shows that the second term of the right side of (3.11) is equal to (3.12). Hence, we established the lemma.

**LEMMA 3.4.** *Let  $f$  be a positive  $P$ -harmonic function on  $R$  which satisfies the same condition as that in Lemma 3.3. Then we have*

$$T^{P_0}(T^{P_0}f) = f \quad \text{on } R .$$

*Proof.* Lemma 3.3 shows the inequality (3.10), and so, Lemma 3.1 and 3.1' imply that  $T^{P_0}(T^{P_0}f)$  is well defined and is  $P$ -harmonic, since

$$T^{P_0}f = \lim_{n \rightarrow +\infty} T_n^{P_0}f$$

and

$$T_n^{P_0}f < T^{P_0}f$$

for every  $n$ . Furthermore,

$$(3.13) \quad T^{0P}(T^{P_0}f) = \lim_{n \rightarrow +\infty} T_n^{0P}(T_n^{P_0}f)$$

on  $R$ .

The definitions of  $T_n^{P_0}$  and  $T_n^{0P}$  give that

$$T_n^{0P}(T_n^{P_0}f)|\partial R_n = T_n^{P_0}f|\partial R_n = f|\partial R_n.$$

Since  $T_n^{0P}(T_n^{P_0}f)$  is  $P$ -harmonic on  $R_n$ , the maximum principle implies that

$$T_n^{0P}(T_n^{P_0}f) = f \quad \text{on } R_n,$$

which completes the proof by (3.13).

LEMMA 3.5. *Let  $g$  be a continuous function on  $R$  such that*

$$(3.14) \quad \int_R P(z)G(z, w_1)|g(z)|dxdy < +\infty$$

*for some point  $w_1$  in  $R$ . Then, it follows that*

$$\int_R P(z)G(z, w_0)|T^{0P}g(z)|dxdy < +\infty,$$

*where  $w_0$  is any point in  $R$ .*

*Proof.* This can be provided in the same way as that of Lemma 3.3.

DEFINITION 3.2. We define the space  $P_0$  (resp.  $H'_P$ ) consisting of positive  $P$ -solutions  $f$  (resp. positive harmonic functions  $g$ ) on  $R$  with the property (3.9) (resp. (3.14)), and define the space  $H_P$  consisting of positive harmonic functions  $g$  on  $R$  such that

$$\int_R P(z)G^P(z, w_1)g(z)dxdy < +\infty$$

for some point  $w_1$  in  $R$ .

LEMMA 3.6. *Let  $g$  be a harmonic function in  $H_P$  such that  $T^{0P}g$  belongs to the space  $P_0$ . Then, it follows that*

$$T^{P_0}(T^{0P}g) = g \quad \text{on } R.$$

*Proof.* Since the function  $T^{0P}g$  satisfies the condition (3.1) in Definition 3.1,  $T^{P_0}(T^{0P}g)$  is well defined, and Lemmas 3.1, 3.1' show

the equality in this lemma by the same way as that in the proof of Lemma 3.4.

LEMMA 3.7.  $H'_P \subset T^{P_0}(P_0) \subset H_P$ .

*Proof.* Lemmas 3.3, 3.5 and 3.6 show this lemma.

THEOREM 3.8.  $T^{P_0}$  is a one-to-one transformation from  $P_0$  onto  $T^{P_0}(P_0)$ , and  $T^{0P}$  coincides with its inverse transformation.

*Proof.* Lemma 3.4 shows this theorem.

LEMMA 3.9. Let  $g$  and  $g_1$  be harmonic functions on  $R$ . If  $g \leq g_1$  on  $R$  and  $g, g_1$  belong to the space  $H_P$ . Then, it follows that

$$T^{0P}g \leq T^{0P}g_1 \quad \text{on } R.$$

*Proof.* Since

$$T_n^{0P}g|_{\partial R_n} = g|_{\partial R_n} \leq g_1|_{\partial R_n} = T_n^{0P}g_1|_{\partial R_n},$$

the maximum principle for  $P$ -solutions shows that

$$T_n^{0P}g \leq T_n^{0P}g_1 \quad \text{on } R_n$$

for each  $n$ . Thus, from Lemma 3.1' it follows that

$$T^{0P}g \leq T^{0P}g_1 \quad \text{on } R.$$

THEOREM 3.10. If a minimal  $P$ -solution  $K_a^P$  belongs to the space  $P_0$  (i.e.,  $a \in \Delta_{P_0}$ ), then  $T^{P_0}K_a^P$  is a minimal harmonic function on  $R$ , that is, there exists a unique point  $b$  in  $\Delta_1$  such that

