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Every homology 3-sphere can be represented as a framed link, in the sense of Kirby, of the following special type. There are k disjoint embeddings in S^3 of a genus one surface with two boundary components. The link is the $2k$ boundary components. If q is the linking number of the ith pair, then one of the components has framing $q+1$, the other $q-1$.

1. Introduction. A very useful and fairly recent method for studying 3-manifolds, in particular for the study of examples, is R. Kirby's "Calculus of framed links" (see [8], [7]). Another, older, method is through the mapping class group of a surface via Heegaard splittings (see [1], [4]). Through the work of Birman [2], Powell [9], and Johnson [6], a geometrically appealing (but infinite) set of generators has been found for group of homeomorphisms of an orientable surface, up to isotopy, that induce the identity on homology. It is the purpose of this paper to use this set of generators to obtain a representation theorem for all homology 3-spheres as special framed links. It turns out that the formula for the μ-invariant of homology 3-spheres represented this way is particularly simple.

2. Notation and conventions. Throughout the paper X_g will be a 3-dimensional genus g handlebody, T_g its boundary, X_g' another such handlebody, and i a PL homeomorphism from T_g to $\partial X_g'$ so that $X_g \cup X_g'$ defines a Heegaard splitting of S^3. Also, A will be an "annulus with a handle"; that is an oriented genus one surface with two boundary components. The boundary components derive orientations from the orientation of A and will be denoted a and b. We assume the reader is familiar with such concepts as "characteristic surface", "intersection matrix", "index", and "μ-invariant" as they apply to three and four manifolds. These terms are defined in [8].

3. Several propositions. In this section we state several results needed for the proof of the main theorem.

PROPOSITION 1. *Let H^3 be a genus g homology 3-sphere. There is a homeomorphism ϕ of T_g such that ϕ induces the identity on the homology of T_g and $H^3 = X_g \bigcup_{i\phi} X_g'$.*

Proof. This is proven by Joan Birman in [3].

Let $f: A \to T_g$ be an embedding. If we do Dehn twists in opposite directions about the boundary curves of $f(A)$, it is not difficult to see that this homeomorphism induces the identity on the homology of T_g. We shall call such a homeomorphism "special" for awhile.

PROPOSITION 2. *Let ϕ be a homeomorphism of T_g that induces the identity on homology. Then ϕ is isotopic to a finite product of "special" homeomorphisms.*

Proof. This follows directly from results of Dennis Johnson ([6]), Jerome Powell ([9]), and Joan Birman ([2]).

PROPOSITION 3. *Let $M^3 = X_g \bigcup_{\alpha_n \cdots \alpha_1 i} X_g'$ where α_j is a homeomorphism of T_g, $1 \leqq j \leqq n$. Then also*

$$M^3 = X_g \bigcup_{id} T_g \times [0, 1] \bigcup_{\hat{\alpha}_1} T_g \times [1, 2] \bigcup_{\hat{\alpha}_2} \cdots \bigcup_{\hat{\alpha}_n} T_g \times [n, n+1] \bigcup_{i} X_g' ,$$

where $\hat{\alpha}_j(x, j) = (\alpha_j(x), j)$.

Proof. We can present M^3 as

$$M^3 = X_g \bigcup_{id} T_g \times [0, 1] \bigcup_{id} T_g \times [1, 2] \cup \cdots \bigcup_{id} T_g \times [n, n+1] \bigcup_{\alpha_n \cdots \alpha_1 i} X_g'$$

and then explicitly define a homeomorphism as below.

$$\begin{array}{ccccccccccc}
X_g & \bigcup_{id} & T_g \times [0, 1] & \bigcup_{\hat{\alpha}_1} & T_g \times [0, 2] & \bigcup_{\hat{\alpha}_2} & \cdots & \bigcup_{\hat{\alpha}_n} & T_g \times [n, n+1] & \bigcup_{i} & X_g' \\
\uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \alpha_1 \times id} & & \uparrow{\scriptstyle \alpha_2\alpha_1 \times id \cdots} & & \uparrow{\scriptstyle \alpha_n \cdots \alpha_1 \times id} & & \uparrow{\scriptstyle id} \\
X_g & \bigcup_{id} & T_g \times [0, 1] & \bigcup_{id} & T_g \times [0, 2] & \cup & \cdots & \bigcup_{id} & T_g \times [n, n+1] & \bigcup_{\alpha_n \cdots \alpha_1 i} & X_g'
\end{array}$$

