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By $C^{(n)}[0, 1]$ (henceforth denoted by $C^{(n)}$) we denote the Banach algebra of complex valued n times continuously differentiable functions on $[0, 1]$ with norm given by

$$\|f\| = \sup_{x \in [0, 1]} \left(\sum_{r=0}^n \frac{|f^{(r)}(x)|}{r!} \right) \quad \text{for } f \in C^{(n)}.$$

By an isometry of $C^{(n)}$ we mean a norm-preserving linear map of $C^{(n)}$ onto itself.

The purpose of this article is to describe the isometries of $C^{(n)}$ for any positive integer n . More precisely, we show that any isometry of $C^{(n)}$ is induced by a point map of the interval $[0, 1]$ onto itself.

The isometries of $C^{(1)}$ (with the same norm as above) are determined by M. Cambern [1]. N. V. Rao and A. K. Roy [2] have also determined the isometries of $C^{(1)}$ with norm of $f \in C^{(1)}$ given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and even for more general norms.

In the proof we shall follow the techniques of [1].

1. Let W denote the compact space $[0, 1] \times [-\pi, \pi]^n$. We prove the following propositions.

PROPOSITION 1.1. *Given $(x, \theta_1, \dots, \theta_n) \in W$, then there exists $h \in C^{(n)}$ such that*

$$\sum_{r=0}^n \frac{|h^{(r)}(x)|}{r!} > \sum_{r=0}^n \frac{|h^{(r)}(y)|}{r!}$$

for $y \in [0, 1]$, $y \neq x$, with $|h(x)| = h(x) > 0$, $|h'(x)| = e^{i\theta_1} h'(x) > 0$, $|h''(x)| = e^{i\theta_2} h''(x) > 0$, \dots , $|h^{(n)}(x)| = e^{i\theta_n} h^{(n)}(x) > 0$.

Proof. Let f_0 be the real valued, nonnegative continuous function on $[0, 1]$ defined as follows

$$f_0(y) = \begin{cases} 0 & \dots (y - x) \leq -\frac{1}{2(n!)} \\ 1 + 2(n!)(y - x) \dots - \frac{1}{2(n!)} & -\frac{1}{2(n!)} < (y - x) \leq 0 \\ 1 - 2(n!)(y - x) \dots 0 & 0 < (y - x) \leq \frac{1}{2(n!)} \\ 0 & \dots \frac{1}{2(n!)} < (y - x). \end{cases}$$

For $1 \leq r \leq n$ define $f_r(y)$ as $f_r(y) = \int_x^y f_{r-1}(t)dt$. It can be easily verified that for $1 \leq r \leq n$, $f_r(y)$ is as follows:

$$f_r(y) = \begin{cases} - \sum_{j=1}^r \frac{1}{(j+1)!(2(n!))^j} \frac{(y-x)^{r-j}}{(r-j)!} \dots (y-x) \leq \frac{-1}{2(n!)} \\ \frac{(y-x)^r}{r!} + \frac{2(n!)(y-x)^{r+1}}{(r+1)!} \dots - \frac{1}{2(n!)} < (y-x) \leq 0 \\ \frac{(y-x)^r}{r!} - \frac{2(n!)(y-x)^{r+1}}{(r+1)!} \dots 0 < (y-x) \leq \frac{1}{2(n!)} \\ \sum_{j=1}^r \frac{(-1)^{j-1}}{(j+1)!(2(n!))^j} \frac{(y-x)^{r-j}}{(r-j)!} \dots \frac{1}{2(n!)} < (y-x) . \end{cases}$$

Now let

$$g(y) = \frac{1}{(2n-1)!} \left[\sum_{j=1}^{(n-1)} e^{i(\theta_1 - \theta_j)} \frac{(y-x)^j}{j!} \right] + e^{i(\theta_1 - \theta_n)} f_n(y) .$$

Clearly, for $1 \leq r \leq n$, $f_n^{(r)} = f_{n-r}$. Therefore $g \in C^{(n)}$ and

$$g^{(r)}(y) = \frac{1}{(2n-1)!} \sum_{j=r}^{(n-1)} e^{i(\theta_1 - \theta_j)} \frac{(y-x)^{j-r}}{(j-r)!} + e^{i(\theta_1 - \theta_n)} f_{n-r}^{(r)}(y) \text{ for } 1 \leq r \leq n .$$

Thus

$$g(x) = 0, g^{(r)}(x) = \frac{1}{(2n-1)!} e^{i(\theta_1 - \theta_r)} \text{ for } 1 \leq r \leq n-1 ,$$

and $g^{(n)}(x) = e^{i(\theta_1 - \theta_n)}$. Therefore

$$\sum_{r=0}^n \frac{|g^{(r)}(x)|}{r!} = \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{r!} + \frac{1}{n!} .$$

Now consider $\sum_{r=0}^n (|g^{(r)}(y)|/r!)$ for $y \in [0, 1]$ and $y \neq x$.

