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The notion of Killing-Ricci forms of Lie triple algebras is introduced as a generalization of both of Killing forms of Lie algebras and the Ricci forms of the tangent Lie triple systems of Riemannian symmetric spaces. For a class of Lie triple algebras $\mathfrak{S}$, it is shown that $\mathfrak{S}$ is decomposed into a direct sum of simple ideals if its Killing-Ricci form is nondegenerate. As an application, structure of the reductive pair consisting of a semi-simple Lie algebra and its semi-simple subalgebra is investigated.

Introduction. The concept of Lie triple algebras has been introduced, originally, by K. Yamaguti [11] as general Lie triple systems, related with locally reductive spaces of K. Nomizu [6], and treated by himself (e.g., [11]-[14]), A. A. Sagle (e.g., [8], [9]) and others. In the articles [2] and [3], the author considered Lie triple algebras as tangent algebras of homogeneous Lie loops or analytic homogeneous systems on manifolds. For the study of such algebraic systems on manifolds it seems to be very important to investigate the structure of real Lie triple algebras of finite dimension, as an extended analogy of the theory of Lie groups and Lie algebras. In this paper we consider the Killing-Ricci form β of a Lie triple algebra $\mathfrak{S}$, a symmetric bilinear form on $\mathfrak{S}$ obtained by restricting the Killing form of the standard enveloping Lie algebra of $\mathfrak{S}$. Then, under an assumption by which β becomes an invariant bilinear form on $\mathfrak{S}$, it is shown that a Lie triple algebra $\mathfrak{S}$ is decomposed into a direct sum of simple Lie triple algebra ideals, if β is nondegenerate (Theorem 2). This result is applied for a reductive pair of semi-simple Lie algebra $\mathfrak{L}$ and semi-simple subalgebra $\mathfrak{K}$ of $\mathfrak{L}$, treated by A. A. Sagle [8], [9]. Then, a direct sum decomposition of $\mathfrak{L}$ into simple Lie triple algebras and semi-simple Lie algebra ideals of $\mathfrak{K}$ is obtained (Theorem 3).

1. Preliminaries. A *Lie triple algebra* $\mathfrak{S}$ over a field F is an anti-commutative algebra over F whose multiplication is denoted by XY for $X, Y \in \mathfrak{S}$, with a trilinear operation $\mathfrak{S} \times \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ denoted by $D(X, Y)Z$ satisfying the following conditions for $X, Y, Z, W \in \mathfrak{S}$:

(i) $D(X, X)Z = 0$,

(ii) $\mathfrak{S}\{(XY)Z + D(X, Y)Z\} = 0$,

(iii) $\mathfrak{S}D(XY, Z)W = 0$,

(iv) $D(X, Y)(ZW) = (D(X, Y)Z)W + Z(D(X, Y)W)$,

(v) $[D(X, Y), D(Z, W)] = D(D(X, Y)Z, W) + D(Z, D(X, Y)W)$,

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. That is, Lie triple algebra is a synonym for *a general Lie triple system* introduced by K. Yamaguti [11]. Throughout this paper, we assume that $\mathfrak{G}$ is a finite dimensional Lie triple algebra over a field of characteristic zero. The endomorphisms $D(X, Y)$ are called *inner derivations* of $\mathfrak{G}$ and the Lie subalgebra $D(\mathfrak{G}, \mathfrak{G})$ of $\mathrm{End}(\mathfrak{G})$ generated by all inner derivations is called the *inner derivation algebra* of $\mathfrak{G}$. The *standard enveloping Lie algebra* of $\mathfrak{G}$ is a Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ whose bracket operation is given as follows; $[X, Y] = XY + D(X, Y)$ for $X, Y \in \mathfrak{G}$, $[U, X] = -[X, U] = UX$ for $U \in D(\mathfrak{G}, \mathfrak{G})$, $X \in \mathfrak{G}$ and $[U, V] = UV - VU$ for $U, V \in D(\mathfrak{G}, \mathfrak{G})$. Thus $D(\mathfrak{G}, \mathfrak{G})$ is a Lie subalgebra of $\mathfrak{A}$ and $(\mathfrak{A}, D(\mathfrak{G}, \mathfrak{G}))$ forms a reductive pair. Conversely, for a Lie algebra $\mathfrak{L}$ and a subalgebra $\mathfrak{K}$, if $(\mathfrak{L}, \mathfrak{K})$ is a reductive pair with the fixed decomposition $\mathfrak{L} = \mathfrak{G} + \mathfrak{K}$, $[\mathfrak{K}, \mathfrak{G}] \subset \mathfrak{G}$, then $\mathfrak{G}$ is a Lie triple algebra ([11]) under the operations $XY = [X, Y]_{\mathfrak{G}}$ and $D(X, Y)Z = [[X, Y]_{\mathfrak{K}}, Z]$ for $X, Y, Z \in \mathfrak{G}$. The Lie triple algebra $\mathfrak{G}$ is reduced to Lie algebra with $[X, Y] = XY$ if $D(\mathfrak{G}, \mathfrak{G}) = \{0\}$, or it is reduced to Lie triple system with $[X, Y, Z] = D(X, Y)Z$ if $\mathfrak{G}\mathfrak{G} = \{0\}$. Conversely, every Lie algebra or Lie triple system is a Lie triple algebra as one of the reduced cases above. If $\mathfrak{G}\mathfrak{G} = \{0\}$ and $D(\mathfrak{G}, \mathfrak{G}) = \{0\}$, $\mathfrak{G}$ is said to be *abelian*. A Lie triple subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is an *invariant subalgebra* if $D(\mathfrak{G}, \mathfrak{G})\mathfrak{H} \subset \mathfrak{H}$, and an *ideal* if $\mathfrak{G}\mathfrak{H} \subset \mathfrak{H}$ and $D(\mathfrak{G}, \mathfrak{H})\mathfrak{G} \subset \mathfrak{H}$. If $\mathfrak{H}$ is an ideal of $\mathfrak{G}$, then it is an invariant subalgebra. Let $\mathfrak{H}$ be an ideal of $\mathfrak{G}$. A chain $\mathfrak{H} = \mathfrak{H}^{(0)} \supset \mathfrak{H}^{(1)} \supset \cdots \supset \mathfrak{H}^{(i)} \supset \mathfrak{H}^{(i+1)} \supset \cdots$ of invariant subalgebras of $\mathfrak{G}$ is defined inductively by $\mathfrak{H}^{(1)} = \mathfrak{H}\mathfrak{H} + D(\mathfrak{G}, \mathfrak{H})\mathfrak{H}$ and $\mathfrak{H}^{(i+1)} = \mathfrak{H}^{(i)}\mathfrak{H}^{(i)} + D(\mathfrak{H}, \mathfrak{H})\mathfrak{H}^{(i)} + D(\mathfrak{G}, \mathfrak{H}^{(i)})\mathfrak{H}^{(i)}$ for positive integers i ([4]). Each $\mathfrak{H}^{(i+1)}$ is an ideal of $\mathfrak{H}^{(i)}$ and the quotient Lie triple algebra $\mathfrak{H}^{(i)}/\mathfrak{H}^{(i+1)}$ is abelian. An ideal $\mathfrak{H}$ of $\mathfrak{G}$ is said to be *solvable* if $\mathfrak{H}^{(i)} = \{0\}$ for some integer i. The *radical* $\mathfrak{r}(\mathfrak{G})$ of $\mathfrak{G}$ is a (unique) maximal solvable ideal of $\mathfrak{G}$. The Lie triple algebra $\mathfrak{G}$ is *semi-simple* if $\mathfrak{r}(\mathfrak{G}) = \{0\}$. The following facts have been shown in [4]:

(1.1) If $\mathfrak{G}$ is solvable then its standard enveloping Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ is a solvable Lie algebra and $D(\mathfrak{G}, \mathfrak{G})$ is a solvable Lie subalgebra of $\mathfrak{A}$.

(1.2) If $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ is a semi-simple Lie algebra then $\mathfrak{G}$ is semi-simple.

A Lie triple algebra $\mathfrak{G}$ is *simple* if it has no nonzero proper ideal. It is easy to show the following:

(1.3) $\mathfrak{G}$ is simple if its standard enveloping Lie algebra is a simple Lie algebra.