$$T^{P_0}K_a^P = T^{P_0}K_a^P(z_0)K_b \quad \text{on } R.$$

*Proof.* Let  $g$  be a positive harmonic function on  $R$  such that

$$(3.15) \quad 0 < g \leq T^{P_0}K_a^P \quad \text{on } R.$$

By Lemmas 3.4 and 3.9 we have

$$0 < T^{0P}g \leq T^{0P}(T^{P_0}K_a^P) = K_a^P \quad \text{on } R,$$

and so,  $T^{0P}g = \alpha K_a^P$  on  $R$ , where  $\alpha$  is a positive constant. Then, since (3.15) implies  $g \in H_P$  by Lemma 3.3, from Lemma 3.6 it follows that

$$g = T^{P_0}(T^{0P}g) = \alpha T^{P_0}K_a^P,$$

which shows that  $T^{P_0}K_a^P$  is a minimal harmonic function on  $R$ .

**LEMMA 3.11.** *Let  $f$  and  $f_1$  be  $P$ -harmonic functions on  $R$ . If  $f \leq f_1$  on  $R$  and  $f, f_1$  belong to the space  $P_0$ , then it follows that*

$$T^{P_0}f \leq T^{P_0}f_1 \quad \text{on } R.$$

*Proof.* By Lemma 3.1 this can be proved similarly as Lemma 3.8.

**THEOREM 3.12.** *If a minimal harmonic function  $K_b$  belongs to  $T^{P_0}(P_0)$ , then  $T^{0P}K_b$  is a minimal  $P$ -harmonic function on  $R$  and is contained in the space  $P_0$ .*

*Proof.* This can be proved similarly as Theorem 3.10 by Lemma 3.11.

Theorems 3.10 and 3.12 can be paraphrased by saying that the transformation  $T^{P_0}: P_0 \rightarrow H_P$  gives a one-to-one mapping from the set of minimal  $P$ -harmonic functions in  $P_0$  onto the set of minimal harmonic functions in  $T^{P_0}(P_0)$ .

The following theorem says that the  $P$ -elliptic measure  $w$  of  $R$  is transformed into the constant function 1 on  $R$  by  $T^{P_0}$ .

**THEOREM 3.13.** *If the pair  $(R, P)$  is hyperbolic, then  $T^{P_0}w = 1$  on  $R$ .*

*Proof.* By  $w = \lim_{n \rightarrow +\infty} w_n$ , Lemma 3.1 implies that

$$T^{P_0}w = \lim_{n \rightarrow +\infty} T_n^{P_0}w_n = 1 \quad \text{on } R.$$

#### 4. Relation between the $P$ -elliptic and harmonic measures.

**DEFINITION 4.1.** Let  $\mathcal{A}_{0P}$  be a set consisting of points  $b$  in  $\mathcal{A}_1$  such that the minimal harmonic function  $K_b$  belongs to the set  $T^{P_0}(P_0)$ :

$$\mathcal{A}_{0P} = \{b \in \mathcal{A}_1: K_b \in T^{P_0}(P_0)\}.$$

In the following, it will be shown that  $\mathcal{A}_{0P}$  is measurable. We shall use the same notations  $\chi_P$  and  $\chi$  for the restrictions of the  $P$ -elliptic and harmonic measures to the measurable sets  $\mathcal{A}_{P_0}$  and  $\mathcal{A}_{0P}$ .

respectively, and consider two measure spaces  $(\Delta_{P_0}, \chi_P)$  and  $(\Delta_{0P}, \chi)$ .

The purpose of this section is to show that there exists a measurability preserving transformation  $t$  from  $(\Delta_{P_0}, \chi_P)$  onto  $(\Delta_{0P}, \chi)$  such that  $\chi_P \circ t^{-1}$  is absolutely continuous with respect to  $\chi$ .

Theorems 3.10 and 3.12 give the following definitions.