Case 1. Let $(y-x) \leq (-1/2(n!))$.

$$(1) \quad \sum_{r=0}^n \frac{|g^{(r)}(y)|}{r!} \leq \frac{1}{(2n-1)!} \sum_{j=1}^{(n-1)} \frac{|y-x|^j}{j!} + \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{r!} \\ \times \left\{ \sum_{j=r}^{(n-1)} \frac{|y-x|^{j-r}}{(j-r)!} \right\} + \sum_{r=0}^n \frac{1}{r!} \left\{ \sum_{j=1}^{(n-r)} \frac{|y-x|^{(n-r-j)}}{(j+1)!(2(n!))^j (n-r-j)!} \right\} .$$

For $n = 1, 2$, it can be easily verified that right hand side of (1) is less than $\sum_{r=0}^n (|g^{(r)}(x)|/r!)$. When $n \geq 3$, denoting $(n!/(n-j)!j!)$ by C_j^n , (1) gives

$$\begin{aligned} \sum_{r=0}^n \frac{|g^{(r)}(y)|}{r!} &\leq \frac{1}{(2n-1)!} \sum_{j=1}^{(n-1)} \frac{1}{j!} + \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{(n-1)}{r!} \\ &\quad + \frac{1}{2(n!)} \sum_{r=0}^{(n-1)} \frac{1}{r!} \sum_{j=1}^{(n-r)} \left\{ \frac{1}{j(j+1)(n-r-1)!} C_{j-1}^{n-r-1} \frac{1}{(2(n!))^{j-1}} \right\}. \end{aligned}$$

Now

$$\frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{(n-1)}{r!} \leq \frac{(n-1)}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{2^{r-1}} < \frac{2(n-1)}{(2n-1)!} < \frac{1}{4(n!)} \quad \text{for all } n \geq 3.$$

Thus we have

$$(2) \quad \text{for } n \geq 3, \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{(n-1)}{r!} < \frac{1}{4(n!)}.$$

Also

$$\begin{aligned} &\frac{1}{2(n!)} \sum_{r=0}^{(n-1)} \frac{1}{r!} \sum_{j=1}^{(n-r)} \left\{ \frac{1}{j(j+1)(n-r-1)!} C_{j-1}^{n-r-1} \frac{1}{(2(n!))^{j-1}} \right\} \\ &\leq \frac{1}{2(n!)} \sum_{r=0}^{(n-1)} \frac{1}{r!} \frac{1}{2(n-r-1)!} \left\{ \sum_{j=1}^{(n-r)} C_{j-1}^{n-r-1} \frac{1}{(2(n!))^{j-1}} \right\} \\ &= \frac{1}{(2(n!))} \cdot \frac{1}{2(n-1)!} \sum_{r=0}^{(n-1)} C_r^{n-1} \left(1 + \frac{1}{2(n!)} \right)^{(n-1-r)} \\ &= \frac{\left(\left(1 + \frac{1}{2(n!)} \right) + 1 \right)^{n-1}}{2(2(n!))(n-1)!} \leq \frac{\left(\frac{9}{4} \right)^{n-1}}{2(2(n!))(n-1)!} < \frac{81}{64} \cdot \frac{1}{2(n!)}. \end{aligned}$$

Thus

$$(3) \quad \begin{aligned} &\frac{1}{2(n!)} \sum_{r=0}^{(n-1)} \frac{1}{r!} \sum_{j=1}^{(n-r)} \left\{ \frac{1}{j(j+1)(n-r-1)!} \cdot C_{j-1}^{n-r-1} \cdot \frac{1}{(2(n!))^{j-1}} \right\} \\ &< \frac{3}{4} \cdot \frac{1}{n!}. \end{aligned}$$

By (2) and (3) it follows immediately that for all $y \in [0, 1]$ and $y \neq x$

$$\sum_{r=0}^n \frac{|g^{(r)}(y)|}{r!} < \sum_{r=0}^n \frac{|g^{(r)}(x)|}{r!}.$$

Case 2. Let $-(1/2(n!)) < (y-x) < 0$

$$\begin{aligned} \sum_{r=0}^n \frac{|g^{(r)}(y)|}{r!} &\leq \frac{1}{(2n-1)!} \sum_{j=1}^{(n-1)} \frac{|y-x|^j}{j!} + \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{r!} \left\{ \sum_{j=r}^{(n-1)} \frac{|y-x|^{j-r}}{(j-r)!} \right\} \\ &\quad + \sum_{r=0}^n \frac{1}{r!} \left| \frac{(y-x)^{n-r}}{(n-r)!} + \frac{2(n!)(y-x)^{n-r+1}}{(n-r+1)!} \right| \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(2n-1)!} \sum_{j=1}^{(n-1)} \frac{(-1)^j (y-x)^j}{j!} + \frac{1}{(2n-1)!} \sum_{r=1}^{n-1} \frac{1}{r!} \\
 &+ \frac{1}{(2n-1)!} \sum_{r=1}^{(n-2)} \frac{1}{r!} \left\{ \sum_{j=r+1}^{(n-1)} \frac{(-1)^{j-r} (y-x)^{j-r}}{(j-r)!} \right\} \\
 &+ \sum_{r=0}^n \frac{(-1)^{n-r}}{r!} \left\{ \frac{(y-x)^{n-r}}{(n-r)!} + \frac{2(n!)(y-x)^{n-r+1}}{(n-r+1)!} \right\} \\
 &= \frac{1}{(2n-1)!} \sum_{r=1}^{(n-1)} \frac{1}{r!} + \frac{1}{n!} + \sum_{s=1}^{(n-1)} (y-x)^s \left\{ \frac{(-1)^s}{s!(2n-1)!} \right. \\
 &+ \left. \frac{(-1)^s}{s!(n-s)!} + \frac{2(n!)(-1)^{s-1}}{s!(n-s+1)!} + \sum_{r=1}^{n-1-s} \frac{(-1)^s}{s!r!} \right\} \\
 &+ (y-x)^n \left\{ \frac{(-1)^n}{n!} + \frac{2(n!)(-1)^{n-1}}{n!} \right\} + \frac{(-1)^n (y-x)^{n+1}}{(n+1)!} \\
 &= \sum_{r=0}^n \frac{|g^{(r)}(x)|}{r!} + \sum_{s=1}^{(n-1)} \frac{(-1)^s (y-x)^s}{s!} \left\{ \frac{1}{(2n-1)!} \right. \\
 &+ \left. \sum_{r=1}^{(n-s)} \frac{1}{r!} - \frac{2(n!)}{(n-s+1)!} \right\} \\
 &+ \frac{(-1)^n (y-x)^n}{n!} \{1 - 2(n!)\} + \frac{(-1)^n (y-x)^{n+1}}{(n+1)!} \\
 &< \sum_{r=0}^n \frac{|g^{(r)}(x)|}{r!}
 \end{aligned}$$