2. Killing-Ricci forms. Let $\{E_1, E_2, \cdots, E_n\}$ be a basis of an n-dimensional Lie triple algebra $\mathfrak{G}$, and $\{D_1, D_2, \cdots, D_N\}$ a basis of the inner derivation algebra $D(\mathfrak{G}, \mathfrak{G})$ if $D(\mathfrak{G}, \mathfrak{G}) \neq \{0\}$. For these bases we express the operations of $\mathfrak{G}$ as follows:

$$E_iE_j = S^k_{ij}E_k\ , \quad D(E_i, E_j)E_k = R^l_{ijk}E_l$$
$$D(E_i, E_j) = D^\alpha_{ij}D_\alpha\ , \quad [D_\alpha, E_i] = D_\alpha E_i = K^j_{\alpha i}E_j\ , \tag{2.1}$$

where the indices run through $1 \leqq i, j, k, l \leqq n$ and $1 \leqq \alpha \leqq N$. Denote by α the Killing form of the standard enveloping Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$. By the *Killing-Ricci form* β of the Lie triple algebra $\mathfrak{G}$ we mean a symmetric bilinear form on $\mathfrak{G}$ determined by restricting α to $\mathfrak{G} \times \mathfrak{G}$.

PROPOSITION 1. *For $X, Y \in \mathfrak{G}$,*

$$\beta(X, Y) = \operatorname{tr} L(X)L(Y) + \operatorname{tr}(r(X, Y) + r(Y, X))\ ,$$

where $L(X)$ and $r(X, Y)$ are endomorphisms of $\mathfrak{G}$ given by

$$L(X)Y = XY\ , \quad r(X, Y)Z = D(Z, X)Y \quad \textit{for} \quad X, Y, Z \in \mathfrak{G}\ .$$

Proof. If we set $S_{ij} = \operatorname{tr} L(E_i)L(E_j)$, $R_{ij} = \operatorname{tr}(r(E_i, E_j))$ and $\beta_{ij} = \beta(E_i, E_j)$, it is sufficient to show the formula

$$\beta_{ij} = S_{ij} + R_{ij} + R_{ji} \quad \text{for} \quad 1 \leqq i, j \leqq n\ . \tag{2.2}$$

The expressions (2.1) with respect to the bases $\{E_1, \cdots, E_n\}$ and $\{D_1, \cdots, D_N\}$ imply the following:

$$[E_i, [E_j, E_k]] = (S^l_{im}S^m_{jk} - R^l_{jki})E_l + S^m_{jk}D^\alpha_{im}D_\alpha\ ,$$
$$[E_i, [E_j, D_\alpha]] = -S^l_{im}K^m_{\alpha j}E_l - D^\beta_{im}K^m_{\alpha j}D_\beta\ .$$

Hence,

$$\beta_{ij} = S^k_{im}S^m_{jk} + R^k_{kji} + D^\alpha_{mi}K^m_{\alpha j}\ . \tag{2.3}$$

On the other hand, the expressions

$$L(E_i)L(E_j)E_k = S^l_{im}S^m_{jk}E_l\ ,$$
$$R^l_{kji}E_l = D(E_k, E_j)E_i = D^\alpha_{kj}K^l_{\alpha i}E_l$$

imply

$$S_{ij} = S^k_{im}S^m_{jk}\ , \quad R_{ij} = R^k_{kij} = D^\alpha_{ki}K^k_{\alpha j}\ . \tag{2.4}$$

Therefore, (2.2) is obtained from (2.3) and (2.4). □

REMARK 1. (1) If $\mathfrak{G}$ is reduced to Lie algebra, then β is the Killing form of the Lie algebra $\mathfrak{G}$. On the other hand, if $\mathfrak{G}$ is reduced to Lie triple system, then $\beta(X, Y)/2 = (r(X, Y) + r(Y, X))/2$ is the Killing form of the Lie triple system $\mathfrak{G}$ introduced by T. Ravisankar [7].

(2) Suppose that $\mathfrak{G}$ is a Malcev algebra. The Killing form θ of $\mathfrak{G}$ introduced by O. Loos [5] is given by $\theta(X, Y) = \mathrm{tr}(\lambda(X)\lambda(Y))$, where $\lambda(X)$ denotes the left translation by X in the Malcev algebra. K. Yamaguti [12] has shown that $\mathfrak{G}$ is a Lie triple algebra (general Lie triple system) under the operations $L(X)=\lambda(X)$, $D(X, Y)=\lambda(\lambda(X)Y) + [\lambda(X), \lambda(Y)]$. The Killing-Ricci form of this Lie triple algebra is equal to 5θ. In [5], O. Loos considered the Malcev algebra $\mathfrak{G}$ as a Lie triple system with the inner derivations $D(X, Y) = 2\lambda(\lambda(X)Y) + [\lambda(X), \lambda(Y)]$. The Killing-Ricci form of this Lie triple system is equal to 6θ ([5, Lemma 6]).