DEFINITION 3.2. We define a transformation

$$t_{P_0}: \Delta_{P_0} \longrightarrow \Delta_{0P}$$

by assigning to  $a$  in  $\Delta_{P_0}$  a point  $b = t_{P_0}(a)$  in  $\Delta_{0P}$  such that  $T^{P_0}K_a^P(z_0)K_b = T^{P_0}K_a^P$  on  $R$ .

DEFINITION 3.3. We define a transformation

$$t_{0P}: \Delta_{0P} \longrightarrow \Delta_{P_0}$$

by assigning, to  $b$  in  $\Delta_{0P}$ , a point  $a = t_{0P}(b)$  in  $\Delta_{P_0}$  such that  $T^{0P}K_b(z_0)K_a^P = T^{0P}K_b$  on  $R$ .

It is clear, by Theorems 3.8, 3.10 and 3.12, that  $t_{0P}$  is the inverse of  $t_{P_0}$ :  $t_{0P} = t_{P_0}^{-1}$ .

LEMMA 4.1. Under Nakai's condition:

$$(4.1) \quad \int_R P(z)G(z, w_1)dx dy < +\infty$$

for some point  $w_1$  in  $R$  (then it holds at all points in  $R$ ), the function  $T^{P_0}K_a^P(w_0)$  of  $a$  in  $\Delta_{P_0}$  is lower semi-continuous on  $\Delta_{P_0}$ , where  $w_0$  is any fixed point in  $R$ .

*Proof.* Let  $D_r(w_0)$  be the disc centered at  $w_0$  and having radius  $r$ . By Harnak's inequality there is a positive constant  $\alpha$  such that, for all  $z$  in  $D_r(w_0)$  and for all points  $a$  in  $\Delta_{P_0}$ ,

$$\alpha^{-1}K^P(w_0, a) \leq K^P(z, a) \leq \alpha K^P(w_0, a).$$

Thus, for each point  $a$  in  $\Delta_{P_0}$

$$(4.2) \quad \begin{aligned} \int_{D_r(w_0)} P(z)G(z, w_0)K^P(z, a)dx dy \\ \leq \alpha K^P(w_0, a) \times \int_{D_r(w_0)} P(z)G(z, w_0)dx dy. \end{aligned}$$

Since (4.1) implies that

$$\lim_{r \rightarrow 0} \int_{D_r(w_0)} P(z)G(z, w_0) dxdy = 0 ,$$

for any  $\varepsilon > 0$  we can find a positive number  $\delta = \delta(\varepsilon)$  such that

$$(4.3) \quad \int_{D_\delta(w_0)} P(z)G(z, w_0) dxdy < \varepsilon .$$

The function  $K^P(z, a)$  is finitely continuous on  $(\bar{R}_n - D_\delta(w_0)) \times \mathcal{A}_{P_0}$ , so that for any  $\varepsilon > 0$  there exists a neighborhood  $U(a)$  of  $a$  such that

$$|K^P(z, a') - K^P(z, a)| < \varepsilon$$

for  $a' \in U(a) \cap \mathcal{A}_{P_0}$  and  $z \in \bar{R}_n - D_\delta(w_0)$ . Therefore, from (4.2) and (4.3) it follows that

$$\begin{aligned} & \left| \int_{R_n} P(z)G(z, w_0)K^P(z, a') dxdy - \int_{R_n} P(z)G(z, w_0)K^P(z, a) dxdy \right| \\ & \leq \int_{R_n - D_\delta(w_0)} P(z)G(z, w_0)|K^P(z, a') - K^P(z, a)| dxdy \\ & \quad + \int_{D_\delta(w_0)} P(z)G(z, w_0)K^P(z, a') dxdy \\ & \quad + \int_{D_\delta(w_0)} P(z)G(z, w_0)K^P(z, a) dxdy \\ & \leq \varepsilon \times \int_{R_n - D_\delta(w_0)} P(z)G(z, w_0) dxdy \\ & \quad + \alpha(K^P(w_0, a') + K^P(w_0, a)) \times \int_{D_\delta(w_0)} P(z)G(z, w_0) dxdy \\ & \leq \varepsilon \times \int_R P(z)G(z, w_0) dxdy + \varepsilon \times \alpha(K^P(w_0, a') + K^P(w_0, a)) . \end{aligned}$$

This inequality shows the continuity of the function on  $\mathcal{A}_{P_0}$ :

$$\int_{R_n} P(z)G(z, w_0)K^P(z, a) dxdy ,$$

by which the relation

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{R_n} P(z)G(z, w_0)K^P(z, a) dxdy \\ & = \int_R P(z)G(z, w_0)K^P(z, a) dxdy \end{aligned}$$

implies that  $T^{P_0}K_a^P(w_0)$  is lower semi-continuous on  $\mathcal{A}_{P_0}$ .