since all the other terms are negative. Verification in cases when $0 < (y-x) \leq (1/2(n!))$ and $(1/2(n!)) < (y-x)$ is similar. From this it follows that the function $h \in C^{(n)}$ defined by $h(y) = 1 + e^{-iy}g(y)$ has the desired properties.

PROPOSITION 1.2. For any $f \in C^{(n)}$

$$\sum_{j=1}^n (-1)^{j-1} C_{j-1}^n (f^{n-j+1})^{(k)}(x) (f(x))^{j-1} = \begin{cases} 0 & \text{if } 1 \leq k < n \\ n!(f'(x))^n & \text{if } k = n \end{cases}$$

where $(f^{n-j+1})^{(k)}(x)$ means the k th derivative of f^{n-j+1} at x .

Proof. We prove this proposition by induction on n . For $n = 1$ it is obvious. Let it be true for $n = r$. Then we have

$$\sum_{j=1}^r (-1)^{j-1} C_{j-1}^r (f^{r-j+1})^{(k)}(x) (f(x))^{j-1} = 0, \quad \text{for } 1 \leq k < r,$$

and

$$\sum_{j=1}^r (-1)^{j-1} C_{j-1}^r (f^{r-j+1})^{(r)}(x) (f(x))^{j-1} = r!(f'(x))^r.$$

Now let $n = r + 1$ and $k = r + 1$.

Since $(f^{r-j+2})'(x) = (r - j + 2)(f^{r-j+1})(x)f'(x)$

$$\begin{aligned}
& \sum_{j=1}^{(r+1)} (-1)^{j-1} C_{j-1}^{r+1} (f^{r-j+2})^{(r+1)}(x) (f(x))^{j-1} \\
&= \sum_{j=1}^{r+1} (-1)^{j-1} C_{j-1}^{r+1} (f(x))^{j-1} \left\{ (r - j + 2) \sum_{s=0}^r C_s^r (f^{r-j+1})^{(r-s)}(x) (f')^{(s)}(x) \right\} \\
&= \sum_{j=1}^{r+1} (-1)^{j-1} (r + 1) C_{j-1}^r (f(x))^{j-1} (f^{r-j+1})^{(r)}(x) f'(x) \\
&\quad + \sum_{j=1}^{r+1} (-1)^{j-1} (r + 1) C_{j-1}^r (f(x))^{j-1} \left\{ \sum_{s=1}^r C_s^r (f^{r-j+1})^{(r-s)}(x) (f')^{(s)}(x) \right\} \\
&= (r + 1) \left\{ \sum_{j=1}^r (-1)^{j-1} C_{j-1}^r (f(x))^{j-1} (f^{r-j+1})^{(r)}(x) \right\} f'(x) \\
&\quad + (r + 1) \sum_{j=1}^{r+1} (-1)^{j-1} (f(x))^{j-1} C_{j-1}^r \left\{ \sum_{s=1}^r c_s^r (f^{r-j+1})^{(r-s)}(x) (f')^{(s)}(x) \right\} \\
&= (r + 1)! (f'(x))^{r+1} + (r + 1) \sum_{s=1}^{r-1} C_s^r (f')^{(s)}(x) \\
&\quad \times \left\{ \sum_{j=1}^r (-1)^j C_{j-1}^r (f^{r-j+1})^{(r-s)}(x) (f(x))^{j-1} \right\} \\
&\quad + (r + 1) \sum_{j=1}^{r+1} (-1)^{j-1} C_{j-1}^r (f(x))^r (f'(x))^{(r)} \\
&= (r + 1)! (f'(x))^{r+1} .
\end{aligned}$$