REMARK 2. (1) Let G be a Riemannian symmetric space. It is well known that the tangent space $\mathfrak{G}$ at $e \in G$ is Lie triple system with respect to the ternary operation $[X, Y, Z] = R_e(X, Y)Z$, where R denotes the curvature tensor of G. Then R^l_{ijk} in (2.1) are the components of R at e, and the Killing-Ricci form β of the Lie triple system $\mathfrak{G}$ has the components $\beta_{ij} = 2R_{ij}$, where $R_{ij} = R^k_{kij}$ are the components of the Ricci tensor of G.

(2) Let G be an analytic homogeneous Lie loop in [2] or, more generally, an analytic homogeneous system in [3], and $\mathfrak{G}$ be its tangent Lie triple algebra at $e \in G$. Then S^k_{ij} and R^l_{ijk} in (2.1) are respectively the components at e of the torsion tensor and the curvature tensor of the canonical connection of G.

Now, let γ be a trilinear form on $\mathfrak{G}$ given by

$$\gamma(X, Y, Z) = \mathrm{tr}(D(X, Y)L(Z)) \quad \text{for} \quad X, Y, Z \in \mathfrak{G}\,. \tag{2.5}$$

It is evident that γ vanishes identically if $\mathfrak{G}$ is reduced to Lie algebra or reduced to Lie triple system.

PROPOSITION 2. *The Killing-Ricci form β of a Lie triple algebra $\mathfrak{G}$ satisfies the followings, for $X, Y, Z \in \mathfrak{G}$:*

$$\beta(XY, Z) + \beta(Y, XZ) = \gamma(Y, X, Z) + \gamma(Z, X, Y) \tag{2.6}$$

$$\begin{aligned} &\beta(X, D(Y, Z)W) - \beta(D(W, X)Y, Z) \\ &\quad = \gamma(Y, Z, WX) - \gamma(W, X, YZ) \end{aligned} \tag{2.7}$$

$$\beta(X, D(Y, Z)W) + \beta(D(Y, Z)X, W) = 0 . \tag{2.8}$$

Proof. The Killing form α of $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ satisfies $\alpha([X, Y], Z) + \alpha(Y, [X, Z]) = 0$ for $X, Y, Z \in \mathfrak{G}$. Since

$$\alpha([X, Y], Z) = \beta(XY, Z) + \gamma(X, Y, Z) , \tag{2.9}$$

the formula (2.6) is obtained. Applying (2.9) for

$$\alpha(X, [[Y, Z], W]) + \alpha([[X, W], Y], Z) = 0$$

we get

$$\begin{aligned} &\beta(X, D(Y, Z)W) + \beta(D(X, W)Y, Z) \\ &\quad = \beta(X, (YZ)W) + \beta((WX)Y, Z) \\ &\qquad + \gamma(ZY, W, X) + \gamma(WX, Y, Z) . \end{aligned} \tag{2.10}$$

By using (2.6) for $\beta(X, W(YZ))$ and $\beta(Y(XW), Z)$, we have

$$\begin{aligned} &\beta(X, (YZ)W) + \beta((WX)Y, Z) \\ &\quad = \gamma(X, W, YZ) + \gamma(Y, Z, WX) \\ &\qquad + \gamma(YZ, W, X) + \gamma(XW, Y, Z) . \end{aligned} \tag{2.11}$$

The formula (2.7) is obtained from (2.10) and (2.11). Setting $X = W$ in (2.7), we get $\beta(X, D(Y, Z)X) = 0$ which implies (2.8). □

3. Semi-simple Lie triple algebras with $\gamma = 0$. By an *invariant form* b on a Lie triple algebra $\mathfrak{G}$ we mean a symmetric bilinear form on $\mathfrak{G}$ satisfying;

$$b(XY, Z) + b(Y, XZ) = 0 \tag{3.1}$$

$$b(X, D(Y, Z)W) - b(D(W, X)Y, Z) = 0 \quad \text{for} \quad X, Y, Z, W \in \mathfrak{G} . \tag{3.2}$$

PROPOSITION 3. *Let b be an invariant bilinear form on $\mathfrak{G}$. If $\mathfrak{H}$ is an invariant subalgebra of G, then $\mathfrak{H}^\perp = \{X \in \mathfrak{G}; b(X, \mathfrak{H}) = 0\}$ is an invariant subalgebra of $\mathfrak{G}$. Moreover, if $\mathfrak{H}$ is an ideal of $\mathfrak{G}$ then $\mathfrak{H}^\perp$ is an ideal.*