**LEMMA 4.2.** *The function  $T^{0P}K_b(w_0)$  of  $b$  in  $\mathcal{A}_{0P}$  is upper semi-continuous on  $\mathcal{A}_{0P}$ , where  $w_0$  is a fixed point in  $R$ .*

*Proof.* Applying the inequality

$$\int_R P(z) G^P(z, w) dx dy \leq 2\pi$$

for all  $w$  in  $R$ , which is stated in Myberg [8], we can prove this lemma in the same way as that of the proof of Lemma 4.1. This Myberg's inequality plays the role of Nakai's condition (4.1) in Lemma 4.1.

Let  $\Delta_P$  and  $\Delta$  be the  $P$ -Martin and Martin ideal boundaries of  $R$ , respectively. We identify these ideal boundaries  $\Delta_P$  and  $\Delta$  with subsets of the product space of the real lines. Let  $\{w_i\}$  be a countable dense set of points in  $R$ . To a point  $a$  in  $\Delta_P$  (resp.  $b$  in  $\Delta$ ) we assign a point  $m_P(a)$  (resp.  $m_0(b)$ ) of the product space  $\prod_{i=1}^{\infty} I_i$  ( $I_i$  is the real line for all  $i$ ) whose  $i$ th coordinate is  $K^P(w_i, a)$  (resp.  $K(w_i, b)$ ) for each  $i$ . Then, the mappings

$$m_P: \Delta_P \longrightarrow \prod_{i=1}^{\infty} I_i$$

and

$$m_0: \Delta \longrightarrow \prod_{i=1}^{\infty} I_i$$

are continuous and one-to-one, and also their inverse mappings

$$m_P^{-1}: m_0(\Delta_P) \longrightarrow \Delta_P$$

and

$$m_0^{-1}: m_0(\Delta) \longrightarrow \Delta$$

are continuous. Therefore, the mappings

$$m_P: \Delta_P \longrightarrow m_P(\Delta_P)$$

and

$$m_0: \Delta \longrightarrow m_0(\Delta)$$

are homeomorphisms.

For a point  $m_P(a)$  in  $m_P(\Delta_{P_0})$  we assign a point in  $m_0(\Delta_{0P})$  whose  $i$ th coordinate is  $K(w_i, t_{P_0}(a))$  for each  $i$ ; this mapping will be denoted by

$$s_{P_0}: m_P(\Delta_{P_0}) \longrightarrow m_0(\Delta_{0P}) .$$



And, the mapping

$$s_{0P}: m_0(\Delta_{0P}) \longrightarrow m_P(\Delta_{P0})$$

is defined by the same way as that of  $s_{P0}$ , that is, for a point  $m_0(b)$  in  $m_0(\Delta_{0P})$  we assign a point in  $m_P(\Delta_{P0})$  whose  $i$ th coordinate is  $K^P(w_i, t_{0P}(b))$  for each  $i$ . It is evident that  $s_{0P}$  is the inverse mapping of  $s_{P0}$ .

In the following we shall always assume Nakai's condition (4.1).

**THEOREM 4.4.** *The mapping*

$$t_{P0}: \Delta_{P0} \longrightarrow \Delta_{0P}$$

*is measurability preserving.*

*Proof.* Since  $m_0^{-1}$  is continuous on  $m_0(\Delta)$  and, by Lemma 4.1, the  $i$ th coordinate of the point  $s_{P0} \circ m_P(a)$ ,  $a \in \Delta_{P0}$ :

$$K(w_i, t_{P0}(a)) = T^{P0} K_a^P(w_i) \times \{T^{P0} K_a^P(z_0)\}^{-1}$$

is a measurable function on  $\Delta_{P0}$  for each  $i$ , that is,  $s_{P0} \circ m_P$  is measurable on  $\Delta_{P0}$ , the relation

$$m_0^{-1} \circ s_{P0} \circ m_P = t_{P0} \quad \text{on} \quad \Delta_{P0}$$

shows that  $t_{P0}$  is measurable on  $\Delta_{P0}$ .

