

# Pacific Journal of Mathematics

**$C^*$ -ALGEBRAS ASSOCIATED WITH IRRATIONAL ROTATIONS**

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## C\*-ALGEBRAS ASSOCIATED WITH IRRATIONAL ROTATIONS

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**For any irrational number  $\alpha$  let  $A_\alpha$  be the transformation group C\*-algebra for the action of the integers on the circle by powers of the rotation by angle  $2\pi\alpha$ . It is known that  $A_\alpha$  is simple and has a unique normalized trace,  $\tau$ . We show that for every  $\beta$  in  $(\mathbb{Z} + \mathbb{Z}\alpha) \cap [0, 1]$  there is a projection  $p$  in  $A_\alpha$  with  $\tau(p) = \beta$ . When this fact is combined with the very recent result of Pimsner and Voiculescu that if  $p$  is any projection in  $A_\alpha$  then  $\tau(p)$  must be in the above set, one can immediately show that, except for some obvious redundancies, the  $A_\alpha$  are not isomorphic for different  $\alpha$ . Moreover, we show that  $A_\alpha$  and  $A_\beta$  are strongly Morita equivalent exactly if  $\alpha$  and  $\beta$  are in the same orbit under the action of  $\text{GL}(2, \mathbb{Z})$  on irrational numbers.**

0. Introduction. Let  $\alpha$  be an irrational number, and let  $S$  denote the rotation by angle  $2\pi\alpha$  on the circle,  $T$ . Then the group of integers,  $\mathbb{Z}$ , acts as a transformation group on  $T$  by means of powers of  $S$ , and we can form the corresponding transformation group C\*-algebra,  $A_\alpha$ , as defined in [8, 19, 30]. If we view  $S$  as also acting on functions on  $T$ , and if  $C(T)$  denotes the algebra of continuous complex-valued functions on  $T$ , then  $S$  acts as an automorphism of  $C(T)$ . This gives an action of  $\mathbb{Z}$  as a group of automorphisms of  $C(T)$ , and  $A_\alpha$  is just the crossed product algebra for this action [19, 30]. A convenient concrete realization of  $A_\alpha$  consists of the norm-closed \*-algebra of operators on  $L^2(T)$  generated by  $S$  together with all the pointwise multiplication operators,  $M_f$ , for  $f \in C(T)$ . It is known [8, 19, 22, 30] that  $A_\alpha$  is a simple C\*-algebra (with identity element) not of type I, and that  $A_\alpha$  has a unique normalized trace,  $\tau$ . In fact, on the dense \*-subalgebra  $C_c(\mathbb{Z}, T, \alpha)$  consisting of finite sums of the form  $\sum M_{f_n} S^n$  the trace is given by

$$\tau(\sum M_{f_n} S^n) = \int_T f_0(t) dt ,$$

where  $dt$  is Lebesgue measure on the circle normalized to give the circle unit measure. (We remark that Theorem 1.1 of [27] can be used to show that this dense subalgebra itself is also simple.)

Little else has been known about the  $A_\alpha$ . In particular, it has not been known whether or not the  $A_\alpha$  are isomorphic as  $\alpha$  varies. An interesting question raised in 7.3 of [8], and again recently in [22], is whether the  $A_\alpha$  contain any projections. But in fact, shortly

after [8] appeared, R. T. Powers showed in unpublished work that there are self-adjoint elements in the  $A_\alpha$  which have disconnected spectrum, from which one can infer that the  $A_\alpha$  contain proper projections.

The main contribution of this paper is to show how to describe very explicitly some projections in the  $A_\alpha$ —so explicitly that it is then obvious what value the trace has on them. Specifically, we show:

**THEOREM 1.** *For each  $\beta \in (\mathbf{Z} + \mathbf{Z}\alpha) \cap [0, 1]$  there is a projection  $p$  in  $A_\alpha$  such that  $\tau(p) = \beta$ .*

This result was announced in [26], together with the conjecture that the trace of any projection in  $A_\alpha$  must be in  $(\mathbf{Z} + \mathbf{Z}\alpha) \cap [0, 1]$ . I was essentially through writing up this work when I received the fascinating preprint [20] of M. Pimsner and D. Voiculescu in which they show that the above conjecture is true. Their ingenious method of proof consists of showing that  $A_\alpha$  can be embedded in one of the special  $AF$  algebras constructed by E. G. Effros and C. L. Shen [10] whose  $K_0$  group is  $\mathbf{Z} + \mathbf{Z}\alpha$ , ordered as a subgroup of the real line  $\mathbf{R}$ . This fact, together with the results of the present paper, show that the range of the trace on the projections in  $A_\alpha$  is exactly  $(\mathbf{Z} + \mathbf{Z}\alpha) \cap [0, 1]$ . And this, in turn, settles the isomorphism question. Specifically, as also stated in [20]:

**THEOREM 2.** *If  $\alpha$  and  $\beta$  are irrational numbers in the interval  $[0, 1/2]$ , and if  $A_\alpha$  and  $A_\beta$  are isomorphic, then  $\alpha = \beta$ . If  $\alpha$  is any irrational number, with fractional part  $\{\alpha\}$ , let  $\beta = \{\alpha\}$  or  $1 - \{\alpha\}$  depending on which is in  $[0, 1/2]$ . Then  $A_\alpha$  and  $A_\beta$  are isomorphic.*

In §1 we also point out that a trivial modification of the result of Pimsner and Voiculescu also settles the isomorphism question for the algebras of  $n \times n$  matrices over the  $A_\alpha$ . Specifically, if  $M_n$  denotes the algebra of complex  $n \times n$  matrices, then:

**THEOREM 3.** *Let  $\alpha$  and  $\beta$  be irrational numbers in  $[0, 1/2]$ , and let  $m$  and  $n$  be positive integers. If  $M_m \otimes A_\alpha$  is isomorphic to  $M_n \otimes A_\beta$ , then  $m = n$  and  $\alpha = \beta$ .*

Finally, in §2 we show how our results together with those of Pimsner and Voiculescu can also be used to settle the question of when the  $A_\alpha$  are strongly Morita equivalent, as defined in [24]. The main result is:

**THEOREM 4.** *The algebras  $A_\alpha$  and  $A_\beta$  are strongly Morita equivalent if and only if  $\alpha$  and  $\beta$  are in the same orbit of the action of  $GL(2, \mathbf{Z})$  on irrational numbers by linear fractional transformations.*

We conclude this paper by pointing out the implications of these theorems for the transformation group  $C^*$ -algebras for flows on the torus at irrational angle, and also a curious consequence for functions of a real variable.

There still remains much that is unknown about the  $A_\alpha$ . Among the few other facts which are known, are that the  $A_\alpha$  are strongly amenable, hence amenable and nuclear—see [28] by J. Rosenberg. I am also familiar with unpublished work of P. Green in which he shows that the group of invertible elements in an  $A_\alpha$  is not connected, so that the  $A_\alpha$  are not themselves  $AF$  algebras. This result also has just appeared at the end of [3]. During second corrections of this paper I received the preprint [21] of Pimsner and Voiculescu in which they show that the  $K_0$  group of  $A_\alpha$  is  $\mathbf{Z} + \mathbf{Z}\alpha$ . They also compute the  $K_1$  group<sup>1</sup>. Also, a very recent combination of arguments of S. Popa and myself [34] show that the strong Ext group of  $A_\alpha$  is  $\mathbf{Z} + \mathbf{Z}$ .

The  $A_\alpha$  occur in a variety of situations. They are exactly the  $C^*$ -algebras generated by any pair of unitary operators  $U$  and  $V$  which satisfy  $UV = \lambda VU$  where  $\lambda = \exp(-2\pi i\alpha)$ . They can be defined as the  $C^*$ -algebras corresponding to appropriate cocycles on  $\mathbf{Z} \times \mathbf{Z}$  as in [30]. They are exactly the simple  $C^*$ -algebras on which the torus group  $T^2$  has ergodic actions [1, 18, 33]. They occur as the simple non-finite-dimensional quotients of the group  $C^*$ -algebra of the Heisenberg group over  $\mathbf{Z}$ , that is, the group of  $3 \times 3$  upper triangular matrices with entries in  $\mathbf{Z}$  and ones on the diagonal [16]. They occur as the quotients by the commutator ideal of certain  $C^*$ -algebras associated to one-parameter semigroups in [7] (see also [11]). They are Morita equivalent to the transformation group  $C^*$ -algebras for flows on the torus at irrational angles. (It was Phil Green who pointed out to me that this is one consequence of the main theorem of [24], and his results in [15] can be used to give more information about the relation between these algebras.) Consequently, the  $A_\alpha$  are strongly Morita equivalent to certain simple quotients of the group  $C^*$ -algebras of various solvable Lie groups (see closing comments in [12, 14]). The  $A_\alpha$  are also related to the work of A. Connes [5] concerning operator algebras associated with foliations<sup>2</sup>.

I am very indebted to R. T. Powers for having pointed out to me at an early stage the benefits of being able to calculate the

<sup>1</sup> See also [32].

<sup>2</sup> See also [31].

trace on projections, namely that if  $B$  is a separable  $C^*$ -algebra with unique normalized trace, then the range of the trace on the projections in  $B$  is a countable subset of the interval  $[0, 1]$  which is an isomorphism invariant of  $B$ . I would also like to thank B. Blackadar and P. Green for helpful comments.

**1. Projections.** For ease of notation we will view the elements of  $C(\mathbf{T})$  as continuous functions on the real line,  $\mathbf{R}$ , which are periodic of period 1. Thus  $S$  just becomes the shift  $S(f)(t) = f(t - \alpha)$  for  $f \in C(\mathbf{T})$  and  $t \in \mathbf{R}$ . Notice that  $SM_f = M_{S(f)}S$ . We will say that an element of  $A_\alpha$  is supported on  $\{-1, 0, 1\}$  if it is of the form

$$M_h S^{-1} + M_f + M_g S$$

for  $h, f, g \in C(\mathbf{T})$ . We have the following slight refinement of Theorem 1:

**THEOREM 1.1.** *For every  $\beta \in (\mathbf{Z} + \mathbf{Z}\alpha) \cap [0, 1]$  there is a projection  $p$  in  $A_\alpha$ , supported on  $\{-1, 0, 1\}$ , such that  $\tau(p) = \beta$ .*

*Proof.* Suppose that  $p$  is a projection supported on  $\{-1, 0, 1\}$ , and expressed, as above, in terms of  $h, f, g$ . Then from the fact that  $p$  is self-adjoint it is easily seen that  $f$  is real-valued, and that  $h = S^*(\bar{g})$ . Combining this with the fact that  $p$  is idempotent, one obtains:

- (1)  $g(t)g(t - \alpha) = 0$ ,
- (2)  $g(t)[1 - f(t) - f(t - \alpha)] = 0$ ,
- (3)  $f(t)[1 - f(t)] = |g(t)|^2 + |g(t + \alpha)|^2$ ,

for  $t \in \mathbf{R}$ . Conversely, it is easily seen that if  $f$  and  $g$  are elements of  $C(\mathbf{T})$  which satisfy these equations, and if we let  $h = S^*(\bar{g})$ , then the corresponding element of  $A_\alpha$  will be a projection. Closer examination then shows that there are myriad choices of  $f$  and  $g$  which satisfy these relations.

