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**RATIOS OF INTERPOLATING BLASCHKE PRODUCTS**

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**Every unimodular function on the unit circle can be uniformly approximated by ratios of interpolating Blaschke products. As a consequence, we show that points of the maximal ideal space of  $H^\infty$  can be separated by interpolating Blaschke products.**

**1. Introduction.** Let  $\Delta$  denote the open unit disc in  $\mathbb{C}$  and let  $H^\infty(\Delta)$  be the Banach algebra of functions bounded and analytic on  $\Delta$ . A sequence of points  $\{z_j\}$  in  $\Delta$  is called an interpolating sequence if for every bounded sequence  $\{\alpha_j\}$  of complex numbers there is a function  $F \in H^\infty(\Delta)$  such that

$$F(z_j) = \alpha_j, \quad j = 1, 2, \dots$$

Lennart Carleson [1] has shown that  $\{z_j\}$  is an interpolating sequence if and only if

$$\inf_j \prod_{\substack{k \\ k \neq j}} \rho(z_j, z_k) > 0.$$

Here  $\rho$  denotes the pseudo hyperbolic metric;  $\rho(w, z) = |(w - z)/(1 - \bar{w}z)|$  for  $w, z \in \Delta$ .

For an arc  $I$  on the unit circle  $T$  let  $|I|$  denote the length of  $I$  and let  $S(I)$  denote the shadow region of  $I$ ,  $S(I) = \{z \in \Delta: (z/|z|) \in I, 1 - |I| < |z| < 1\}$ . A positive measure  $\mu$  on  $\Delta$  is called a Carleson measure if

$$\sup_I \frac{1}{|I|} \mu(S(I)) = \|\mu\|_C < \infty,$$

where the above supremum is taken over all arcs  $I$  of  $T$ . There is also a characterization of interpolating sequences in terms of Carleson measures. A sequence  $\{z_j\}$  is an interpolating sequence if and only if

$$(i) \quad \inf_{\substack{j \neq k \\ j, k}} \rho(z_j, z_k) > 0$$

and

$$(ii) \quad \sum (1 - |z_j|) \delta_{z_j} \text{ is a Carleson measure,}$$

where  $\delta_z$  denotes the Dirac  $\delta$  measure at  $z$ .

A Blaschke product with simple zeros lying on an interpolating sequence is called an interpolating Blaschke product. The purpose

of this paper is to study ratios of interpolating Blaschke products and then use the information gathered to prove a theorem about the maximal ideal space of  $H^\infty$ . A complex valued function  $u$  on  $T$  is called unimodular if  $|u(e^{i\theta})| = 1$  for almost all  $\theta$ . Our first result asserts that every unimodular function on  $T$  can be uniformly approximated by ratios of interpolating Blaschke products. (Recall that every Blaschke product has nontangential boundary values satisfying  $|B(e^{i\theta})| = 1$  almost everywhere.)

**THEOREM 1.** *If  $u$  is a unimodular function on  $T$  and  $\varepsilon > 0$ , there are interpolating Blaschke products  $B_1$  and  $B_2$  such that*

$$\left\| u - \frac{B_1}{B_2} \right\|_\infty < \varepsilon.$$

Theorem 1 is a refined version of a theorem of Douglas and Rudin [5]. They proved Theorem 1 with  $B_1$  and  $B_2$  (not necessarily interpolating) Blaschke products. Theorem 1 will be proved in §2.

Our next result answers a question of John Garnett and Donald Marshall.

**THEOREM 2.** *Interpolating Blaschke products separate the points of the maximal ideal space of  $H^\infty(\mathcal{A})$ .*

Theorem 2 follows rather easily from Theorem 1 and known results. The strongest result previously known is Ziskind's theorem [13]: If  $m_1$  and  $m_2$  are homomorphisms in the maximal ideal space of  $H^\infty(\mathcal{A})$  with  $m_1$  lying on the Silov boundary and  $m_2$  lying off the Silov boundary, there is an interpolating Blaschke  $B$  such that  $m_1(B) \neq m_2(B)$ . Theorem 2 will be proved in §3.

Section 4 will be devoted to some comments and open questions.