Now let $n = r + 1$ and $k < (r + 1)$. Then

$$\begin{aligned}
& \sum_{j=1}^{r+1} (-1)^{j-1} C_{j-1}^{r+1} (f^{r-j+2})^{(k)}(x) (f(x))^{j-1} \\
&= \sum_{j=1}^{r+1} (-1)^{j-1} C_{j-1}^{r+1} (r - j + 2) (f(x))^{j-1} \left\{ \sum_{s=0}^{k-1} C_s^{k-1} (f^{r-j+1})^{(k-1-s)}(x) (f')^{(s)}(x) \right\} \\
&= (r + 1) \sum_{s=0}^{k-2} C_s^{k-1} (f')^{(s)}(x) \left\{ \sum_{j=1}^r (-1)^{j-1} C_{j-1}^r (f(x))^{j-1} (f^{r-j+1})^{(k-1-s)}(x) \right\} \\
&\quad + (r + 1) \sum_{j=1}^{r+1} (-1)^{j-1} C_{j-1}^r (f(x))^{k-1} (f')^{(k-1)}(x) \\
&= 0 .
\end{aligned}$$

Hence the proposition follows by mathematical induction.

2. If X is any compact Hausdorff space, we will denote by $C(X)$ the Banach algebra of continuous complex functions defined on X with norm $\| \cdot \|_{\infty}$ determined by $\|g\|_{\infty} = \sup_{x \in X} |g(x)|$ for $g \in C(X)$.

Given $f \in C^{(n)}$, we define $\tilde{f} \in C(W)$ by

$$\begin{aligned}
\tilde{f}(x, \theta_1, \dots, \theta_n) &= f(x) + e^{i\theta_1} f'(x) + \frac{e^{i\theta_2}}{2!} f''(x) + \dots + \frac{e^{i\theta_n}}{n!} f^{(n)}(x) , \\
& \qquad \qquad \qquad (x, \theta_1, \dots, \theta_n) \in W .
\end{aligned}$$

The following lemma is then obvious.

LEMMA 2.1. *The mapping $f \rightarrow \tilde{f}$ establishes a linear and norm-preserving correspondence between $C^{(n)}$ and the closed subspace S of $C(W)$, $S = \{\tilde{f}: f \in C^{(n)}\}$.*

Next given $(x, \theta_1, \dots, \theta_n) \in W$, we define a continuous linear functional $L(x, \theta_1, \dots, \theta_n)$ on $C^{(n)}$ by

$$L_{(x, \theta_1, \dots, \theta_n)}(f) = \tilde{f}(x, \theta_1, \dots, \theta_n), \quad f \in C^{(n)}.$$

In view of Proposition 1.1 the proof of the following lemma is analogous to the proof of Lemma 1.2 in [1].

LEMMA 2.2. *An element of $C^{(n)*}$ is an extreme point of the unit ball U^* of $C^{(n)*}$ if and only if f^* is of the form $e^{i\eta}L_{(x, \theta_1, \dots, \theta_n)}$ for some $\eta \in [-\pi, \pi]$, $(x, \theta_1, \dots, \theta_n) \in W$.*

We now suppose that T is an isometry of $C^{(n)}$. The adjoint T^* is then an isometry of $C^{(n)*}$, and thus carries extreme points of U^* onto itself.

LEMMA 2.3. *The image by T of the constant function $\mathbf{1}$ of $C^{(n)}$ is a constant function $e^{i\lambda}$, $\lambda \in [-\pi, \pi]$.*

Proof. For each extreme point $e^{i\eta}L_{(x, \theta_1, \dots, \theta_n)}$ of U^* ,

$$|(e^{i\eta}L_{(x, \theta_1, \dots, \theta_n)})(\mathbf{1})| = 1.$$

Thus for each extreme point $|T^*(e^{i\eta}L_{(x, \theta_1, \dots, \theta_n)})(\mathbf{1})| = 1$. Therefore, $|L_{(x, \theta_1, \dots, \theta_n)}(T(\mathbf{1}))| = 1$. Thus for a fixed x , $|(T(\mathbf{1}))(x) + e^{i\theta_1}(T(\mathbf{1}))'(x) + \dots + (e^{i\theta_n}/n!)(T(\mathbf{1}))^{(n)}(x)| = 1$ for all $(\theta_1, \dots, \theta_n) \in [-\pi, \pi]^n$. Choosing $\theta_1, \theta_2, \dots, \theta_n$, so that

$$\arg((T(\mathbf{1}))(x)) = \arg(e^{i\theta_1}(T(\mathbf{1}))'(x)) = \dots = \arg\left(\frac{e^{i\theta_n}}{n!}(T(\mathbf{1}))^{(n)}(x)\right)$$

we get

$$|(T(\mathbf{1}))(x)| + |(T(\mathbf{1}))'(x)| + \dots + \frac{|(T(\mathbf{1}))^{(n)}(x)|}{n!} = 1.$$

Again by choosing $\theta_1, \dots, \theta_n$, so that

$$\arg((T(\mathbf{1}))(x)) = \pi + \arg(e^{i\theta_1}(T(\mathbf{1}))'(x)) = \dots = \pi + \arg(e^{i\theta_n}(T(\mathbf{1}))^{(n)}(x))$$

we get

$$\left| |(T(\mathbf{1}))(x)| - \left\{ |(T(\mathbf{1}))'(x)| + \dots + \frac{|(T(\mathbf{1}))^{(n)}(x)|}{n!} \right\} \right| = 1.$$