Since translation by  $\alpha$  is the same on  $C(\mathbf{T})$  as translation by the fractional part of  $\alpha$ , we assume now that  $\alpha \in [0, 1]$ . Furthermore since  $S^*$  is translation by  $1 - \alpha$ , so that  $A_\alpha \cong A_{1-\alpha}$ , we can assume that  $\alpha \in [0, 1/2]$ . With this assumption, let us show first how to construct a projection  $p$  such that  $\tau(p) = \alpha$ . For this, let  $f$  be almost the characteristic function of  $[0, \alpha]$ , but rounded at the ends in a somewhat careful way. Specifically, we notice that equation (3) says that if  $f(t)$  is not 0 or 1, then either  $g(t)$  or  $g(t + \alpha)$  is non-zero (while equation (1) says that not both can be nonzero simultaneously). Then equation (2) says that if  $g(t) \neq 0$ , then  $f(t) + f(t - \alpha) = 1$ . Choose any  $\varepsilon > 0$  such that  $\varepsilon < \alpha$  and  $\alpha + \varepsilon < 1/2$ . On  $[0, \varepsilon]$  let  $f$  be any continuous function with values in  $[0, 1]$  and

with  $f(0) = 0$  and  $f(\varepsilon) = 1$ . On  $[\alpha, \alpha + \varepsilon]$  define  $f$  by  $f(t) = 1 - f(t - \alpha)$ , while on  $[\varepsilon, \alpha]$  and  $[\alpha + \varepsilon, 1]$  let  $f$  have values 1 and 0 respectively. Finally, on  $[\alpha, \alpha + \varepsilon]$  define  $g$  by

$$g(t) = (f(t)(1 - f(t)))^{1/2},$$

and let  $g$  have value zero elsewhere on  $[0, 1]$ . Then  $f$  and  $g$  satisfy relations (1), (2) and (3) above and so define a projection, whose trace is  $\int_0^1 f(t)dt = \alpha$ .

To handle the general case, note first that, for any positive integer  $m$ , the algebra  $C(T)$  contains the algebra  $C_m(T)$  of continuous functions on  $R$  periodic of period  $1/m$ . On  $C_m(T)$  the shift by  $\alpha$  looks like the shift on  $C(T)$  by  $\{m\alpha\}$ , the fractional part of  $m\alpha$ . What this means is that  $A_{\{m\alpha\}}$  is embedded as a subalgebra of  $A_\alpha$ , with the same identity element. The restriction to  $A_{\{m\alpha\}}$  of the trace on  $A_\alpha$  will be the trace on  $A_{\{m\alpha\}}$ , and so a projection in  $A_{\{m\alpha\}}$  of trace  $\{m\alpha\}$ , constructed as above, will be a projection in  $A_\alpha$  of same trace. Furthermore, elements of  $A_{\{m\alpha\}}$  which are supported on  $\{-1, 0, 1\}$  will also be supported there when viewed as elements of  $A_\alpha$ .

Finally, we must treat values of form  $\{-m\alpha\}$  for  $m$  positive. But for these it suffices to find projections of form  $1 - \{-m\alpha\} = \{m\alpha\}$ , and this is handled above. □

If we combine this theorem with that of Pimsner and Voiculescu [20] described earlier, we obtain:

**THEOREM 1.2.** *The range of the trace on projections in  $A_\alpha$  is exactly  $(Z + Z\alpha) \cap [0, 1]$ .*

To view this result in a wider context, let  $p$  and  $q$  be projections in a  $C^*$ -algebra  $A$ . We say they are unitarily equivalent if there is a unitary  $u$  in  $A$  such that  $q = upu^*$ . It can be shown that if  $\|p - q\| < 1$ , then  $p$  and  $q$  are unitarily equivalent [19]. If  $A$  is separable, it then follows that there is only a countable number of unitary equivalence classes of projections in  $A$ . Now any trace on  $A$  will be constant on unitary equivalence classes, and so the range of the trace when restricted to projections will be a countable set of positive numbers. If  $A$  has a unique normalized trace, then the range of this trace on projections will be an isomorphism invariant for  $A$ . All of this was pointed out to me by Robert T. Powers.

Now if  $(Z + Z\alpha) \cap [0, 1] = (Z + Z\beta) \cap [0, 1]$ , with  $\alpha, \beta \in [0, 1]$ , then a quick calculation shows that  $\beta = \alpha$  or  $1 - \alpha$ . Since, as noted above,  $A_\alpha \cong A_{1-\alpha}$  (and, of course,  $A_\alpha \cong A_{\alpha+n}$  for all  $n \in Z$ ), we see that we have arrived at a proof of Theorem 2.