Proof. From (3.2) the following is obtained:

$$b(D(Y, Z)W, X) + b(W, D(Y, Z)X) = 0 . \tag{3.3}$$

Therefore, if $\mathfrak{H}$ is an invariant subalgebra of $\mathfrak{G}$, then $D(\mathfrak{G}, \mathfrak{G})\mathfrak{H}^\perp \subset \mathfrak{H}^\perp$. If $X, Z \in \mathfrak{G}$ and $Y \in \mathfrak{H}^\perp$ then $b(XY, \mathfrak{H}) = b(Y, X\mathfrak{H})$ and $b(D(X, Y)Z, \mathfrak{H}) = b(D(Z, \mathfrak{H})X, Y)$, which imply that $\mathfrak{H}^\perp$ is an ideal (resp. subalgebra) if $\mathfrak{H}$ is an ideal (resp. subalgebra) of the Lie triple algebra $\mathfrak{G}$. □

REMARK 3. If $\mathfrak{G}$ is reduced to Lie triple system, then from

(3.2) and (3.3) it follows that a symmetric bilinear form b on $\mathfrak{S}$ is an invariant form if and only if b is an invariant form of Lie triple system, in the sense of J. Wolf [10, (10.11)].

By a similar consideration of invariant bilinear forms on $\mathfrak{S}$ as one for the cases of Lie triple system [10, §10] and nonassociative algebras [1, p. 71], we have;

PROPOSITION 4. *Let b be an invariant bilinear form on $\mathfrak{S}$. Suppose that b is nondegenerate. Then,*

(1) *The ideal $\mathfrak{Z}^{\perp}$ for the center $\mathfrak{Z}=\{X \in \mathfrak{S};\ X\mathfrak{S}=0$ and $D(X, \mathfrak{S})=0\}$ of $\mathfrak{S}$ is equal to $\mathfrak{S}^{(1)} = \mathfrak{S}\mathfrak{S} + D(\mathfrak{S}, \mathfrak{S})\mathfrak{S}$;*

(2) *If $\mathfrak{S}$ has no nonzero ideal $\mathfrak{H}$ satisfying $\mathfrak{H}\mathfrak{H} = \{0\}$ and $D(\mathfrak{S}, \mathfrak{H})\mathfrak{H} = \{0\}$, then $\mathfrak{S}$ is decomposed into*

$$\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots + \mathfrak{S}_r\,, \quad b = b_1 + b_2 + \cdots + b_r\,,$$

where $\mathfrak{S}_i (1 \leqq i \leqq r)$ are simple ideals of $\mathfrak{S}$ and $b_i = b \,|\, \mathfrak{S}_i \times \mathfrak{S}_i$, which is an invariant bilinear form on $\mathfrak{S}_i$ for each i.

Proof. (1) is obtained from (3.1) and (2.2), and (2) is shown as follows. Let $\mathfrak{S}_1$ be a minimal nonzero ideal in $\mathfrak{S}$. By Proposition 3, $\mathfrak{S}_1^{\perp}$ is an ideal of $\mathfrak{S}$ and so $\mathfrak{H} = \mathfrak{S}_1 \cap \mathfrak{S}_1^{\perp}$ is an ideal of $\mathfrak{S}$ contained in $\mathfrak{S}_1$. If $X, Y \in \mathfrak{S}$ and $Z, W \in \mathfrak{H}$, then from (3.1) and (3.2) we have

$$b(X, ZW) = b(XZ, W) \in b(\mathfrak{H}, \mathfrak{H}) = \{0\}\,,$$
$$b(X, D(Y, Z)W) = b(D(W, X)Y, Z) \in b(\mathfrak{H}, \mathfrak{H}) = \{0\}\,.$$

Since b is supposed to be nondegenerate, $ZW = D(Y, Z)W = \{0\}$ for $Y \in \mathfrak{S}$ and $Z, W \in \mathfrak{H}$; i.e., $\mathfrak{H}\mathfrak{H} = \{0\}$ and $D(\mathfrak{S}, \mathfrak{H})\mathfrak{H} = \{0\}$. Hence, by hypothesis, $\mathfrak{H} = \mathfrak{S}_1 \cap \mathfrak{S}_1^{\perp} = \{0\}$ and so $\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_1^{\perp}$. It is clear that the bilinear form $b_1 = b \,|\, \mathfrak{S}_1 \times \mathfrak{S}_1$ is a nondegenerate invariant form on $\mathfrak{S}_1$. The proposition is then established by induction on dim $\mathfrak{S}$. □

Now, let β be the Killing-Ricci form of $\mathfrak{S}$. If the trilinear form γ given by (2.5) vanishes identically on $\mathfrak{S}$, then from (2.6) and (2.7) it follows that β is an invariant bilinear form on $\mathfrak{S}$.