Similarly, from

$$t_{P0}^{-1} = t_{0P} = m_P^{-1} \circ s_{0P} \circ m_0 \quad \text{on} \quad \Delta_{0P}$$

and Lemma 4.2, it follows that  $t_{P0}^{-1}$  is measurable on  $\Delta_{0P}$ . Then the transformation  $t_{P0}: \Delta_{P0} \rightarrow \Delta_{0P}$  is measurability preserving.

**LEMMA 4.4.**  $\Delta_{0P}$  is measurable, so that  $(\Delta_{0P}, \chi)$  is a measure space, where  $\chi$  also denotes the restriction to  $\Delta_{0P}$  of the harmonic measure on  $\Delta_1$ .

*Proof.* Since  $\Delta_{P0}$  is measurable, this follows from the preceding lemma and the fact that  $\Delta_{0P} = t_{0P}^{-1}(\Delta_{P0})$ .

**THEOREM 4.5.** *The set  $\Delta_1 - \Delta_{0P}$  is of harmonic measure zero:*

$$\chi(\Delta_1 - \Delta_{0P}) = 0.$$

*Proof.* Since  $\Delta'_{0P}$  consists of points  $b$  in  $\Delta_1$  such that the minimal harmonic function  $K_b$  belongs to the set  $H'_P$  (where  $\Delta'_{0P}$  and  $H'_P$  are defined in §§2 and 3, respectively), Lemma 3.7 shows that

$$\Delta_1 - \Delta_{0P} \subset \Delta_1 - \Delta'_{0P} ,$$

which gives  $\chi(\Delta_1 - \Delta_{0P}) = 0$  by Theorem 2.4.

**LEMMA 4.6.** *Let  $u$  be a  $P$ -harmonic function in  $P_0$ , and let  $\mu$  be the canonical measure representing  $u$ :*

$$u(z) = \int_{\Delta_{P1}} K^P(z, a) d\mu(a) .$$

Then,

$$T^{P_0}u(z) = \int_{\Delta_{P0}} T^{P_0}K_a^P(z) d\mu(a) , \quad z \in R .$$

*Proof.* For a point  $z$  in  $R$ , let  $F_z$  be a function defined by

$$F_z(w, a) = P(w)G(w, z)K^P(w, a)$$

for  $(w, a)$  in  $R \times \Delta_{P0}$ . Since Lemma 2.1 shows that  $\Delta_1 - \Delta_{P0}$  has  $\mu$ -measure zero, it follows that

$$\int_R \left\{ \int_{\Delta_{P0}} F_z(w, a) d\mu(a) \right\} dudv = \int_R P(w)G(w, z)u(w) dudv < +\infty .$$

Then Fubini's theorem gives that

$$\begin{aligned} T^{P_0}u(z) &= \int_{\Delta_{P0}} K^P(z, a) d\mu(a) + \int_R \left\{ \int_{\Delta_{P0}} \frac{1}{2\pi} F_z(w, a) d\mu(a) \right\} dudv \\ &= \int_{\Delta_{P0}} \left\{ K^P(z, a) + \frac{1}{2\pi} \int_R P(w)G(w, z)K^P(w, a) dudv \right\} d\mu(a) \\ &= \int_{\Delta_{P0}} T^{P_0}K_a^P(z) d\mu(a) , \quad z \in R . \end{aligned}$$

By the uniqueness in the Martin integral representation, we obtain the following usefull theorem:

**THEOREM 4.7.** *Let  $u$  be a  $P$ -harmonic function in  $P_0$ , and let  $v$  denote the harmonic function  $T^{P_0}u$ . If the measures which represent  $u$  and  $v$  are denoted by  $\mu_u$  and  $\mu_v$ , respectively:*

$$u(z) = \int_{\Delta_{P1}} K^P(z, a) d\mu_u(a) ,$$

$$v(z) = \int_{\Delta_1} K(z, b) d\mu_v(b) ,$$

then the measure assigned to the restricted measure  $\mu_u|_{\Delta_{P0}}$  of  $\mu_u$  by the measurability preserving transformation  $t_{P_0}$  is absolutely con-

tinuous with respect to  $\mu_v$ ;  $d\mu = T^{P_0} K_{t_{P_0}^{-1}(f)}^P(z_0) d\mu \circ t_{P_0}^{-1}(b)$ .