We turn now to the proof of Theorem 3. Let  $B_\alpha$  denote the  $AF$  algebra constructed by Effros and Shen [10] whose  $K_0$  group is  $\mathbf{Z} + \mathbf{Z}\alpha$ , and into which Pimsner and Voiculescu [20] show that  $A_\alpha$  can be embedded (with same identity element). As they emphasize,  $B_\alpha$  has a unique normalized trace whose range on projections is  $(\mathbf{Z} + \mathbf{Z}\alpha) \cap [0, 1]$ . Now  $M_n \otimes B_\alpha$  will have the same  $K_0$  group as  $B_\alpha$ , and will also have a unique normalized trace, but this trace is easily seen now to have  $(n^{-1}(\mathbf{Z} + \mathbf{Z}\alpha)) \cap [0, 1]$  as its range on projections. Since  $M_n \otimes A_\alpha$  can be embedded in  $M_n \otimes B_\beta$ , it follows that the range of the trace for  $M_n \otimes A_\alpha$  on projections must be contained in  $(n^{-1}(\mathbf{Z} + \mathbf{Z}\alpha)) \cap [0, 1]$ . But if  $0 < j + k\alpha < n$ , and if we let  $m$  denote the integer part of  $j + k\alpha$ , then  $(j - m) + k\alpha$  is in  $[0, 1]$  so that there is a projection,  $q$ , in  $A_\alpha$  with  $\tau(q) = (j - m) + k\alpha$ . Since  $m < n$ , we can form a projection in  $M_n \otimes A_\alpha$  which has  $q$  as one diagonal entry, 1's in  $m$  other diagonal entries, and 0's elsewhere. It is clear that the normalized trace for  $M_n \otimes A_\alpha$  on this projection will be  $n^{-1}(j + k\alpha)$ . Consequently:

**PROPOSITION 1.3.** *The range of the normalized trace for  $M_n \otimes A_\alpha$  on projections is exactly  $(n^{-1}(\mathbf{Z} + \mathbf{Z}\alpha)) \cap [0, 1]$ .*

*Proof of Theorem 3.* The range of the trace of  $M_n \otimes A_\alpha$  and  $M_n \otimes A_\beta$  on projections must clearly contain  $1/m$  and  $1/n$  respectively. From Proposition 1.3 it follows that  $m = n$ . Then again from Proposition 1.3,  $n^{-1}\alpha = n^{-1}(p + q\beta)$  and  $n^{-1}\beta = n^{-1}(r + s\alpha)$ . It follows that  $\alpha = \beta$ .  $\square$

For the purposes of the next section, let us now interpret the above results at the level of  $K_0$  groups, as defined in [9]. Let  $A$  be a  $C^*$ -algebra which has a faithful trace,  $\tau$ . Then  $K_0(A)$  will be a partially ordered group for the reasons given in [6, 12]. Furthermore,  $\tau$  defines an evident homomorphism,  $\hat{\tau}$ , from  $K_0(A)$  to  $\mathbf{R}$ , and  $\hat{\tau}$  will be order preserving. From the earlier results one quickly obtains:

**PROPOSITION 1.4.** *As ordered group,  $\hat{\tau}(K_0(A_\alpha))$  is just  $\mathbf{Z} + \mathbf{Z}\alpha$  ordered as a subgroup of  $\mathbf{R}$ .*

As mentioned in the introduction, Pimsner and Voiculescu have gone on to show [21] that  $\hat{\tau}$  is in fact an isomorphism of  $K_0(A_\alpha)$  with  $\mathbf{Z} + \mathbf{Z}\alpha$ .

**2. Morita equivalence.** Let  $G$  be a locally compact group, and let  $H$  and  $K$  be closed subgroups of  $G$ . Then  $G$  acts by translation

on  $G/H$  and  $G/K$ , and we can restrict this action to  $K$  and  $H$  respectively, so that  $K$  acts on  $G/H$  while  $H$  acts on  $G/K$ . The main theorem of [24] then says that the corresponding transformation group  $C^*$ -algebras  $C^*(K, G/H)$  and  $C^*(H, G/K)$  are strongly Morita equivalent.

If we apply the above to the case in which  $G = \mathbf{R}$ ,  $H = \mathbf{Z}$  and  $K = \mathbf{Z}\alpha$ , we find that  $C^*(\mathbf{Z}\alpha, \mathbf{R}/\mathbf{Z})$  is strongly Morita equivalent to  $C^*(\mathbf{Z}, \mathbf{R}/\mathbf{Z}\alpha)$ . Now the first of these algebras is just  $A_\alpha$ . But if we apply the homeomorphism  $t \rightarrow t\alpha^{-1}$  to  $\mathbf{R}$ , we find that the second of these algebras is isomorphic to  $C^*(\mathbf{Z}\alpha^{-1}, \mathbf{R}/\mathbf{Z})$ . That is,  $A_\alpha$  is strongly Morita equivalent to  $A_{(\alpha^{-1})}$ . (Of course, if we want to restrict to  $\alpha$  in  $[0, 1]$  we need to take the fractional part of  $\alpha^{-1}$ , but for present purposes it is simpler not to make this restriction.)

As indicated earlier,  $A_\alpha$  is obviously isomorphic to  $A_{(\alpha+n)}$  for any  $n \in \mathbf{Z}$ . Let  $\text{GL}(2, \mathbf{Z})$  denote the group of  $2 \times 2$  matrices with entries in  $\mathbf{Z}$  and with determinant  $\pm 1$ , and let  $\text{GL}(2, \mathbf{Z})$  act on the set of irrational numbers by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \frac{a\alpha + b}{c\alpha + d}.$$

It is well-known (see Appendix B of [17]) that  $\text{GL}(2, \mathbf{Z})$  is generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

But these are just the matrices which carry  $\alpha$  to  $\alpha^{-1}$ , and  $\alpha + 1$  respectively. It follows that if  $\alpha$  and  $\beta$  are irrational numbers which are in the same orbit of the action of  $\text{GL}(2, \mathbf{Z})$ , then  $A_\alpha$  and  $A_\beta$  are strongly Morita equivalent. We will now see that by using the results of Pimsner and Voiculescu we can show the converse, thus obtaining a proof of Theorem 4.