THEOREM 1. *Let $\mathfrak{S}$ be a finite dimensional Lie triple algebra over a field of characteristic zero, on which the trilinear form γ defined by* (2.5) *vanishes identically. Then;*

(1) *The Killing-Ricci form β of $\mathfrak{S}$ is nondegenerate if and only if the standard enveloping Lie algebra $\mathfrak{A} = \mathfrak{S} + D(\mathfrak{S}, \mathfrak{S})$ is a semi-simple Lie algebra.*

(2) *If β is nondegenerate, then $\mathfrak{S}$ is a semi-simple Lie triple algebra and $\mathfrak{S} = \mathfrak{S}^{(1)}$.*

Proof. Since the Killing form α of the Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ satisfies $\alpha(D(X, Y), Z) = \operatorname{tr} D(X, Y)L(Z) = \gamma(X, Y, Z)$ for $X, Y, Z \in \mathfrak{G}$, the condition $\gamma = 0$ implies that $\mathfrak{G}$ and $D(\mathfrak{G}, \mathfrak{G})$ are orthogonal to each other with respect to α. Hence,

$$(3.4)\quad \begin{aligned}\alpha(D(X, Y), D(Z, W)) &= \alpha([X, Y], D(Z, W)) \\ &= \beta(Y, D(Z, W)X) \quad \text{for} \quad X, Y, Z, W \in \mathfrak{G}\,.\end{aligned}$$

Therefore, if β is nondegenerate so is the restriction of α on $D(\mathfrak{G}, \mathfrak{G}) \times D(\mathfrak{G}, \mathfrak{G})$, whence α itself is nondegenerate.

Conversely, if β is degenerate, then, since β is an invariant form on $\mathfrak{G}$, $\mathfrak{G}^{\perp} = \{X \in \mathfrak{G};\ \beta(X, \mathfrak{G}) = 0\}$ is a nonzero ideal of $\mathfrak{G}$ by Proposition 3, and $B = \mathfrak{G}^{\perp} + D(\mathfrak{G}, \mathfrak{G}^{\perp})$ is a Lie algebra ideal of $\mathfrak{A}$. By (3.4) we get

$$\begin{aligned}\alpha(\mathfrak{B}, \mathfrak{A}) &= \alpha(\mathfrak{G}^{\perp}, \mathfrak{G}) + \alpha(D(\mathfrak{G}, \mathfrak{G}^{\perp}), D(\mathfrak{G}, \mathfrak{G})) \\ &= \beta(\mathfrak{G}^{\perp}, \mathfrak{G}) + \beta(D(\mathfrak{G}, \mathfrak{G}^{\perp})\mathfrak{G}, \mathfrak{G}) = \{0\}\,.\end{aligned}$$

This shows that the Killing form α is degenerate, that is, $\mathfrak{A}$ is not semi-simple. Thus (1) is proved.

By (1.2), $\mathfrak{G}$ is a semi-simple Lie algebra and so its center $\mathfrak{Z} = \{0\}$. Then (1) of Proposition 4 implies $\mathfrak{G} = \mathfrak{G}^{(1)}$. □

THEOREM 2. *Let $\mathfrak{G}$ be the same Lie triple algebra as in Theorem 1. Assume that the Killing-Ricci form β of $\mathfrak{G}$ is nondegenerate. Then $\mathfrak{G}$ is decomposed into a direct sum of simple Lie triple algebra ideals $\mathfrak{G}_i (1 \leqq i \leqq r)$ as follows;*

$$\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 + \cdots + \mathfrak{G}_r;\ \beta = \beta_1 + \beta_2 + \cdots + \beta_r\,,$$

where β_i is the Killing-Ricci form of $\mathfrak{G}_i$ for each i. Moreover, the standard enveloping Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ is a direct sum of the standard enveloping Lie algebras $\mathfrak{A}_i = \mathfrak{G}_i + D(\mathfrak{G}_i, \mathfrak{G}_i)$ of $\mathfrak{G}_i$, each of which is a semi-simple Lie algebra ideal of $\mathfrak{A}$.