Thus either

$$\left\{ |(T(\mathbf{1}))(x)| = 1 \quad \text{and} \quad |(T(\mathbf{1}))'(x)| + \dots + \frac{|(T(\mathbf{1}))^{(n)}(x)|}{n!} = 0 \right\}$$

or

$$(4) \quad \left\{ |(T(\mathbf{1}))(x)| = 0 \quad \text{and} \quad |(T(\mathbf{1}))'(x)| + \dots + \frac{|(T(\mathbf{1}))^{(n)}(x)|}{n!} = 1 \right\}.$$

Therefore, for any $x \in [0, 1]$, $|(T(\mathbf{1}))(x)| = 1$ or $|(T(\mathbf{1}))(x)| = 0$. But since $|T(\mathbf{1})|$ is a continuous function on $[0, 1]$ we have

$$|(T(\mathbf{1}))(x)| \equiv 0 \quad \text{or} \quad |(T(\mathbf{1}))(x)| \equiv 1.$$

Now $|(T(\mathbf{1}))(x)| \equiv 0$ implies that $(T(\mathbf{1}))(x) \equiv (T(\mathbf{1}))'(x) \equiv (T(\mathbf{1}))''(x) \equiv \dots \equiv (T(\mathbf{1}))^{(n)}(x) \equiv 0$ which contradicts (4).

Hence $|(T(\mathbf{1}))(x)| \equiv 1$ from which it follows that $(T(\mathbf{1}))'(x) \equiv 0$ and hence

$$T(\mathbf{1}) \equiv e^{i\lambda} \quad \text{for some fixed } \lambda \in [-\pi, \pi].$$

We denote $T^*(L_{(x, \theta_1, \dots, \theta_n)})$ by

$$e^{i\lambda(x, \theta_1, \dots, \theta_n)} L_{(y(x, \theta_1, \dots, \theta_n), \psi_1(x, \theta_1, \dots, \theta_n), \dots, \psi_n(x, \theta_1, \dots, \theta_n))}.$$

The above Lemma 2.3, shows that $\lambda(x, \theta_1, \dots, \theta_n) \equiv \lambda$ for all $(\theta_1, \dots, \theta_n) \in [-\pi, \pi]$. For

$$(T^*(L_{(x, \theta_1, \dots, \theta_n)}))(1) = e^{i\lambda(x, \theta_1, \dots, \theta_n)} L_{(y(x, \theta_1, \dots, \theta_n), \psi_1(x, \theta_1, \dots, \theta_n), \dots, \psi_n(x, \theta_1, \dots, \theta_n))}(1),$$

so that $L_{(x, \theta_1, \dots, \theta_n)}(T(\mathbf{1})) = e^{i\lambda(x, \theta_1, \dots, \theta_n)}$ and thus $L_{(x, \theta_1, \dots, \theta_n)}(e^{i\lambda}) = e^{i\lambda(x, \theta_1, \dots, \theta_n)}$. Hence $\lambda(x, \theta_1, \dots, \theta_n) \equiv \lambda$.

LEMMA 2.4. *If $x \in [0, 1]$, then for all $(\theta_1, \dots, \theta_n) \in [-\pi, \pi]^n$,*

$$y_{(x, \theta_1, \dots, \theta_n)} = y_{(x, 0, \dots, 0)}.$$

Proof. For fixed $x \in [0, 1]$, we consider the map $\rho: [-\pi, \pi]^n \rightarrow [0, 1]$ given by

$$\rho(\theta_1, \theta_2, \dots, \theta_n) = y_{(x, \theta_1, \dots, \theta_n)}.$$

It is easy to verify that this mapping is continuous. Hence the image of $[-\pi, \pi]^n$ in $[0, 1]$ is a connected subset of $[0, 1]$. It is, in fact, a singleton. For otherwise we could find g in $C^{(n)}$ such that $g \equiv g' \equiv \dots \equiv g^{(n)} \equiv 0$ on an open subinterval $I \subset \rho([-\pi, \pi]^n)$ while for some $y_{(x, \varphi_1, \dots, \varphi_n)} \notin I$,

$$\left| g(y_{(x, \varphi_1, \dots, \varphi_n)}) + e^{i\psi_1(x, \varphi_1, \dots, \varphi_n)} g'(y_{(x, \varphi_1, \dots, \varphi_n)}) \right|$$

$$\begin{aligned}
 &+ e^{i\psi_2(x, \varphi_1, \dots, \varphi_n)} \cdot \frac{1}{2!} g''(y_{(x, \varphi_1, \dots, \varphi_n)}) + \dots \\
 &+ e^{i\psi_{(n-1)}(x, \varphi_1, \dots, \varphi_n)} \cdot \frac{1}{(n-1)!} g^{(n-1)}(y_{(x, \varphi_1, \dots, \varphi_n)}) \Big| \\
 &< \left| \frac{1}{n!} g^{(n)}(y_{(x, \varphi_1, \dots, \varphi_n)}) \right|.
 \end{aligned}$$