If  $A$  and  $B$  are  $C^*$ -algebras with identity elements which are strongly Morita equivalent, then they are stably isomorphic [4], and from this it is known that  $A$  and  $B$  will have isomorphic  $K_0$  groups. Now, as mentioned earlier, traces on a  $C^*$ -algebra define homomorphisms from the  $K_0$  group of the algebra into  $\mathbf{R}$ . For  $C^*$ -algebras which are strongly Morita equivalent and have unique traces, the ranges of the corresponding homomorphisms from the  $K_0$  groups will be isomorphic as groups. But note from Proposition 1.4 that the  $\hat{\tau}(K_0(A_\alpha))$ , as abstract groups, are all isomorphic anyway for different  $\alpha$ . So in order to gain significant information, what we need to show is that for algebras which are strongly Morita equivalent, the isomorphisms which one obtains between the  $\hat{\tau}(K_0(A_\alpha))$  are in fact order



isomorphisms for the order obtained from being subgroups of  $R$ . To do this we must carefully relate traces to Morita equivalence.

Recall [4] that by a corner of a  $C^*$ -algebra  $C$  with identity element we mean a subalgebra of form  $pCp$  where  $p$  is a projection in  $C$ , and that a corner is said to be full if it is not contained in any proper two-sided ideal. Now for  $C^*$ -algebras with identity elements, strong Morita equivalence is essentially the same as purely algebraic Morita equivalence. In particular, in analogy with 22.7 of [2], we have:

**PROPOSITION 2.1.** *If  $C$  and  $D$  are  $C^*$ -algebras which are strongly Morita equivalent, and if they both have identity elements, then each is a full corner of the algebra of  $n \times n$  matrices over the other, for suitable  $n$ .*

*Proof.* Let  $X$  be a  $C$ - $D$ -equivalence bimodule (i.e., imprimitivity bimodule—see 6.10 of [23]). By the definition of  $X$ , the range of  $\langle \ , \ \rangle_D$  spans a dense ideal of  $D$ . But since  $D$  has an identity element, this range must in fact coincide with  $D$ . Consequently, we can find  $2n$  elements,  $x_1, \dots, x_n, y_1, \dots, y_n$  of  $X$  such that

$$\sum \langle x_i, y_i \rangle_D = 1.$$

Let  $M_n$  denote the algebra of  $n \times n$  complex matrices and let  $E = M_n \otimes C$ . Consider  $X^n$  as an  $E$ - $D$ -equivalence bimodule in the evident way, and let  $x = \{x_1, \dots, x_n\}$  and  $y = \{y_1, \dots, y_n\}$ , which are elements of  $X^n$ . Then  $\langle x, y \rangle_D = 1$ . Consequently,  $\langle y, x \rangle_D = 1$  also, so that

$$\begin{aligned} 1 &= \langle x, y \rangle_D \langle y, x \rangle_D = \langle x, y \langle y, x \rangle_D \rangle_D \\ &= \langle x, \langle y, y \rangle_E x \rangle_D = \langle z, z \rangle_D, \end{aligned}$$

where  $z = \langle y, y \rangle_E^{1/2} x$ . Then

$$\begin{aligned} \langle z, z \rangle_E \langle z, z \rangle_E &= \langle \langle z, z \rangle_E z, z \rangle_E \\ &= \langle z \langle z, z \rangle_D, z \rangle_E = \langle z, z \rangle_E, \end{aligned}$$

so that  $\langle z, z \rangle_E$  is a projection, which we will denote by  $p$ . Simple calculations show that the map  $\phi$  of  $D$  into  $E$  defined by  $\phi(d) = \langle zd, z \rangle_E$  is a  $*$ -homomorphism which is injective and into the corner  $pEp$ . Finally, since the range of  $\langle \ , \ \rangle_E$  is dense in  $E$ , the corner  $pEp$  will be densely spanned by elements of form  $p \langle x, y \rangle_E p$  for  $x, y \in X^n$ . But a simple calculation shows that

$$p \langle x, y \rangle_E p = \phi(\langle z, x \rangle_D \langle y, z \rangle_D).$$

Thus the range of  $\phi$  is exactly the corner  $pEp$ . By reversing the

roles of  $C$  and  $D$  one finds in the same way that  $C$  is isomorphic to a corner in matrices over  $D$ . It is easily seen that the corners must be full. □

Now if  $C$  and  $D$  are  $C^*$ -algebras and if  $X$  is a  $C$ - $D$ -equivalence bimodule, then every trace on  $C$  can be induced by  $X$  to give a trace on  $D$ . For the general case of possibly unbounded traces, this is implicit in Proposition 28 of [14]. But in the present case of  $C^*$ -algebras with identity elements and of finite traces, the situation is very simple:

**PROPOSITION 2.2.** *Let  $C$  and  $D$  be  $C^*$ -algebras with identity elements, and let  $X$  be a  $C$ - $D$ -equivalence bimodule. Then there is a bijection between the (nonnormalized) finite traces on  $C$  and those on  $D$ , under which to a trace on  $C$  there is associated a trace  $\tau_x$  on  $D$  satisfying*

$$\tau_x(\langle x, y \rangle_D) = \tau(\langle y, x \rangle_C)$$

for all  $x, y \in X$ .