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**2. Proof of Theorem 1.** To prove Theorem 1 it is sufficient to show that for any unimodular function  $u$  there are interpolating Blaschke products  $B_1$  and  $B_2$  such that

$$\left\| \arg \left( \frac{UB_2}{B_1} \right) \right\|_\infty < \varepsilon$$

on  $T$ . Here we define the function  $\arg$  by

$$\arg(e^{i(\theta+2\pi k)}) = \theta, \quad -\pi < \theta \leq \pi, \quad k \in \mathbb{Z}.$$

Our proof of Theorem 1 uses an approximation argument due to A. M. Davie which appears in [4]. The idea of this approximation argument is quite simple. For the rest of this section  $\varepsilon > 0$  will denote a small positive constant;  $\varepsilon \leq 1/10$  will do. Suppose  $\alpha$  is a real number,  $\varepsilon/2 < \alpha \leq \pi$ , and set  $\beta = 4\pi\varepsilon^3/\alpha$ . Let  $r = 1 - \eta$ , where  $0 < \eta < 1$  is a small number, and let  $A_1$  be the Blaschke product with simple zeros at the points

$$\{re^{i\theta_j}: \theta_j = j\beta\eta - \varepsilon^3\eta, 0 \leq j \leq \left[\frac{2\pi}{\beta\eta}\right] - 1\}.$$

Also let  $A_2$  be the Blaschke product with simple zeros at the points

$$\{re^{i\theta_k}: \theta_k = k\beta\eta - \varepsilon^3\eta, 0 \leq k \leq \left[\frac{2\pi}{\beta\eta}\right] - 1\}.$$

Then one can check by hand that

$$\left| \alpha - \arg \frac{A_1(x)}{A_2(x)} \right| < \frac{\varepsilon}{2}, \quad x \in T,$$

if  $A_1$  and  $A_2$  are first multiplied by suitable unimodular constants. Suppose now that  $I$  is a subarc of  $T$  and  $\eta$  is very small with respect to  $|I|$ . Let  $B_1$  and  $B_2$  be the Blaschke products having simple zeros at  $\{z: A_1(z) = 0, z/|z| \in I\}$  and  $\{z: A_2(z) = 0, z/|z| \in I\}$  respectively. Then  $\arg B_1(x)/B_2(x) \approx \alpha$  for most  $x$  which lie in  $I$  and  $\arg B_1(x)/B_2(x) \approx 0$  for most  $x$  which lie off  $I$ . More precisely, the following lemma holds.

**LEMMA 2.1.** *Suppose  $I$  is a subarc of  $T$  and  $\alpha$  is a real number,  $|\alpha| \leq \pi$ . Then if  $\varepsilon, \delta, \eta > 0$  are three positive numbers, there are finite Blaschke products  $A_1$  and  $A_2$  with simple zeros such that*

$$(2.1) \quad \left| \left\{ x \in I: \left| \alpha - \arg \frac{A_1(x)}{A_2(x)} \right| \geq \frac{\varepsilon}{2} \right\} \right| < \eta$$

$$(2.2) \quad \left| \left\{ x \notin I: \left| \arg \frac{A_1(x)}{A_2(x)} \right| \geq \delta \right\} \right| < \eta$$

$$(2.3) \quad \frac{z}{|z|} \in I \text{ if } z \text{ is a zero of } A_1 \text{ or } A_2$$

$$(2.4) \quad \rho(z_1, z_2) \geq \varepsilon^3 \text{ if } z_1 \text{ and } z_2 \text{ are two zeros } A_1 \text{ (} A_2 \text{)}.$$

Furthermore, there is a positive constant  $\gamma_0 = \gamma_0(\varepsilon, \delta, \eta, I) > 0$  such that whenever  $0 < \gamma < \gamma_0$  we can choose  $A_1$  and  $A_2$  as above so that  $1 - |z| = \gamma$  for all zeros  $z$  of  $A_1$  and  $A_2$ .