4. Statement and proof of the main theorem.

MAIN THEOREM. *Let f_i, $1 \leqq i \leqq n$ be a set of smooth embeddings of A in S^3 with pairwise disjoint images. Let q_i be the linking number of $f_i(a)$ and $f_i(b)$. Frame the link $\{f_1(a), f_1(b); f_2(a), f_2(b), \cdots, f_n(b)\}$ with framing numbers $\{-q_1+1, -q_1-1; -q_2+1, \cdots, -q_n-1\}$ respectively and let M^3 be the 3-manifold constructed from this framed link using the Kirby calculus* ([8]).

Then M^3 is a homology 3-sphere and every homology 3-sphere can be constructed this way. The μ-invariant of M^3 (as an integer mod 2) is the mod 2 sum of the *Arf* invariants of the surfaces $f_i(A)$ that have q_i even.

Proof. We begin by showing that an arbitrary homology 3-sphere H^3 has this type of representation. It follows directly from Propositions 1, 2 and 3 that

$$H^3 = X_g \bigcup_{id} T_g \times [0, 1] \bigcup_{\hat{\alpha}_1} T_g \times [1, 2] \bigcup_{\hat{\alpha}_2} \cdots \bigcup_{\hat{\alpha}_n} T_g \times [n, n+1] \bigcup_{i} X_g'$$

where $S^3 = X_g \bigcup_{id} T_g \times [0, 1] \bigcup_{id} T_g \times [1, 2] \cup \cdots \bigcup_{id} T_g \times [n, n+1] \bigcup_i X_g'$ and $\hat{\alpha}_j\colon T_g \times \{j\} \to T_g \times \{j\}$ is "special". Thus there are embeddings $f_j\colon A \to T_g \times \{j\}$, $1 \leqq j \leqq n$ and $\hat{\alpha}_j$ consists of simultaneous Dehn twists in opposite directions about the bounding curves of $f_j(A)$ in the surface $T_g \times \{j\}$. Denote $f_j(a)$ and $f_j(b)$ by k_1 and k_2 respectively.

For $i = 1, 2$ let U_i be a tubular neighborhood of k_i, let m_i be a meridian, let l_i be a longitude, and let $s_i = f_j(A) \cap \partial U_i$. Assume m_i, l_i, and s_i lie in ∂U_i. Orient s_i and l_i parallel to k_i and orient m_i using k_i and the "right hand rule". To do a Dehn twist in the surface $T_g \times \{j\}$ along the curve k_i we split S^3 along $T_g \times \{j\}$, do a full twist in one of the two annuli bounded by the copies of k_i and s_i and sew S^3 back together along $T_g \times \{j\}$.

This is equivalent to removing U_i and sewing it back in so that a meridian is sewn to $m_i \pm s_i$ where the sign is determined by the direction of the twist. To see this, before removing U_i just push it down a little into the surface $T_g \times [j-1, j]$, so that the annulus bounded by k_i and s_i lies in the boundary of U_i.

If the meridian of U_1 is sewn to $m_1 \pm s_1$, then the meridian of U_2 is sewn to $m_2 \mp s_2$ since the twists are in opposite directions. We assume (by renumbering if necessary) that the meridian of U_1 is sewn to $m_1 + s_1$.

In the homology of the complement of k_1, we have $s_1 = l_1 + cm_1$ for some integer c then $L(s_1, k_1) = L(l_1, k_1) + cL(m_1, k_1)$ where $L(\ ,\ )$ stands for linking number. Thus $c = L(s_1, k_1) = L(-s_2, k_1) = L(-k_2, k_1) = -L(k_2, k_1)$. Similarly $s_2 = l_2 - L(k_2, k_1)m_2$.