Proof. If $\mathfrak{H}$ is an ideal of $\mathfrak{G}$ satisfying $\mathfrak{H}\mathfrak{H} = \{0\}$ and $D(\mathfrak{G}, \mathfrak{H})\mathfrak{H} = \{0\}$, then $\mathfrak{H}^{(1)} = \mathfrak{H}\mathfrak{H} + D(\mathfrak{G}, \mathfrak{H})\mathfrak{H} = \{0\}$ so $\mathfrak{H}$ is solvable. Since β is assumed to be nondegenerate, $\mathfrak{G}$ is semi-simple by (2) in Theorem 1, whence such an ideal $\mathfrak{H}$ must be $\{0\}$. Thus the assumptions of (2) in Proposition 4 are satisfied for the invariant bilinear form β. Therefore, $\mathfrak{G}$ is a direct sum of simple ideals $\mathfrak{G}_i (1 \leqq i \leqq r)$ and $\beta = \beta_1 + \cdots + \beta_r$, $\beta_i = \beta \,|\, \mathfrak{G}_i \times \mathfrak{G}_i$. If $X_i \in \mathfrak{G}_i$, $Y_j \in \mathfrak{G}_j$ and $Z_k \in \mathfrak{G}_k$, then $D(X_i, Y_j)Z_k \in \mathfrak{G}_i \cap \mathfrak{G}_j \cap \mathfrak{G}_k$. Hence $D(X_i, Y_j) = 0$ and $D(X_i, Y_i) \,|\, \mathfrak{G}_j = 0$ for $i \neq j$. Thus we get $D(\mathfrak{G}, \mathfrak{G}) = D(\mathfrak{G}_1, \mathfrak{G}_1) + \cdots + D(\mathfrak{G}_r, \mathfrak{G}_r)$, $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G}) = \mathfrak{A}_1 + \cdots + \mathfrak{A}_r$, where $\mathfrak{A}_i = \mathfrak{G}_i + D(\mathfrak{G}_i, \mathfrak{G}_i)$ are ideals of $\mathfrak{A}$, and $\beta_i = \beta \mid \mathfrak{G}_i \times \mathfrak{G}_i = \alpha \mid \mathfrak{G}_i \times \mathfrak{G}_i = \alpha_i \mid \mathfrak{G}_i \times \mathfrak{G}_i$, where α_i denotes the Killing form of the Lie algebra $\mathfrak{A}_i$ for each i. □