*Proof.* This is essentially proved in Lemma 12.3 of [4]. Only (2.2) requires verification. Let  $C_1$  and  $C_2$  be the two Blaschke factors given by  $C_1(z) = c_1 \cdot (z - z_1)/(1 - \bar{z}_1 z)$ ,  $C_2(z) = c_2 \cdot (z - z_2)/(1 - \bar{z}_2 z)$ , where

$$z_1 = (1 - \gamma)e^{i(\theta_0 - \varepsilon^3 \gamma)}, \quad z_2 = (1 - \gamma)e^{i(\theta_0 + \varepsilon^3 \gamma)},$$

where  $|c_1| = |c_2| = 1$  are chosen so that  $C_1(e^{i(\theta_0 + \pi)}) = C_2(e^{i(\theta_0 + \pi)})$ . Then

$$(2.5) \quad \left| \arg \frac{C_2(x)}{C_1(x)} \right| \leq 10\varepsilon^3 \left( 1 + \frac{|x - e^{i\theta_0}|}{\gamma^2} \right)^{-1}.$$

Since the ratio  $A_1/A_2$  given by the construction in Lemma 12.3 of [6] is a product of factors like  $C_1/C_2$  or  $C_2/C_1$ , inequality (2.2) follows from (2.3), (2.4), and (2.5), if all zeros  $z$  of  $A_1$  and  $A_2$  have modulus  $|z| = 1 - \gamma$  where  $\gamma$  is sufficiently small.

Repeating the argument of Lemma 2.1  $N$  times, we obtain the following lemma.

**LEMMA 2.2.** *Suppose  $I_1, I_2, \dots, I_N$  are a finite number of disjoint subarcs of  $T$  and  $\alpha_1, \dots, \alpha_N$  are  $N$  real numbers,  $|\alpha_j| \leq \pi$ ,  $1 \leq j \leq N$ . Then if  $\varepsilon, \delta, \eta > 0$  are three positive numbers, there are Blaschke products  $A_1$  and  $A_2$  with simple zeros such that*

$$(2.6) \quad \left| \left\{ x \in I_j : \left| \alpha_j - \arg \frac{A_1(x)}{A_2(x)} \right| \geq \frac{\varepsilon}{2} \right\} \right| < \frac{\eta}{4N}, \quad 1 \leq j \leq N.$$

$$(2.7) \quad \left| \left\{ x \notin \bigcup_{j=1}^N I_j : \left| \arg \frac{A_1(x)}{A_2(x)} \right| \geq \delta \right\} \right| < \frac{\eta}{4},$$

$$(2.8) \quad \frac{z}{|z|} \in \bigcup_{j=1}^N I_j \text{ if } z \text{ is a zero of } A_1 \text{ or } A_2.$$

$$(2.9) \quad \rho(z_1, z_2) \geq \varepsilon^3 \text{ if } z_1 \text{ and } z_2 \text{ are two zeros of } A_1(A_2).$$

Furthermore, there is a positive constant  $\gamma_0 = \gamma_0(\varepsilon, \delta, \eta, I_1, \dots, I_N) > 0$  such that whenever  $0 < \gamma < \gamma_0$  we can choose  $A_1$  and  $A_2$  as above so that  $1 - |z| = \gamma$  for all zeros  $z$  of  $A_1$  and  $A_2$ .

We now construct the Blaschke products  $B_1$  and  $B_2$  of Theorem 1. Let  $u$  be the unimodular function in the statement of the theorem and at stage one put  $v_1 = \arg u$ . Then  $\|v_1\|_\infty \leq \pi$ . Let  $\eta_{-1} = 2\pi$ ,  $\eta_0 = 1$ , and let  $E_1 = \{x : |v_1(x)| \geq \varepsilon\}$ . Find  $\delta_1 > 0$  such that

$$|\{x : \varepsilon - \delta_1 \leq |v_1(x)| < \varepsilon\}| < \frac{\eta_0}{4} = \frac{1}{4}.$$

Find also a finite collection of disjoint subarcs of  $T$ ,  $I_1^1, I_2^1, \dots, I_N^1$ , and  $N_1$  real numbers  $\alpha_1, \alpha_2, \dots, \alpha_{N_1}$ ,  $|\alpha_j| \leq \pi$ ,  $1 \leq j \leq N_1$ , such that

$$\left| \left\{ x: \left| v_1(x) \chi_{E_1}(x) - \sum_{j=1}^{N_1} \alpha_j \chi_{I_j^1}(x) \right| \geq \frac{\varepsilon}{4} \right\} \right| < \frac{\eta_0}{4} = \frac{1}{4}.$$