For instance, one may take

$$g(y) = \begin{cases} 0 & y \leq y_1 \\ (y - y_1)^{(n+1)} & y > y_1 \end{cases}$$

where y_1 is least upper bound of I and $y_{(x, \varphi_1, \dots, \varphi_n)}$ sufficiently near to y_1 . Thus for an infinite number of $(\theta_1, \theta_2, \dots, \theta_n) \in [-\pi, \pi]^n$ with $y_{(x, \theta_1, \dots, \theta_n)} \in I$,

$$\begin{aligned}
 L_{(x, \theta_1, \dots, \theta_n)}(T(g)) &= T^* L_{(x, \theta_1, \dots, \theta_n)}(g) \\
 &= e^{i\lambda} L_{(y_{(x, \theta_1, \dots, \theta_n)}, \psi_1(x, \theta_1, \dots, \theta_n), \dots, \psi_n(x, \theta_1, \dots, \theta_n))}(g) \\
 &= 0
 \end{aligned}$$

while

$$\begin{aligned}
 L_{(x, \varphi_1, \dots, \varphi_n)}(T(g)) \\
 = e^{i\lambda} L_{(y_{(x, \varphi_1, \dots, \varphi_n)}, \psi_1(x, \varphi_1, \dots, \varphi_n), \dots, \psi_n(x, \varphi_1, \dots, \varphi_n))}(g) \neq 0.
 \end{aligned}$$

Since ρ is continuous, $\rho^{-1}(I)$ is open in $[-\pi, \pi]^n$ and therefore for each $i = 1, 2, \dots, n$ there exist an infinite number of θ_i 's such that

$$(5) \quad L_{(x, \theta_1, \dots, \theta_n)}(T(g)) = 0 \quad \text{while} \quad L_{(x, \varphi_1, \dots, \varphi_n)}(T(g)) \neq 0.$$

Therefore $(T(g))(x) + e^{i\theta_1}(T(g))'(x) + \dots + (e^{i\theta_n}/n!)(T(g))^{(n)}(x) = 0$.

For any j with $1 \leq j \leq n$, by keeping θ_i constant for $i \neq j$ and varying θ_j we can see that $(T(g))^{(j)}(x) = 0$. Thus $L_{(x, \varphi_1, \dots, \varphi_n)}(T(g)) = 0$ which contradicts (5).

Hence $y_{(x, \theta_1, \dots, \theta_n)} = y_{(x, 0, \dots, 0)}$ for all $(\theta_1, \dots, \theta_n) \in [-\pi, \pi]^n$.

Finally, we define a point map τ of $[0, 1]$ to $[0, 1]$ by

$$\tau(x) = y_{(x, 0, \dots, 0)}.$$

Consideration of $(T^{-1})^*$ shows that τ is onto, and, applying Lemma 2.4, one-one.

THEOREM 2.5. *Let T be an isometry of $C^{(n)}$. Then, for $f \in C^{(n)}$,*

$$(T(f))(x) = e^{i\lambda} f(\tau(x))$$

with $e^{i\lambda} = T(1)$. Moreover, τ is one of the two functions $F, 1 - F$ where F is the identity mapping of $[0, 1]$ onto itself.

Proof. Given $x \in [0, 1]$ and $\theta \in [-\pi, \pi]$, consider the function g of the Proposition 1.1 constructed for $(x, \theta, \dots, \theta)$. Clearly, g does not depend on θ ; $g(x) = 0$; $g'(x), g''(x), \dots, g^{(n)}(x)$ are positive reals and $\sum_{r=1}^n (g^{(r)}(x)/r!) > \sum_{r=0}^n (|g^{(r)}(y)|/r!)$ for all $y \in [0, 1], y \neq x$. Therefore,

$$\begin{aligned} \|g\| &= g'(x) + \frac{1}{2!}g''(x) + \dots + \frac{1}{n!}g^{(n)}(x) \\ &= e^{-i\theta}L_{(x, \theta, \dots, \theta)}(g) \\ &= e^{-i\theta}T^*L_{(x, \theta, \dots, \theta)}(T^{-1}(g)) \\ &= e^{i(\lambda-\theta)}L_{(\tau(x), \psi_1(x, \theta, \dots, \theta), \dots, \psi_n(x, \theta, \dots, \theta))}(T^{-1}(g)). \end{aligned}$$

Thus we have for all $\theta \in [-\pi, \pi]$

$$(6) \quad \|g\| = e^{i(\lambda-\theta)}[(T^{-1}(g))(\tau(x)) + e^{i\psi_1(x, \theta, \dots, \theta)}(T^{-1}(g))'(\tau(x)) + \dots + \frac{1}{n!}e^{i\psi_n(x, \theta, \dots, \theta)}(T^{-1}(g))^{(n)}(\tau(x))].$$

Since

$$\begin{aligned} \|g\| &= \|T^{-1}(g)\| \\ &= \text{Sup}_{y \in [0, 1]} \sum_{r=0}^n \frac{|(T^{-1}(g))^{(r)}(y)|}{r!}, \end{aligned}$$

by (6) we have

$$\|g\| = |(T^{-1}(g))(\tau(x))| + |(T^{-1}(g))'(\tau(x))| + \dots + \frac{1}{n!}|(T^{-1}(g))^{(n)}(\tau(x))|.$$

Again since g is independent of θ ,

$$(T^{-1}(g))(\tau(x)), (T^{-1}(g))'(\tau(x)), \dots, (T^{-1}(g))^{(n)}(\tau(x))$$

are independent of θ but

$$A(\theta) = \left\{ e^{i\psi_1(x, \theta, \dots, \theta)}(T^{-1}(g))'(\tau(x)) + \dots + \frac{1}{n!}e^{i\psi_n(x, \theta, \dots, \theta)}(T^{-1}(g))^{(n)}(\tau(x)) \right\}$$

depends on θ for otherwise (6) cannot be true. In other words, $A(\theta)$ is not constant. Now by (6) $A(\theta)$ must be on a circle with center as $\{-(T^{-1}(g))(\tau(x))\}$ and radius equal to $\|g\|$.