*Proof.* Since  $t_{P_0}: \Delta_{P_0} \rightarrow \Delta_{0P}$  is measurable, from Lemma 4.6 and the definition of  $t_{P_0}$ , it follows that

$$\begin{aligned} v(z) &= T^{P_0} u(z) = \int_{\Delta_{P_0}} T^{P_0} K_a^P(z) d\mu_u(a) \\ &= \int_{\Delta_{P_0}} K(z, t_{P_0}(a)) T^{P_0} K_a^P(z_0) d\mu_u(a) = \int_{\Delta_{0P}} K(z, b) T^{P_0} K_{t_{P_0}^{-1}(b)}^P(z_0) d\mu_u \circ t_{P_0}^{-1}(b). \end{aligned}$$

From the uniqueness of the canonical measure which represents  $v$ , this theorem follows.

Theorems 4.7 and 4.3 are reduced to the following theorem:

**THEOREM 4.8.** *Let  $\mu_u$  and  $\mu_v$  be measures defined in Theorem 4.7. Then,  $t_{P_0}$  is a measurability preserving transformation from the measure space  $(\Delta_{P_0}, \mu_u)$  onto the measure space  $(\Delta_{0P}, \mu_v)$  such that  $d\mu_v = T^{P_0} K_{t_{P_0}^{-1}(b)}^P(z_0) d\mu_u \circ t_{P_0}^{-1}(b)$ .*

**COROLLARY 4.9.** *Let  $(R, P)$  be a hyperbolic pair.  $t_{P_0}$  is a measurability preserving transformation from the measure space  $(\Delta_{P_0}, \chi_P)$  onto the measure space  $(\Delta_{0P}, \chi)$  such that  $d\chi = T^{P_0} K_{t_{P_0}^{-1}(b)}^P(z_0) d\chi_P \circ t_{P_0}^{-1}(b)$ .*

*Proof.* By Theorem 3.13, Theorem 4.8 shows this corollary.

**5. Comparisons between relative Hardy spaces.** In [13] Parreau gave a characterization for harmonic functions in Hardy space on a Riemann surface, using the Martin boundary ([7]) and related kernel; in [9] L. L. Naim proved the similar results for the axiomatic functions of Brelot, using essentially Gowrisankaran's results ([3]) on axiomatic Martin boundary and fine limits. Since typical examples of Brelot's axiomatic setting are given by harmonic functions and by solutions of the differential equation  $\Delta u = Pu$  ( $P \geq 0$ ) on an open Riemann surface  $R$ , any result established in [9] for Brelot's axiomatic setting holds for each of these two special cases without further verification. Restating definitions and theorems in [9] in the case of harmonic functions and  $P$ -solutions, we recall the definitions of Hardy spaces, the relative Hardy spaces and some theorems for functions in these spaces.

For an exhaustion  $\{R_n\}$  of  $R$  and a fixed point  $z_0$  in  $R$ , we denote by  $\mu_{n, z_0}^P$  and  $\mu_{n, z_0}$  the  $P$ -elliptic measure and harmonic measure on  $\partial R_n$  relative to  $z_0$  and  $R_n$ , respectively. Clearly,

$$\int_{\partial R_n} d\mu_{n,z_0}^P \leq 1 \quad \text{and} \quad \int_{\partial R_n} d\mu_{n,z_0} = 1$$

for all positive integers  $n$ .

**DEFINITION 5.1.** A harmonic function  $g$  on  $R$  belongs to the Hardy space  $H^p$ ,  $1 \leq p \leq +\infty$ , if and only if the  $L^p$ -norms with respect to the harmonic measures  $\mu_{n,z_0}$  of the restrictions of  $g$  to the boundaries  $\partial R_n$ , are uniformly bounded in  $n$ . In other words,  $g$  belongs to  $H^p$  if and only if there exists a constant  $M$ , independent of  $n$ , such that  $\|g_{p,n}\| \leq M$  for all  $n$ , where

$$\|g\|_{p,n} = \left\{ \int_{\partial R_n} |g|^p d\mu_{n,z_0} \right\}^{1/p}, \quad 1 \leq p < +\infty,$$

and

$$\|g\|_{\infty,n} = \sup_{\partial R_n} |g|.$$

We proceed to define the relative Hardy spaces for the equation  $\Delta u = Pu$  and harmonic functions. For a fixed positive  $P$ -harmonic function  $u$  on  $R$  we define the relative  $u$ - $P$ -elliptic measure with respect to  $z_0 \in R_n$  and  $R_n$  by