Applying Lemma 2.2 with  $\delta = \delta_1$  and  $\eta = \eta_0 = 1$ , we can find finite Blaschke products  $A_{1,1}$  and  $A_{2,1}$  with simple zeros such that

$$\left| \left\{ x: \left| v_1(x) - \arg \frac{A_{1,1}(x)}{A_{2,1}(x)} \right| \geq \varepsilon \right\} \right| < \eta_0 = 1.$$

If  $z$  is a zero of  $A_{1,1}$  or  $A_{2,1}$  then  $z/|z| \in \bigcup_{j=1}^{N_1} I_j^1$ . If  $z_1$  and  $z_2$  are two zeros of  $A_1$  ( $A_2$ ) then

$$\rho(z_1, z_2) \geq \varepsilon^3.$$

Furthermore we may choose  $A_{1,1}$  and  $A_{2,1}$  so that  $1 - |z| = \eta_1$  whenever  $z$  is a zero of  $A_{1,1}$  or  $A_{2,1}$ , and we may choose  $\eta_1 \leq \eta_0/4 = 1/4$ . The Blaschke product  $A_{1,1}$ , satisfies

$$\sum_{A_{1,1}(z)=0} (1 - |z|) \leq \eta_{-1} \varepsilon^{-3} = 2\pi \varepsilon^{-3}.$$

$A_{2,1}$  satisfies a similar inequality.

At stage two set

$$v_2(x) = \arg \left\{ u(x) \frac{A_{2,1}(x)}{A_{1,1}(x)} \right\}$$

and let  $E_2 = \{x: |v_2(x)| \geq \varepsilon\}$ . Find  $\delta_2 > 0$  such that

$$|\{x: \varepsilon - \delta_2 \leq |v_2(x)| < \varepsilon\}| < \frac{\eta_1}{4}.$$

Find also a finite collection of disjoint subarcs of  $T$ ,  $I_1^2, I_2^2, \dots, I_{N_2}^2$ , and  $N_2$  real numbers  $\alpha_1, \alpha_2, \dots, \alpha_{N_2}$ ,  $|\alpha_j| \leq \pi$ ,  $1 \leq j \leq N_2$ , such that

$$\left| \left\{ x: \left| v_2(x) \chi_{E_2}(x) - \sum_{j=1}^{N_2} \alpha_j \chi_{I_j^2}(x) \right| \geq \frac{\varepsilon}{4} \right\} \right| < \frac{\eta_1}{4}.$$

Since  $|\{x: |v_2(x)| \geq \varepsilon\}| < \eta_0 = 1$ , we can pick our intervals so that

$$\sum_{j=1}^{N_2} |I_j^2| < \eta_0 = 1.$$

Applying Lemma 2.2 with  $\delta = \delta_2$  and  $\eta = \eta_1$ , we can find Blaschke products  $A_{1,2}$  and  $A_{2,2}$  with simple zeros such that

$$\left| \left\{ x: \left| v_2(x) - \arg \frac{A_{1,2}(x)}{A_{2,2}(x)} \right| \geq \varepsilon \right\} \right| < \eta_1.$$

If  $z$  is a zero of  $A_{1,2}$  or  $A_{2,2}$  then  $z/|z| \in \bigcup_{j=1}^{N_2} I_j^2$ . If  $z_1$  and  $z_2$  are two zeros of  $A_{1,2}$  ( $A_{2,2}$ ) then

$$\rho(z_1, z_2) \geq \varepsilon^3.$$

Therefore  $A_{1,2}$  satisfies

$$\sum_{A_{1,2}(z)=0} (1 - |z|) \leq \eta_0 \varepsilon^{-3} = \varepsilon^{-3},$$

and  $A_{2,2}$  satisfies a similar inequality. We may choose  $A_{1,2}$  and  $A_{2,2}$  so that  $1 - |z| = \eta_2$  whenever  $z$  is a zero of  $A_{1,2}$  or  $A_{2,2}$ , and we may choose  $\eta_2 \leq \eta_1/4$ .