*Proof.* Since the span of the range of  $\langle \ , \ \rangle_D$  is all of  $D$ , it is clear that, if  $\tau_x$  exists, then  $\tau_x$  is uniquely determined by the above condition. Let  $n, z, E$  and  $\phi$  be defined as in the proof of Proposition 2.1. Let  $\tau$  also denote the corresponding (nonnormalized) trace on  $M_n \otimes C = E$ , and let  $\tau_x$  be the trace on  $D$  defined by  $\tau_x(d) = \tau(\phi(d))$ . We show that  $\tau_x$  satisfies the above condition. Let  $x, y \in X$ , and view them as the elements  $(x, 0, \dots, 0)$  and  $(y, 0, \dots, 0)$  of  $X^n$ , so that

$$\tau(\langle y, x \rangle_E) = \tau(\langle y, x \rangle_C).$$

Then

$$\begin{aligned} \tau_x(\langle x, y \rangle_D) &= \tau(\langle z \langle x, y \rangle_D, z \rangle_E) \\ &= \tau(\langle \langle z, x \rangle_E y, z \rangle_E) = \tau(\langle z, x \rangle_E \langle y, z \rangle_E) \\ &= \tau(\langle y, z \rangle_E \langle z, x \rangle_E) = \tau(\langle y \langle z, z \rangle_D, x \rangle_E) \\ &= \tau(\langle y, x \rangle_E) = \tau(\langle y, x \rangle_C). \end{aligned}$$

□

Let  $C, D$  and  $X$  still be as above, and let  $A$  be the linking algebra for  $X$  as defined on page 350 of [4]. If  $\tau$  is a trace on  $C$  and if  $\tau_x$  is defined as in Proposition 2.2, then a straightforward calculation shows that the functional on  $A$  defined by using  $\tau$  and  $\tau_x$  to evaluate on the diagonal of elements of  $A$  will be a trace. In fact one quickly sees in this way that:

PROPOSITION 2.3. *Let  $C$ ,  $D$  and  $X$  be as above, and let  $A$  be the linking algebra for  $X$ . Then each trace on  $C$  has a unique extension to a trace on  $A$ . The restriction to  $D$  of this trace on  $A$  will be  $\tau_X$ .*

By construction,  $C$  and  $D$  sit as complementary full corners of the linking algebra  $A$ . We recall that if  $\psi$  is any homomorphism between  $C^*$ -algebras (possibly not preserving identity elements), then  $\psi$  induces a homomorphism between the corresponding  $K_0$  groups. This homomorphism is described in [9], and is denoted by  $\tilde{\psi}_*$ . We now need the following fact, which is undoubtedly familiar to other workers in this area:

PROPOSITION 2.4. *Let  $A$  be a  $C^*$ -algebra with identity element, let  $pAp$  be a full corner of  $A$ , and let  $\psi$  be the injection of  $pAp$  into  $A$ . Then  $\tilde{\psi}_*$  is an isomorphism of  $K_0(pAp)$  with  $K_0(A)$ .*

*Proof.* View  $X = pA$  as a  $pAp$ - $A$ -equivalence bimodule (6.8 of [23]). Then, as in the proof of Proposition 2.1, we can find  $a_1, \dots, a_n \in A$  such that  $\sum a_i^* p a_i = 1$ . Let  $\phi$  be the corresponding map of  $A$  into  $M_n(pAp)$ , so that  $\phi(a) = (p a_i a_j^* p)_{i,j}$ , and  $\phi$  maps  $A$  onto the corner of  $M_n(pAp)$  defined by the projection  $(p a_i a_j^* p)_{i,j} = P$ . Let  $V$  be the element of  $M_n(A)$  whose first column consists of  $p a_1, \dots, p a_n$ , and all of whose other entries are 0. Then a simple calculation shows that  $VV^* = P$ , while  $V^*V$  is the matrix with 1 in the upper left corner, and 0 elsewhere. Thus, "conjugation" of  $M_n(A)$  by  $V$  carries  $\phi(A)$  onto the corner of  $M_n(A)$  consisting of the matrices all of whose entries are zero except that in the upper left corner. If we view  $\psi$  as giving also the inclusion of  $M_n(pAp)$  into  $M_n(A)$ , we see in this way that  $\tilde{\psi}_* \circ \tilde{\phi}_*$  is the identity map on  $K_0(A)$ . (We use here the fact that, as remarked in [9] immediately after the proof of Lemma 3.6, it does not matter whether one uses unitary or Murray-von Neumann equivalence in defining  $K_0$ .) It follows that  $\tilde{\psi}_*$  is surjective. Thus we have shown that the inclusion map of a full corner into an algebra induces a surjection of  $K_0$  groups. But  $\phi(A)$  is a full corner of  $M_n(pAp)$ , and  $\phi$  is an isomorphism of  $A$  with  $\phi(A)$ . It follows that  $\tilde{\phi}_*$  is surjective. Since  $\tilde{\psi}_* \circ \tilde{\phi}_*$  is an isomorphism, it follows that  $\tilde{\psi}_*$  must be injective, and so is an isomorphism.  $\square$

Now let again  $X$  be a  $C$ - $D$ -equivalence bimodule, and let  $A$  be the linking algebra for  $X$ . Let  $\tilde{\psi}_*$  and  $\tilde{\delta}_*$  denote the isomorphisms of  $K_0(C)$  and  $K_0(D)$  with  $K_0(A)$  obtained from the inclusions of  $C$  and  $D$  as corners of  $A$ . We will let  $\Phi_X$  denote the isomorphism  $(\tilde{\delta}_*)^{-1} \circ \tilde{\psi}_*$  of  $K_0(C)$  with  $K_0(D)$ . (We remark in passing that by this means

one can see that the Picard group of a C\*-algebra  $B$ , as defined in [4], will act as a group of automorphisms of  $K_0(B)$ .