$$\mu_{n,z_0}^{P,u} = \frac{u}{u(z_0)} \times \mu_{n,z_0}^P,$$

and for a fixed positive harmonic function  $v$  on  $R$  we define the relative  $v$ -harmonic measure with respect to  $z_0 \in R_n$  and  $R_n$  by

$$\mu_{n,z_0}^v = \frac{v}{v(z_0)} \times \mu_{n,z_0}.$$

For the positive  $P$ -harmonic function  $u$   $u$ - $P$ -harmonic functions are quotients of  $P$ -harmonic functions on  $R$  by  $u$ , and for the positive harmonic function  $v$   $v$ -harmonic functions are quotients of harmonic functions on  $R$  by  $v$ .

**DEFINITIONS 5.2.** A  $u$ - $P$ -harmonic function  $f'$  belongs to the relative Hardy class  $PH_u^p$ ,  $1 \leq p \leq +\infty$ , if and only if the  $L^p$ -norms with respect to the relative  $u$ - $P$ -elliptic measure  $\mu_{n,z_0}^{P,u}$ , of the restrictions of  $f'$  to the boundaries  $\partial R_n$  are uniformly bounded in  $n$ . In other words,  $f'$  belongs to  $PH_u^p$  if and only if there exists a constant  $M$ , independent of  $n$ , such that  $\|f'\|_{p,n}^P \leq M$  for positive integers, where

$$\|f'\|_{p,n}^P = \left\{ \int_{\partial R_n} |f'|^p d\mu_{n,z_0}^{P,u} \right\}^{1/P}, \quad 1 \leq p < +\infty,$$

$$\|f'\|_{\infty,n}^P = \sup_{\partial R_n} |f'|^P.$$

DEFINITION 5.3. A  $v$ -harmonic function  $g'$  belongs to the relative Hardy space  $H_v^p$ ,  $1 \leq p \leq +\infty$ , if and only if the  $L^p$ -norms with respect to the relative  $v$ -harmonic measure  $\mu_{n, z_0}^v$ , of the restrictions of  $g'$  to the boundaries  $\partial R_n$  are uniformly bounded in  $n$ . In other words,  $g'$  belongs to  $H_v^p$  if and only if there exists a constant  $M$ , independent of  $n$ , such that  $\|g'\|_{p,n} \leq M$  for all  $n$ , where

$$\|g'\|_{p,n} = \left\{ \int_{\partial R_n} |g'|^p d\mu_{n, z_0}^v \right\}^{1/p}, \quad 1 \leq p < +\infty,$$

$$\|g'\|_{\infty,n} = \sup_{\partial R_n} |g'|.$$

Naim gave the extended characterization of functions in Hardy spaces, showing the role of uniform integrability. We shall recall her theorems and restate them in our case. In the following, fine filters defined by the  $P$ -Martin compactification and minimal  $P$ -harmonic functions  $K_a^P$ ,  $a \in \Delta_{P1}$ , is called  $P$ -fine filters.

THEOREM 5.1. Let  $u$  be a fixed positive  $P$ -harmonic function on  $R$ . A  $u$ - $P$ -harmonic function  $f'$  belongs to the space  $PH_u^p$ ,  $1 < p \leq +\infty$ , if and only if  $f'$  is the solution of a Dirichlet problem relative to  $u$  with the  $P$ -minimal boundary  $\Delta_{P1}$ , the  $P$ -fine filters in  $R$  and boundary function  $\bar{f}'$  in  $L^p(\Delta_{P1}, \mu_u)$ , where  $\mu_u$  represents  $u$  in the integral representation

$$u(z) = \int_{\Delta_{P1}} K^P(z, a) d\mu_u(a).$$

And, the correspondence  $f' \rightarrow \bar{f}'$  is an isometric isomorphism of the Banach space  $PH_u^p$  onto  $L^p(\Delta_{P1}, \mu_u)$ .

THEOREM 5.2. A harmonic function  $g$  belongs to the space  $H^p$ ,  $1 < p \leq +\infty$ , if and only if  $g$  is the solution of a Dirichlet problem with the minimal boundary  $\Delta_1$ , the fine filters in  $R$  and boundary function  $\bar{g}$  in  $L^p(\Delta_1, \chi)$ . And, the correspondence  $g \rightarrow \bar{g}$  is an isometric isomorphism of the Banach space  $H^p$  onto  $L^p(\Delta_1, \chi)$ .