Suppose by induction that we have found Blaschke products with simple zeros  $A_{1,1}, A_{1,2}, \dots, A_{1,n-1}$  and  $A_{2,1}, A_{2,2}, \dots, A_{2,n-1}$  having the following properties. If  $z$  is a zero of  $A_{1,k}$  or  $A_{2,k}$  then

$$(2.10) \quad 1 - |z| = \eta_k \leq \frac{\eta_{k-1}}{4}, \quad 1 \leq k \leq n-1.$$

If  $z_1$  and  $z_2$  are two zeros of  $A_{1,k}$  ( $A_{2,k}$ ) then

$$(2.11) \quad \rho(z_1, z_2) \geq \varepsilon^3.$$

The zeros of  $A_{1,k}$  satisfy

$$(2.12) \quad \sum_{A_{1,k}(z)=0} (1 - |z|) < \eta_{k-2} \varepsilon^{-3}, \quad 1 \leq k \leq n-1,$$

and  $A_{2,k}$  satisfies a similar inequality. If  $B_{1,k} = \prod_{j=1}^k A_{1,j}$ ,  $B_{2,k} = \prod_{j=1}^k A_{2,j}$  and  $v_{k+1} = \arg\{u \cdot (B_{2,k}/B_{1,k})\}$  then

$$(2.13) \quad |\{x: |v_{k+1}(x)| \geq \varepsilon\}| < \eta_{k-1}, \quad 1 \leq k \leq n-1.$$

At stage  $n$  let  $E_n = \{x: |v_n(x)| \geq \varepsilon\}$ . Find  $\delta_n > 0$  such that

$$|\{x: \varepsilon - \delta_n \leq |v_n(x)| < \varepsilon\}| < \frac{\eta_{n-1}}{4}.$$

Find also a finite collection of disjoint subarcs of  $T$ ,  $I_1^n, I_2^n, \dots, I_{N_n}^n$ , and  $N_n$  real numbers  $\alpha_1, \alpha_2, \dots, \alpha_{N_n}$ ,  $|\alpha_j| \leq \pi$ ,  $1 \leq j \leq N_n$ , such that

$$\left| \left\{ x: \left| v_n(x) \chi_{E_n}(x) - \sum_{j=1}^{N_n} \alpha_j \chi_{I_j^n}(x) \right| \geq \frac{\varepsilon}{4} \right\} \right| < \frac{\eta_{n-1}}{4}.$$

Since  $|E_n| < \eta_{n-2}$  by the induction hypothesis, we can pick our intervals so that

$$\sum_{j=1}^{N_n} |I_j^n| < \eta_{n-2}.$$

Applying Lemma 2.2 with  $\delta = \delta_n$  and  $\eta = \eta_{n-1}$ , we can find Blaschke products  $A_{1,n}$  and  $A_{2,n}$  with simple zeros such that

$$\left| \left\{ x: \left| v_n(x) - \arg \frac{A_{1,n}(x)}{A_{2,n}(x)} \right| \geq \varepsilon \right\} \right| < \eta_{n-1}.$$

If  $x$  is a zero of  $A_{1,n}$  or  $A_{2,n}$  then  $z/|z| \in \bigcup_{j=1}^N I_j^n$ . If  $z_1$  and  $z_2$  are two zeros of  $A_{1,n}$  ( $A_{2,n}$ ) then

$$\rho(z_1, z_2) \geq \varepsilon^3.$$

Therefore  $A_{1,n}$  satisfies

$$\sum_{A_{1,n}(z)=0} (1 - |z|) \leq \eta_{n-2} \varepsilon^{-3},$$

and  $A_{2,n}$  satisfies a similar inequality. We may choose  $A_{1,n}$  and  $A_{2,n}$  so that  $1 - |z| = \eta_n$  whenever  $z$  is a zero of  $A_{1,n}$  or  $A_{2,n}$ , and we may choose  $\eta_n \leq \eta_{n-1}/4$ .

Working formally for a moment, put  $B_1 = \prod_{k=1}^{\infty} A_{1,k}$ ,  $B_2 = \prod_{k=1}^{\infty} A_{2,k}$ . Let  $\{z_{1,j}\} = \{z: A_{1,k}(z) = 0 \text{ for some } k \geq 1\}$  and let  $\{z_{2,j}\} = \{z: A_{2,k}(z) = 0 \text{ for some } k \geq 1\}$ . Notice that by (2.10) every point in  $\{z_{1,j}\}$  or  $\{z_{2,j}\}$  appears with multiplicity one. Suppose we know that  $\{z_{1,j}\}$  and  $\{z_{2,j}\}$  are interpolating sequences. Then  $\{z_{1,j}\}$  and  $\{z_{2,j}\}$  are Blaschke sequences and the partial products in the definition of  $B_1$  and  $B_2$  converge to interpolating Blaschke products. Inequality (2.13) then shows that