**PROPOSITION 2.5.** *Let  $C$  and  $D$  be C\*-algebras with identity, let  $X$  be a C-D-equivalence bimodule, let  $\tau$  be a finite trace on  $C$  and let  $\tau_x$  be the corresponding (nonnormalized) trace on  $D$  defined above. Let  $\Phi_x$  denote the isomorphism of  $K_0(C)$  onto  $K_0(D)$  determined by  $X$  as above, and let  $\hat{\tau}$  and  $\hat{\tau}_x$  be the homomorphisms of  $K_0(C)$  and  $K_0(D)$  into  $\mathbf{R}$  determined by  $\tau$  and  $\tau_x$ . Then*

$$\hat{\tau}_x \circ \Phi_x = \hat{\tau} .$$

*Proof.* This follows immediately from the definitions and the fact that  $\tau_x$  is the restriction to  $D$  of the unique extension of  $\tau$  to the linking algebra. □

**COROLLARY 2.6.** *Let  $C$ ,  $D$  and  $X$  be as above, let  $\tau$  be a trace on  $C$ , and let  $\tau_x$  be the corresponding trace on  $D$ . Then the ranges of  $\hat{\tau}$  and  $\hat{\tau}_x$  are the same.*

*Proof of Theorem 4.* Suppose that  $A_\alpha$  and  $A_\beta$  are strongly Morita equivalent, with equivalence bimodule  $X$ . Let  $\tau$  be the normalized trace on  $A_\alpha$ , and let  $\tau_x$  be the corresponding (nonnormalized) trace on  $A_\beta$ , so that  $\hat{\tau}_x(K_0(A_\beta)) = \hat{\tau}(K_0(A_\alpha))$ . Now  $\tau_x$  differs from the normalized trace on  $A_\beta$  only by a scalar multiple. From this and Proposition 1.4 it follows that there is a positive real number  $r$  such that  $\mathbf{Z} + \mathbf{Z}\beta = r(\mathbf{Z} + \mathbf{Z}\alpha)$ . In particular, there are  $j, k, m, n \in \mathbf{Z}$  such that  $j + k\beta = r$  and  $1 = r(m + n\alpha)$ . On eliminating  $r$  from these equations one finds that  $\alpha$  and  $\beta$  are in the same orbit of  $GL(2, \mathbf{Z})$ . This is a special case of the fact that if  $\mathbf{Z} + \mathbf{Z}\alpha$  and  $\mathbf{Z} + \mathbf{Z}\beta$  are isomorphic as ordered groups, then  $\alpha$  and  $\beta$  are in the same orbit of  $GL(2, \mathbf{Z})$ , as mentioned in [7, 10] and shown in Lemma 4.7 of [29]. □

We remark that the situation described in the first two paragraphs of this section is also interesting at the von Neumann algebra level. Specifically, let  $M$  and  $N$  denote the von Neumann algebras on  $L^2(\mathbf{R})$  generated by  $C^*(\mathbf{Z}\alpha, \mathbf{R}/\mathbf{Z})$  and  $C^*(\mathbf{Z}, \mathbf{R}/\mathbf{Z}\alpha)$  respectively. Then  $M$  and  $N$  are finite factors which are each other's commutants. Thus the coupling constant between them is defined, and a simple calculation shows that this coupling constant is just  $\alpha$ . I plan to discuss this matter and its generalizations in a future paper [35, 36].

Let  $\mathbf{R}$  act on the torus  $T^2$  by the flow at an irrational angle,  $\alpha$ , and let  $C_\alpha$  denote the corresponding transformation group C\*-

algebra. As mentioned in the introduction, Philip Green pointed out to me some years ago that one consequence of the main theorem of [24] is that  $C_\alpha$  is Morita equivalent to  $A_\alpha$ , if the bookkeeping is done correctly. It is well-known that the flow at an irrational angle is the “flow under the constant function” corresponding to the rotation on  $T$  by angle  $\alpha$ , and Phil Green has shown, in as of yet unpublished work, that quite generally the transformation group  $C^*$ -algebra for the flow under a constant function is strongly Morita equivalent to that for the original transformation<sup>3</sup>. Moreover, from the results in [15] he can conclude even more, namely that the  $C^*$ -algebra for the flow is isomorphic to the tensor product of the algebra of compact operators with the  $C^*$ -algebra for the transformation. In particular,  $C_\alpha$  will be stable [4]. It follows then from [4] that if  $C_\alpha$  and  $C_\beta$  are strongly Morita equivalent, then they are in fact isomorphic.

Now the group of automorphisms of  $T^2$  is  $GL(2, \mathbf{Z})$ , via its evident action as automorphisms of  $\mathbf{Z}^2$ , the dual group of  $T^2$ . Furthermore, the corresponding action of  $GL(2, \mathbf{Z})$  on the one-parameter subgroups of  $T^2$  of irrational slope is according to the action on irrational numbers by fractional linear transformations described earlier. It follows that  $C_\alpha$  and  $C_\beta$  are isomorphic if  $\alpha$  and  $\beta$  are in the same orbit under the action of  $GL(2, \mathbf{Z})$ . With hindsight, this might be viewed as the reason that the corresponding  $A_\alpha$  and  $A_\beta$  are strongly Morita equivalent. Now if  $\alpha$  and  $\beta$  are not in the same orbit of  $GL(2, \mathbf{Z})$  then we have seen that  $A_\alpha$  and  $A_\beta$  are not strongly Morita equivalent. Consequently:

**THEOREM 2.7.** *The algebras  $C_\alpha$  and  $C_\beta$  are isomorphic if and only if  $\alpha$  and  $\beta$  are in the same orbit of  $GL(2, \mathbf{Z})$ . If  $\alpha$  and  $\beta$  are not in the same orbit, then  $C_\alpha$  and  $C_\beta$  are not even strongly Morita equivalent.*