THEOREM 5.3. Let  $v$  be a fixed positive harmonic function on  $R$ . A  $v$ -harmonic function  $g'$  belongs to the space  $H_v^p$ ,  $1 < p \leq +\infty$ , if and only if  $g'$  is the solution for a Dirichlet problem relative to  $v$  with the minimal boundary  $\Delta_1$ , the fine filters in  $R$  and boundary function  $\bar{g}'$  in  $L^p(\Delta_1, \mu_v)$ , where  $\mu_v$  represents  $v$  in the integral representation

$$v(z) = \int_{\Delta} K(z, b) d\mu_v(b).$$

And,  $H_v^p$  is a Banach space isometrically isomorphic to  $L^p(\Delta_1, \mu_v)$ .

**THEOREM 5.4.** *Let  $u$  and  $v$  be functions on  $R$  satisfying the same conditions as those in Theorem 4.7. If Nakai's condition:*

$$(5.1) \quad \int_R P(z)G(z, w_0)dx dy < +\infty$$

for some point  $w_0$  in  $R$  is satisfied, then the Banach space  $PH_u^p$ ,  $1 < p \leq +\infty$ , is isometrically isomorphic to the Banach space  $H_v^p$ .

*Proof.* For a function  $f'$  in  $PH_u^p$ ,  $1 < p \leq +\infty$ , there exists a boundary function  $\bar{f}'$  in  $L^p(\Delta_{P1}, \mu_u)$ , with which the relative Dirichlet problem gives  $f'$ . The function  $\{T^{0P}K_b(z_0)\}^{1/p}\bar{f}' \circ t_{P0}^{-1}$  is defined on  $\Delta_{0P}$  and satisfies, by Lemma 2.1 and Theorem 4.8, that

$$\begin{aligned} \int_{\Delta_{0P}} T^{0P}K_b(z_0) |\bar{f}' \circ t_{P0}^{-1}|^p d\mu_v &= \int_{\Delta_{P0}} |\bar{f}'|^p d\mu_u = \int_{\Delta_{P1}} |\bar{f}'|^p d\mu_u, \\ \text{ess. sup}_{\Delta_{0P}} |\bar{f}' \circ t_{P0}^{-1}| &= \text{ess. sup}_{\Delta_{P1}} |\bar{f}'|, \end{aligned}$$

where the essential supremums are taken with respect to  $\mu_u$  and  $\mu_v$  respectively. This shows that  $\bar{f}' \circ t_{P0}^{-1}$  belongs to  $L^p(\Delta_1, \mu_v)$ , since  $L^p(\Delta_1, \mu_v) = L^p(\Delta_{0P}, \mu_v)$  by Lemma 2.2.

To a  $u$ - $P$ -harmonic function  $f'$  in  $PH_u^p$  we assign the solution for the Dirichlet problem relative to  $v$  with the boundary function  $\{T^{0P}K_b(z_0)\}^{1/p}\bar{f}' \circ t_{P0}^{-1}$ . Then, this solution is a function in the space  $H_v^p$  by Theorem 5.3. Denoting this function by  $\bar{T}_{P0}(f')$ , we define a linear transformation

$$\bar{T}_{P0}: PH_u^p \longrightarrow H_v^p.$$

The fact that  $\bar{T}_{P0}$  is an isometric isomorphism from  $PH_u^p$  onto  $H_v^p$  is easily verified by theorems prepared in §4.

**THEOREM 5.5.** *Let  $(R, P)$  be a hyperbolic pair. If Nakai's condition (5.1) is satisfied, then the Banach space  $PH_w^p$ ,  $1 < p \leq +\infty$ , is isometrically isomorphic to the Banach space  $H^p$ , where  $w$  is the  $P$ -elliptic measure.*

*Proof.* By Corollary 4.9 we can prove this theorem by the same way as that in the proof of Theorem 5.4.

Since  $PH_w^\infty = PB$  and  $H^\infty = HB$ , it is clear that this theorem contains Nakai's result ([11]): under the condition (5.1)  $PB$  and  $HB$  are isometrically isomorphic.

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