$$\left\| \arg u - \arg \frac{B_1}{B_2} \right\|_{\infty} \leq \varepsilon.$$

We now verify that  $\{z_{1,j}\}$  is an interpolating sequence. The proof for  $\{z_{2,j}\}$  is exactly the same. If  $k \neq j$ , inequalities (2.10) and (2.11) show

$$\rho(z_{1,j}, z_{1,k}) \geq \varepsilon^3.$$

To show that  $\mu = \sum (1 - |z_{1,j}|) \delta_{z_{1,j}}$  is a Carleson measure, it is only necessary to verify the Carleson condition for arcs  $I$  of length  $2\eta_n$ ,  $n \geq 1$ . So suppose  $|I| = 2\eta_n$ . Then by (2.11),

$$\sum_{\substack{z_{1,j} \in \tilde{I} \\ (1 - |z_{1,j}|) = \eta_n}} (1 - |z_{1,j}|) \leq 2\varepsilon^{-3} |I|$$

and

$$\sum_{\substack{z_{1,j} \in \tilde{I} \\ (1 - |z_{1,j}|) = \eta_{n+1}}} (1 - |z_{1,j}|) \leq 2\varepsilon^{-3} |I|.$$

If  $k \geq 2$ , then by (2.10) and (2.12),

$$\sum_{\substack{z_{1,j} \in \tilde{I} \\ (1 - |z_{1,j}|) = \eta_{n+k}}} (1 - |z_{1,j}|) \leq \eta_{n+k-2} \varepsilon^{-3}$$

$$\begin{aligned} &\leq \eta_n \varepsilon^{-3} 4^{2-k} \\ &= 8\varepsilon^{-3} 4^{-k} |I|. \end{aligned}$$

Summing on  $k \geq 2$  we obtain

$$\sum_{z_1, j \in \tilde{I}} (1 - |z_{1,j}|) \leq 5\varepsilon^{-3} |I|,$$

so  $\sum (1 - |z_{1,j}|) \delta_{z_{1,j}}$  is a Carleson measure. The proof of Theorem 1 is complete.

**3. Proof of Theorem 2.** The maximal ideal space of  $H^\infty$ ,  $\mathcal{M}_{H^\infty}$ , is divided into three disjoint parts, which we denote by  $\omega$ ,  $G$ , and  $H$ .  $\omega$  is the Šilov boundary of  $H^\infty$ . See [5] for properties of  $\omega$ .  $G$  is the set of all homomorphisms which lie in the closure (in the topology of  $\mathcal{M}_{H^\infty}$ ) of an interpolating sequence in the disk.  $H$  is the set of homomorphisms which lie in neither  $\omega$  nor  $G$ . To study  $H$  we will have need of L. Carleson's corona theorem [2]:

If  $m \in \mathcal{M}_{H^\infty}$  then there is a net  $\{z_\alpha\}$  of points lying in the unit disk which converge to  $m$  in the topology of  $\mathcal{M}_{H^\infty}$ .

Let  $m_1 \neq m_2$  be two homomorphisms in  $\mathcal{M}_{H^\infty}$ . To separate  $m_1$  and  $m_2$  by an interpolating Blaschke product there are four cases we must treat.

*Case I.* Either  $m_1 \in G$  or  $m_2 \in G$ . Let us suppose  $m_1 \in G$ . Then there is an interpolating sequence  $\{z_j\}$  such that  $m_1$  is in the closure of the points  $\{z_j\}$ , but  $m_2$  is not. If  $B$  is the interpolating Blaschke product with simple zeros at the points  $\{z_j\}$ , then  $m_1(B) = 0$  and  $m_2(B) \neq 0$ .

*Case II.*  $m_1 \in \omega$  and  $m_2 \notin \omega$ . This case has been treated previously by Ziskind [13]. Theorem 1 can also be used to treat this case. There is a Blaschke product  $B$  such that  $m_1(B) = 1$  and  $m_2(B) = 0$ . (See e.g., [7].) Let  $B_1$  and  $B_2$  be two interpolating Blaschke products such that

$$\left\| B - \frac{B_1}{B_2} \right\|_\infty = \|BB_2 - B_1\|_\infty < \frac{1}{2}.$$

Then  $|m_2(B_1)| = |m_2(B_1 - BB_2)| < 1/2$ . On the other hand,  $|m_1(B_1)| = 1$  because  $m_1 \in \omega$ .