We conclude with a curiosity. By specializing the formulas of [24], the strong Morita equivalence of  $A_\alpha$  with  $A_{(\alpha-1)}$  can be described quite explicitly. Specifically,  $C_c(\mathbf{R})$ , the space of continuous functions of compact support on  $\mathbf{R}$ , forms the natural equivalence (i.e., imprimitivity) bimodule between the pre- $C^*$ -algebras  $C_c(\mathbf{Z}\alpha, \mathbf{R}/\mathbf{Z})$  and  $C_c(\mathbf{Z}, \mathbf{R}/\mathbf{Z}\alpha)$ . The actions of these algebras on  $C_c(\mathbf{R})$  come from the corresponding “covariant representations” obtained from translation on  $C_c(\mathbf{R})$  by  $\mathbf{Z}\alpha$  and  $\mathbf{Z}$ , and pointwise multiplication by functions of period 1 and  $\alpha$  respectively. If we denote the above algebras by  $C$  and  $D$  respectively, as in [24], then the algebra-valued inner products are given by

$$\langle F, G \rangle_c(m, r) = \sum_n F(r - n) \bar{G}(r - n - m\alpha)$$

<sup>3</sup> See also [37].

$$\langle F, G \rangle_D(m, r) = \sum_n \bar{F}(r - n\alpha)G(r - n\alpha - m)$$

for  $F, G \in C_c(\mathbf{R})$ ,  $m \in \mathbf{Z}$ ,  $r \in \mathbf{R}$ , where we write  $f(m, r)$  instead of the  $f_m(r)$  used earlier in this paper.

Let us consider projections in  $C$  which are of the form  $\langle F, F \rangle_C$ . First, notice that there are many of them. For according to Proposition 2.1,  $D$  can be embedded as a corner in  $n \times n$  matrices over  $C$ . But in the present situation something special happens, namely that in one direction matrices of size  $1 \times 1$  will work. To see this, assume that  $\alpha \in [0, 1]$ , and let  $G$  be any nonnegative function in  $C_c(\mathbf{R})$  which is supported in an interval of length strictly smaller than 1, but is strictly positive on an interval of length greater than  $\alpha$ . Then from the first of these conditions it follows that  $\langle G, G \rangle_D(m, r) = 0$  if  $m \neq 0$ , whereas from the second condition it follows that  $\langle G, G \rangle_D(0, r) > 0$  for all  $r \in \mathbf{R}$ . In other words,  $\langle G, G \rangle_D$  is in the (Cartan) subalgebra  $C(\mathbf{R}/\mathbf{Z}\alpha)$  of  $D$ , and is invertible there. Let  $H = G^* \langle G, G \rangle_D^{-1/2}$ . Then  $H \in C_c(\mathbf{R})$  and  $\langle H, H \rangle_D = 1_D$ . As seen earlier, it follows that  $\langle H, H \rangle_C$  is a projection in  $C$ . In fact, it was exactly this observation which led me to discover the projections described in § 1. Now, again as seen earlier, the map  $f \rightarrow \langle H^*f, H \rangle_C$  will be an isomorphism of  $D$  onto a corner of  $C$  (except for the fact that  $C$  is not complete). In particular if  $p$  is any projection in  $D$ , then  $\langle H^*p, H^*p \rangle_C$  will be a projection in  $C$ . Since we saw in § 1 that  $D$  contains many projections, of different sizes, it follows that many  $\langle F, F \rangle_C$  are projections, of many sizes.

There is a simple abstract characterization of those  $F$  which give projections:

**PROPOSITION 2.8.** *Let  $X$  be an  $A$ - $B$ -equivalence bimodule and let  $x \in X$ . Then  $\langle x, x \rangle_A$  is a projection iff  $x \langle x, x \rangle_B = x$ .*

*Proof.* Suppose this last equation holds. Then from the fact that  $x \langle x, x \rangle_B = \langle x, x \rangle_A x$  it is easily seen that  $\langle x, x \rangle_A$  is idempotent, and so is a projection since it is self-adjoint. Conversely, suppose that  $\langle x, x \rangle_A$  is a projection. Then a simple calculation shows that

$$\langle x \langle x, x \rangle_B - x, x \langle x, x \rangle_B - x \rangle_A = 0,$$

so that  $x \langle x, x \rangle_B = x$ . □

Now the trace of  $\langle F, F \rangle_C$  is given by

$$\begin{aligned} \tau(\langle F, F \rangle_C) &= \int_0^1 \sum_n F(r - n) \bar{F}(r - n) dr \\ &= \int_{-\infty}^{\infty} |F(r)|^2 dr. \end{aligned}$$

Putting together this observation with Theorem 1.2 and Proposition 2.11 we obtain:

**PROPOSITION 2.9.** *Let  $\alpha$  be an irrational number, and let  $F$  be an element of  $C_c(\mathbf{R})$  which satisfies the functional equation*

$$F(r) = \sum_{m,n} F(r-m) \bar{F}(r-m-n\alpha) F(r-n\alpha).$$

(There are many such  $F$ .) Then

$$\int_{-\infty}^{\infty} |F(r)|^2 dr \in (\mathbf{Z} + \mathbf{Z}\alpha) \cap [0, 1].$$

It is an interesting challenge to find a proof of this result concerning functions of a real variable which does not use  $C^*$ -algebra techniques.

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Received January 2, 1980. This work was partially supported by National Science Foundation grant MCS 77-13070.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.),  
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