On the other hand $A(\theta)$ must be on or within the circle with center as origin and radius equal to $\rho = \sum_{r=1}^n (|(T^{-1}(g))^{(r)}(x)|/r!) = \|g\| - |(T^{-1}(g))(\tau(x))|$. This implies that $(T^{-1}(g))(\tau(x)) = 0$ for otherwise $A(\theta)$ has to be a constant (see Figure 2.1) which is false.

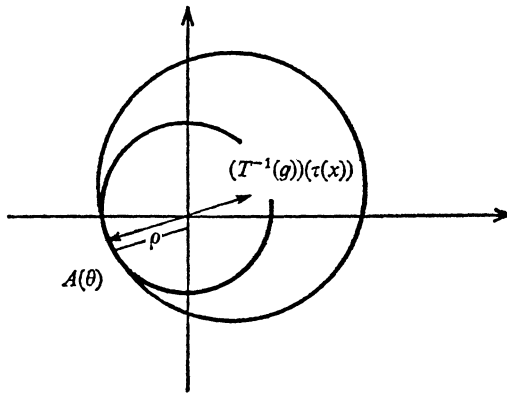


FIGURE 2.1.

Therefore, we have

$$\begin{aligned} \arg e^{i\psi_1(x, \theta, \dots, \theta)}(T^{-1}(g))'(\tau(x)) &= \arg \cdot \frac{1}{2!} e^{i\psi_2(x, \theta, \dots, \theta)}(T^{-1}(g))''(\tau(x)) = \dots \\ &= \arg \cdot \frac{1}{n!} e^{i\psi_n(x, \theta, \dots, \theta)}(T^{-1}(g))^{(n)}(\tau(x)). \end{aligned}$$

Thus for all $\theta \in [-\pi, \pi]$, $1 \leq k \leq n$, $1 \leq j \leq n$

$$\psi_k(x, \theta, \dots, \theta) - \psi_j(x, \theta, \dots, \theta) = \psi_k(x, 0, \dots, 0) - \psi_j(x, 0, \dots, 0).$$

Also by (6)

$$\begin{aligned} \|g\| &= e^{i(\lambda - \theta)} \left[\sum_{k=1}^n \frac{1}{k!} e^{i\psi_k(x, \theta, \dots, \theta)}(T^{-1}(g))^{(k)}(\tau(x)) \right] \\ &= e^{i(\lambda - \theta + \psi_j(x, \theta, \dots, \theta))} \left[\sum_{k=1}^n \frac{1}{k!} e^{i(\psi_k(x, \theta, \dots, \theta) - \psi_j(x, \theta, \dots, \theta))}(T^{-1}(g))^{(k)}(\tau(x)) \right] \\ &= e^{i(\lambda - \theta + \psi_j(x, \theta, \dots, \theta))} \left[\sum_{k=1}^n \frac{1}{k!} e^{i(\psi_k(x, 0, \dots, 0) - \psi_j(x, 0, \dots, 0))}(T^{-1}(g))^{(k)}(\tau(x)) \right]. \end{aligned}$$

Since the left hand side is independent of θ , we have

$$\lambda - \theta + \psi_j(x, \theta, \dots, \theta) = \lambda + \psi_j(x, 0, \dots, 0).$$

Hence for all $\theta \in [-\pi, \pi]$, $1 \leq j \leq n$

$$\psi_j(x, \theta, \dots, \theta) = \psi_j(x, 0, \dots, 0) + \theta.$$

Now let f be any element of $C^{(n)}$ such that $f(x) = 0$ then for all $\theta \in [-\pi, \pi]$

$$f'(x) + \frac{1}{2!}f''(x) + \dots + \frac{1}{n!}f^{(n)}(x)$$

$$\begin{aligned}
&= e^{-i\theta} \mathbf{L}_{(x, \theta, \dots, \theta)}(f) \\
&= e^{-i\theta} T^* \mathbf{L}_{(x, \theta, \dots, \theta)}(T^{-1}(f)) \\
&= e^{i(\lambda - \theta)} \mathbf{L}_{(\tau(x), \psi_1(x, \theta, \dots, \theta), \dots, \psi_n(x, \theta, \dots, \theta))}(T^{-1}(f)) \\
&= e^{i(\lambda - \theta)} \left[(T^{-1}(f))(\tau(x)) + \sum_{k=1}^n \frac{1}{k!} e^{i\psi_k(x, \theta, \dots, \theta)} (T^{-1}(f))^{(k)}(\tau(x)) \right] \\
&= e^{i\lambda} \left[e^{-i\theta} (T^{-1}(f))(\tau(x)) + \sum_{k=1}^n \frac{1}{k!} e^{i\psi_k(x, 0, \dots, 0)} (T^{-1}(f))^{(k)}(\tau(x)) \right]
\end{aligned}$$