4. Applications for pairs of semi-simple Lie algebras. Let $\mathfrak{K}$ be a semi-simple Lie subalgebra of a semi-simple Lie algebra $\mathfrak{L}$ over a field of characteristic zero, where $\mathfrak{L}$ is assumed to be of finite dimension. A. Sagle [8, 9] has shown that the pair $(\mathfrak{L}, \mathfrak{K})$ is then a reductive pair with a decomposition $\mathfrak{L} = \mathfrak{G} + \mathfrak{K}$, where $\mathfrak{G} = \{X \in \mathfrak{L}; \mathrm{Kill}_{\mathfrak{L}}(X, \mathfrak{K}) = 0\}$ and $\mathrm{Kill}_{\mathfrak{L}}$ denotes the Killing form of $\mathfrak{L}$, and the bilinear form β on $\mathfrak{G}$ given by $\beta(X, Y) = \mathrm{Kill}_{\mathfrak{L}}(X, Y)$ for $X, Y \in \mathfrak{G}$ is nondegenerate. For brevity of discussion assume that $\mathfrak{K}$ contains no nonzero ideal of $\mathfrak{L}$. Then $\mathfrak{K}$ may be identified with a Lie subalgebra of the Lie algebra $\mathrm{End}(\mathfrak{G})$. For $X, Y \in \mathfrak{G}$ let $[X, Y] = XY + D(X, Y)$, where $XY = [X, Y]_{\mathfrak{G}}$ and $D(X, Y) = [X, Y]_{\mathfrak{K}}$ are the projections of $[X, Y]$ into $\mathfrak{G}$ and $\mathfrak{K}$, respectively. Since $(\mathfrak{L}, \mathfrak{K})$ is a reductive pair, $\mathfrak{G}$ is a Lie triple algebra with the operations XY and $D(X, Y)Z = [D(X, Y), Z]$ for $X, Y, Z \in \mathfrak{G}$ ([13]). The standard enveloping Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ of this Lie triple algebra $\mathfrak{G}$ is an ideal of $\mathfrak{L}$ and the inner derivation algebra $D(\mathfrak{G}, \mathfrak{G})$ is an ideal of $\mathfrak{K}$. Hence the Killing form α of $\mathfrak{A}$ is the restriction of the Killing form of $\mathfrak{L}$ to $\mathfrak{A} \times \mathfrak{A}$, and so $\gamma(X, Y, Z) = \alpha(D(X, Y), Z) \in \mathrm{Kill}_{\mathfrak{L}}(\mathfrak{G}, \mathfrak{K}) = \{0\}$ for $X, Y, Z \in \mathfrak{G}$. The nondegenerate bilinear form β is equal to the Killing-Ricci form of the Lie triple algebra $\mathfrak{G}$. Therefore, from Theorem 2 it follows that $\mathfrak{G}$ is decomposed into a direct sum of simple ideals $\mathfrak{G}_i (1 \leqq i \leqq r)$ of $\mathfrak{G}$. Each $\mathfrak{G}_i$ is ad $\mathfrak{K}$-invariant since $\beta(\mathfrak{G}_i, [\mathfrak{K}, \mathfrak{G}_j]) = 0$ for $i \neq j$. In fact, by using Jacobi's identity in $\mathfrak{K}$ and the fact $D(\mathfrak{G}_i, \mathfrak{G}_j) = \{0\}$ for $i \neq j$, we can show $\mathrm{ad}\, \mathfrak{G}_i\, \mathrm{ad}\, [\mathfrak{K}, G_j](\mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})) \subset \mathfrak{G}_i \cap \mathfrak{G}_j + D(\mathfrak{G}, \mathfrak{G}_i) \cap D(\mathfrak{G}, \mathfrak{G}_j) = \{0\}$ for $i \neq j$. If $\mathfrak{G}_i$ is reduced to Lie algebra for some i, then $D(\mathfrak{G}_i, \mathfrak{G}) = \{0\}$ and so $\mathfrak{G}_i$ is a Lie algebra ideal of $\mathfrak{L}$. Let $\mathfrak{K}_0$ be an ideal of $\mathfrak{K}$ such that $\mathfrak{K} = D(\mathfrak{G}, \mathfrak{G}) + \mathfrak{K}_0$. In case $\mathfrak{K}$ contains nonzero ideals of $\mathfrak{L}$, $\mathfrak{L}$ and $\mathfrak{K}$ should be factored by the maximal ideal $\mathfrak{L}_0$ among them.

Summing up the arguments above, we obtain from Theorem 2 the following;

THEOREM 3. *Let $\mathfrak{K}$ be a semi-simple Lie subalgebra of a finite dimensional semi-simple Lie algebra $\mathfrak{L}$ over a field of characteristic zero. Then $\mathfrak{L}$ is decomposed as follows:*

$$\mathfrak{L} = \mathfrak{G}_1 + \cdots + \mathfrak{G}_r + \mathfrak{D}_1 + \cdots + \mathfrak{D}_r + \mathfrak{K}_0 + \mathfrak{L}_0 ,$$

where $\mathfrak{G}_i (1 \leqq i \leqq r)$ are simple Lie triple algebras which are ad $\mathfrak{K}$-*invariant in $\mathfrak{L}$, $\mathfrak{D}_i = D(\mathfrak{G}_i, \mathfrak{G}_i)$ are their inner derivation algebras some of which might be zero that are reduced to Lie algebra ideals* *of* $\mathfrak{L}$, $\mathfrak{L}_0$ *is the maximal ideal of* $\mathfrak{L}$ *contained in* $\mathfrak{K}$, *and* $\mathfrak{K}_0$ *is the complmeentary ideal of the ideal* $\mathfrak{D}_1 + \cdots + \mathfrak{D}_r + \mathfrak{L}_0$ *in* $\mathfrak{K}$, *each of* $\mathfrak{D}_i$ *being an ideal of* $\mathfrak{K}$.

For each i, *the standard enveloping Lie algebra* $\mathfrak{A}_i = \mathfrak{S}_i + \mathfrak{D}_i$ *is a semi-simple ideal of the Lie algebra* $\mathfrak{K}$.

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