If we do framed surgery, say on k_1, with framing t, then the meridian is sewn to $l_1 + tm_1 = s_1 + L(k_2, k_1)m_1 + tm_1$. If we choose $t = -L(k_2, k_1) + 1$, then the Dehn surgery and the framed surgery have the same effect. By an analogous argument, the Dehn surgery at k_2 has the same effect as framed surgery at k_2 with framing $-L(k_2, k_1) - 1$. Thus we have shown that every homology 3-sphere has the asserted representation.

Now consider the manifold M^3 defined by any framed link satisfying the hypothesis of the theorem. Following Kirby [8], we construct a simply connected 4-manifold W^4 that M^3 bounds. The intersection matrix of W^4, with respect to the basis $\{f_1(a), f_1(b), \cdots, f_n(b)\}$, as in the hypothesis of the theorem, is a $2n \times 2n$ matrix with framing numbers on the diagonal and linking numbers off the diagonal. We can think of this matrix as an $n \times n$ matrix with 2×2 matrix entries. Off diagonal entries have the form $\begin{bmatrix} x & -x \\ -x & x \end{bmatrix}$ and diagonal entries have the form $\begin{bmatrix} -q+1 & q \\ q & -q-1 \end{bmatrix}$. Using symmetric row and column operations, we see that this matrix is equivalent to a matrix with off diagonal entries $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and diagonal entries $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. (Add the first column to the second column and then the first row to the second row to get first column off diagonal entries like $\begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix}$, first row off diagonal entries like $\begin{bmatrix} x & -x \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -q+1 & 1 \\ 1 & 0 \end{bmatrix}$ on the diagonal, etc.) Thus the index of this matrix and the index of W^4 is zero. Since the matrix is unimodular, M^3 is a homology sphere (see [8]).

There is the following method for computing the μ-invariant of a homology 3-sphere (see [5]). Let W^4 be any simply connected 4-manifold that M^3 bounds and let F^2 be a characteristic surface. Then $\mu(M^3)$, as an integer mod $2 = 1/8$ (index $W^4 - F \cdot F + 8$ (Kervaire invariant of F)). In our case index $W^4 = 0$, and a characteristic element in $H_2(W^4)$ corresponds to the sum of the basis elements with odd framing numbers (so that g_j is even). This can be verified by direct computation. Thus a characteristic surface F can be taken as a disjoint union of tori, each torus consisting of the union of an $f_j(A)$ and two discs in the attached 2-handles. If F_j is a torus component of F corresponding to $f_j(A)$, then $F_j \cdot F_j = (f_j(a) + f_j(b)) \cdot (f_j(a) + f_j((b)) = f_j(a) \cdot f_j(a) + 2f_j(a) \cdot f_j(b) + f_j(b) \cdot f_j(b) = -q_j + 1 + 2q_j - q_j - 1 = 0$. Thus $F \cdot F = 0$. The Kervaire invariant of a component F_j is just the *Arf* invariant of $f_j(A)$ as it is embedded in S^3 (see [5] for a definition of Kervaire invariant). Thus two of the three terms in the formula for $\mu(M^3)$ vanish and we are done.

REFERENCES

1. J. Birman, *On the equivalence of Heegaard splittings of closed, orientable* 3-*manifolds*, preprint.
2. ———, *On Siegel's modular group*, Math. Ann., **191** (1971), 59-68.
3. ———, *Poincaré's conjecture and the homotopy group of a closed orientable* 2-*manifold*, preprint.
4. J. Birman and H. Hilden, *Heegaard splittings of branched coverings of* S^3, Trans. Amer. Math. Soc., **213** (1975), 315-352.
5. S. Cappell and J. Shaneson, *Some new four manifolds*, Ann. of Math., (2) **104** (1976), 61-72.
6. D. Johnson, *Homeomorphisms of a surface which art trivially on homology*, to appear

in the Proc. Amer. Math. Soc.
7. S. Kaplan, Lecture Notes Prepared in Connection with the Summer Institute on Algebraic and Geometric Topology held at Stanford University, Stanford, CA, August 2-21, 1976, S1-S47.
8. R. Kirby, *A calculus for framed links in S^3*, Invent. Math., **45** (1978), 35-56.
9. J. Powell, *Two theorems on the mapping class group of surfaces,* to appear in the Proc. Amer. Math. Soc.

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