*Case III.*  $m_1, m_2 \in H$ . By theorem of Hoffman [7], every point of  $G$  is a one-point Gleason part. Thus for every  $\varepsilon > 0$  we can find  $f_\varepsilon \in H^\infty$ ,  $\|f_\varepsilon\|_\infty = 1$ , such that  $m_1(f_\varepsilon) = 0$  and  $|m_2(f_\varepsilon)| > 1 - \varepsilon$ . The unit ball of  $H^\infty$  is the norm closed convex hull of the Blaschke

products [10], so we can find an inner function  $u_\varepsilon$  such that  $m_1(u_\varepsilon) = 0$  and  $|m_2(u)| > 1 - \varepsilon$ . We now invoke a theorem of Ziskind [13].

**THEOREM.** (*Ziskind*) *Let  $u$  be an inner function. There are universal constants  $0 < \varepsilon_0 < 1$ ,  $c_1$ , and  $c_2$ , and there is an interpolating Blaschke product  $B$  such that*

$$(3.1) \quad \text{if } |u(z)| \leq \frac{1}{10} \text{ then } |B(z)| \leq \frac{1}{10}.$$

$$(3.2) \quad \text{if } B(z) = 0 \text{ then } |u(z)| \leq 1 - \varepsilon_0.$$

$$(3.3) \quad \rho(z_1, z_2) \geq c_1 \text{ whenever } z_1 \text{ and } z_2 \text{ are two zeros of } B.$$

$$(3.4) \quad \sum_{\substack{B(z_j) = 0 \\ z_j \in I}} (1 - |z_j|) \leq c_2 |I| \text{ for all subarcs } I \text{ of } T.$$

For each  $\varepsilon > 0$  find  $B_\varepsilon$  as in Ziskind's theorem, corresponding to the inner function  $u_\varepsilon$ . For a set  $E \subset \mathcal{A}$  define  $E^* \subset T$  by

$E^* = \{e^{i\theta} : re^{i\theta_0} \in E \text{ for some } r \text{ and } \theta_0, |\theta - \theta_0| \leq 1 - r\}$ .  $E^*$  is the nontangential projection of  $E$  onto  $T$ .

Now fix  $\varepsilon > 0$ . There are nets  $\{w_\alpha\}$  and  $\{z_\beta\}$  of points of  $\mathcal{A}$  such that

$$w_\alpha \longrightarrow m_1$$

and

$$z_\beta \longrightarrow m_2.$$

Inequality (3.1) shows  $|m_1(B_\varepsilon)| \leq 1/10$ . By taking a subnet we can assume that  $|u_\varepsilon(z_\beta)| > 1 - \varepsilon$  for all  $\beta$ . Let  $\tau_\beta$  be a Mobius transformation which sends 0 to  $z_\beta$ , and let  $v_{\varepsilon,\beta} = u_\varepsilon \circ \tau_\beta$ . Then  $|v_{\varepsilon,\beta}(0)| > 1 - \varepsilon$ . Let  $E_{\varepsilon,\beta} = \{z : |v_{\varepsilon,\beta}(z)| \leq 1 - \varepsilon_0\}$ , where  $\varepsilon_0$  is the constant in Ziskind's theorem. Then

$$|E_{\varepsilon,\beta}^*| \leq c_3(\varepsilon)$$

where

$$c_3(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

This is essentially proved in §4 of [11]. Thus by (3.2)-(3.4),

$$|B_\varepsilon \circ \tau_\beta(0)| = |B_\varepsilon(z_\beta)| \geq 1 - c_4(\varepsilon)$$

where

$$c_4(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Picking  $\varepsilon$  so small that  $c_4(\varepsilon) \leq 1/2$  we see that  $|m_2(B)| \geq 1/2$ .