so that $(T^{-1}(f))(\tau(x)) = 0$. For an arbitrary $f \in C^{(n)}$, define $g(y) = f(y) - f(x)$, $y \in [0, 1]$ then $g(x) = 0$ and so

$$\begin{aligned}
0 &= (T^{-1}(g))(\tau(x)) = (T^{-1}(f))(\tau(x)) - f(x)(T^{-1}(1))(\tau(x)) \\
&= (T^{-1}(f))(\tau(x)) - e^{-i\lambda} f(x).
\end{aligned}$$

Thus, replacing f by $T(f)$, it follows that for all $x \in [0, 1]$ and $f \in C^{(n)}$,

$$(T(f))(x) = e^{i\lambda} f(\tau(x)).$$

Now if, for $0 \leq r \leq n - 1$, F_r is the mapping of $[0, 1]$ onto itself given by $F_r(x) = x^{r+1}$ (where F_0 is the identity map F), we have

$$(T(F_r))(x) = e^{i\lambda} (\tau(x))^{r+1} = e^{i\lambda} (\tau^{r+1})(x), \quad 0 \leq r \leq n - 1.$$

Therefore $(T(F_r))(x) = (T(F_{r-1}))(x) \cdot \tau(x)$. Now

$$\begin{aligned}
\sum_{k=0}^n \frac{1}{k!} (T(F_r))^{(k)}(x) &= \mathbf{L}_{(x, 0, \dots, 0)}(T(F_r)) \\
&= T^* \mathbf{L}_{(x, 0, \dots, 0)}(F_r) \\
&= e^{i\lambda} \mathbf{L}_{(\tau(x), \psi_1(x, 0, \dots, 0), \dots, \psi_n(x, 0, \dots, 0))}(F_r) \\
&= e^{i\lambda} \left[F_r(\tau(x)) + \sum_{j=1}^n \frac{1}{j!} e^{i\psi_j(x, 0, \dots, 0)} F_r^{(j)}(\tau(x)) \right] \\
&= e^{i\lambda} \left[(\tau(x))^{r+1} + \sum_{j=1}^{r+1} e^{i\psi_j(x, 0, \dots, 0)} C_j^{r+1}(\tau(x))^{r+1-j} \right].
\end{aligned}$$

Thus for $0 \leq r \leq n - 1$

$$(7) \quad \sum_{k=1}^n \frac{1}{k!} (T(F_r))^{(k)}(x) = e^{i\lambda} \sum_{j=1}^{r+1} e^{i\psi_j(x, 0, \dots, 0)} C_j^{r+1}(\tau(x))^{r+1-j}.$$

Taking $r = 0$ in (7), we get

$$\sum_{k=1}^n \frac{1}{k!} (T(F))^{(k)}(x) = e^{i(\lambda + \psi_1(x, 0, \dots, 0))}.$$

Taking $r = 1$, we get

$$\sum_{r=1}^n \frac{1}{k!} (T(F_1))^{(k)}(x) = C_1^2(\tau(x)) e^{i(\lambda + \psi_1(x, 0, \dots, 0))} + e^{i(\lambda + \psi_2(x, 0, \dots, 0))}.$$

Hence

$$e^{i(\lambda + \psi_2(x, 0, \dots, 0))} = \sum_{k=1}^n \frac{1}{k!} (T(F_1))^{(k)}(x) - C_1^2(\tau(x)) \sum_{k=1}^n \frac{1}{k!} (T(F))^{(k)}(x).$$

Thus by successive iterations we get for $1 \leq r \leq n$

$$\begin{aligned} e^{i(\lambda + \psi_r(x, 0, \dots, 0))} &= \sum_{k=1}^n \frac{1}{k!} \left\{ \sum_{j=1}^r (-1)^{j-1} C_{j-1}^r (T(F_{r-j}))^{(k)}(x) (\tau(x))^{j-1} \right\} \\ &= e^{i\lambda} \sum_{k=1}^n \frac{1}{k!} \left\{ \sum_{j=1}^r (-1)^{j-1} C_{j-1}^r (\tau^{r-j+1})^{(k)}(x) (\tau(x))^{j-1} \right\}. \end{aligned}$$

Therefore,

$$e^{i\psi_n(x, 0, \dots, 0)} = \sum_{k=1}^n \frac{1}{k!} \left\{ \sum_{j=1}^n (-1)^{j-1} C_{j-1}^n (\tau^{n-j+1})^{(k)}(x) (\tau(x))^{j-1} \right\}.$$

Applying Proposition 1.2 to the function τ which clearly belongs to $C^{(n)}$ we get

$$e^{i\psi_n(x, 0, \dots, 0)} = \{\tau'(x)\}^n.$$

Thus $\tau'(x)$ is an n th root of a complex number of absolute value one.

But since $\tau'(x)$ is real valued and continuous we have $\tau'(x) \equiv 1$ or $\tau'(x) \equiv -1$ and, therefore, $\tau(x) \equiv F$ or $1 - F$.

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