*Case IV.*  $m_1, m_2 \in \mathfrak{M}$ . Let  $u$  be an inner function such that  $m_1(u) = 1$ ,  $m_2(u) = -1$ . By Theorem 1 there are interpolating Blaschke products  $B_1$  and  $B_2$  so that  $\|u - B_1/B_2\|_\infty \leq 1/4$ . Then

$$|m_1(B_2 u) - m_1(B_1)| = |m_1(B_2) - m_1(B_1)| \leq \frac{1}{4}$$

and

$$|m_2(B_2 u) - m_2(B_1)| = |m_2(B_2) + m_2(B_1)| \leq \frac{1}{4}.$$

If  $m_1(B_1) = m_2(B_1)$  the above inequalities show  $m_1(B_2) \neq m_2(B_2)$ . (Since  $m_1, m_2 \in \mathfrak{M}$ ,  $|m_1(v)| = |m_2(v)| = 1$  for all inner functions  $v$ .)

REMARKS 4. We conclude with some remarks and questions. It is possible that Theorem 1 could be derived from the Douglas-Rudin theorem but our attempts at this have been unsuccessful. In particular, a positive answer to the following question would suffice.

*Question 1.* If  $B_0$  is a Blaschke product and  $\varepsilon > 0$ , is there an interpolating Blaschke product  $B_1$  with  $\|B_0 - B_1\|_\infty < \varepsilon$ ? One might hope that for a given  $B_0$  there is a complex number  $\alpha$ ,  $|\alpha| < \varepsilon$ , such that  $B_0 + \alpha/1 + \bar{\alpha}B_0$  is an interpolating Blaschke product. Unfortunately, this is not the case. Kahane [9] and Piranian [12] have shown that there is a Blaschke product  $B$  such that

$$|B'(z)| = o(1 - |z|),$$

while  $B$  has infinitely many zeros. On the other hand, if  $B$  is an interpolating Blaschke product with infinitely many zeros,

$$|B'(z)| = O(1 - |z|).$$

Theorem 2 shows that the uniform algebra generated by interpolating Blaschke products (call it  $J$ ) is large in some sense. A natural question is just how large is  $J$ ?

*Question 2.* Does  $J = H^\infty$ ?

We do not even know if  $J$  and  $H^\infty$  have the same maximal ideal space. Note that by Marshall's theorem [10], an affirmative answer to Question 1 would give an affirmative answer to Question 2. By using Theorem 1 and the machinery in [10] one can reduce Question 2 to a problem concerning interpolation of bounded sequences by interpolating Blaschke products. An indication that  $J$  might be equal to  $H^\infty$  comes from the Chang-Marshall theorem [3], [11].

Finally, we note that BMO can be related to ratios of interpolating Blaschke products through the results of [8]. Perhaps arguments using BMO could be used to answer Questions 1 and 2.

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George E. Andrews, The Rogers-Ramanujan reciprocal and Minc's partition function .....	251
Allan Calder, William H. Julian, Ray Mines, III and Fred Richman, $\varepsilon$ -covering dimension .....	257
Thomas Curtis Craven and George Leslie Csordas, An inequality for the distribution of zeros of polynomials and entire functions .....	263
Thomas Jones Enright and R. Parthasarathy, The transfer of invariant pairings to lattices .....	281
Allen Roy Freedman and John Joseph Sember, Densities and summability .....	293
Robert Heller and Francis Aubra Roach, A generalization of a classical necessary condition for convergence of continued fractions .....	307
Peter Wilcox Jones, Ratios of interpolating Blaschke products .....	311
V. J. Joseph, Smooth actions of the circle group on exotic spheres .....	323
Mohd Saeed Khan, Common fixed point theorems for multivalued mappings .....	337
Samuel James Lomonaco, Jr., The homotopy groups of knots. I. How to compute the algebraic 2-type .....	349
Louis Magnin, Some remarks about $C^\infty$ vectors in representations of connected locally compact groups .....	391
Mark Mandelker, Located sets on the line .....	401
Murray Angus Marshall and Joseph Lewis Yucas, Linked quaternionic mappings and their associated Witt rings .....	411
William Lindall Paschke, $K$ -theory for commutants in the Calkin algebra .....	427
W. J. Phillips, On the relation $PQ - QP = -iI$ .....	435
Ellen Elizabeth Reed, A class of Wallman-type extensions .....	443
Sungwoo Suh, The space of real parts of algebras of Fourier transforms .....	461
Antonius Johannes Van Haagen, Finite signed measures on function spaces .....	467
Richard Hawks Warren, Identification spaces and unique uniformity .....	483