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By iterated forcing we create generic Souslin sets, which we use to answer questions of Ulam, Hansell, and Mauldin. For X a topological space a set $Y \subseteq X$ is analytic in X (also called Souslin in X or Σ_1^1 in X) iff there are Borel sets B_s for $s \in \omega^{<\omega}$ such that:

$$Y = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} B_{f \restriction n} .$$

For $X = 2^\omega$ (the Cantor space) a set $Y \subseteq X$ is analytic iff it is the projection of a Borel subset of $2^\omega \times 2^\omega$. Given $R \subseteq P(X)$ (the power set of X) let $B(R)$ be the smallest family of subsets of X including R and closed under countable union and complementation (i.e., the σ-algebra generated by R). If X is a topological space and R the family of open sets then $B(R)$ is the family of Borel subsets of X. The following question was raised by Ulam.

(1) Does there exist $R \subseteq P(2^\omega)$ such that R is countable and every analytic set in 2^ω is an element of $B(R)$?

Rothberger showed that assuming CH there is such a R. We will show that it is consistent with ZFC that there is no such R.

(2) Does there exist a separable metric space X in which every subset is analytic but not every subset is Borel?

This was raised by R. W. Hansell. Clearly CH implies no such X exists. We show that it is consistent with ZFC that such a X exists.

Let $R = \{A \times B\colon A, B \subseteq 2^\omega\}$, the abstract rectangles in the plane. Let $S(R)$ be the family of subsets of $2^\omega \times 2^\omega$ obtained by applying the Souslin operation to sets in $B(R)$. The next question was asked by D. Mauldin.

(3) Does $S(R) = P(2^\omega \times 2^\omega)$ imply $B(R) = P(2^\omega \times 2^\omega)$?

We show that the answer to this question is no.

Preliminaries. Recall the following definitions:

(1) $\omega = \{0, 1, 2, \cdots\}$ and $\forall n < \omega$, $n = \{m \mid m < n\}$;

(2) $\omega^n = \{s \mid s\colon n \to \omega\}$;

(3) for $s \in \omega^m$ and $n < \omega$, $s^\frown n$ is that $t \in \omega^{m+1}$ such that $t \restriction m = s$ and $t(m) = n$;

(4) ϕ denotes the empty sequence;

(5) $\omega^{<\omega} = \bigcup \{\omega^n\colon n < \omega\}$;

(6) $T \subseteq \omega^{<\omega}$ is a tree iff $\forall s, t \in \omega^{<\omega}(s \subseteq t \in T \to s \in T)$;

(7) T is a well founded tree iff $\forall f \in \omega^\omega \exists n < \omega f \restriction n \notin T$;

(8) for $s \in T$ a well founded tree $|s|_T$ is defined inductively by:

$$|s|_T = \sup\{|s^\frown n|_T + 1 : \exists n\, s^\frown n \in T\}\ ;$$

(9) for $\alpha < \omega_1$, T is a normal α-tree iff

(a) T is a well founded tree such that $|\phi|_T = \alpha$;

(b) if $s \in T$ and $|s|_T > 0$, then $\forall n\ s^\frown n \in T$;

(c) if $s \in T$ and $|s|_T = \beta + 1$, then $\forall n\, |s^\frown n|_T = \beta$;

(d) if $s \in T$ and $|s|_T = \lambda$ where λ is a limit ordinal, then $\forall \beta < \lambda$, $\{n: |s^\frown n|_T < \beta\}$ is finite (see [9]);

(10) for $T \subseteq \omega^{<\omega}$ a tree define:
$\boldsymbol{P}(T) = \{p \mid \exists F \in [T]^{<\omega},\ p: F \to 2,\ \forall n < \omega,\ \forall s \in \omega^{<\omega},$ if $s, s^\frown n \in F$, then $p(s) = 1$ implies $p(s^\frown n) = 0\}$, $\boldsymbol{P}(T)$ is ordered by inclusion.

(11) A notion of rank on a partial order $\boldsymbol{P}$ is a function whose domain is a subset of $\boldsymbol{P}$ and whose range is the ordinals. For α an ordinal and $p \in \boldsymbol{P}$, we let $|p| = \alpha$ mean that p is in the domain of this function and its value is α. The following property must be satisfied. For every $p \in \boldsymbol{P}$ and $\beta \geqq 1$, there exists $\hat{p} \in \boldsymbol{P}$ compatible with p such that $|\hat{p}| \leqq \beta$ and for every $q \in \boldsymbol{P}$ if $|q| < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.

(12) Given a notion of rank on $\boldsymbol{P}$ if τ is a term such that $\Vdash$ "$\tau \in 2^\omega$", then we say that $|\tau| = 0$ iff for any $p \in \boldsymbol{P}$ and $n < \omega$ there exists $q \in \boldsymbol{P}$ compatible with p such that $|q| = 0$ and $s \in 2^n$ such that $q \Vdash$ "$\check{s} \subseteq \tau$".

(13) For T a normal α-tree and $p \in \boldsymbol{P}(T)$ define $|p|$ to be the maximum $|s|_T$ for $s \in \mathrm{dom}(p)$.

(14) $T^* = \{s \in T: |s|_T = 0\}$.

The following lemma is key. It implies that $|p|$ is a rank on $\boldsymbol{P}(T)$.

LEMMA 1. *$\forall \beta \geqq 1\ \forall p \in \boldsymbol{P}(T) \exists \hat{p} \in \boldsymbol{P}(T)$ such that*

(a) *p and $\hat{p}$ are compatible;*

(b) *$p \restriction T^* = \hat{p} \restriction T^*$;*

(c) *$|\hat{p}| \leqq \beta$;*

(d) *$\forall q \in \boldsymbol{P}(T)$ if $|q| < \beta$, then $\hat{p}$ and q are compatible implies p and q are compatible.*

Proof. This is essentially Lemma 2 of [10]. We reprove it here for completeness. Let $F = \{s^\frown n: s \in \mathrm{dom}(p),\ p(s) = 1,\ |s|_T = \lambda$ a limit ordinal $> \beta$, and $|s^\frown n|_T < \beta\}$. By normality of T, F is finite, and $\forall t \in F$, $|t|_T \geqq 2$. Thus we can find $r \geqq p\ \forall t \in F \exists m\ t^\frown m \in \mathrm{dom}(r)$ and $r(t^\frown m) = 1$. Let $D = \{s \in \mathrm{dom}(r): |s|_T \leqq \beta\}$ and $\hat{p} = r \restriction D$. p and $\hat{p}$ are compatible since r extends them both. $p \restriction T^* = \hat{p} \restriction T^*$ since $\forall t \in F \forall m\, |t^\frown m|_T \geqq 1$.

Now we check (d). Suppose $|q| < \beta$ and p and q are not compatible. Then there are $s \in \mathrm{dom}(p)$ and $t \in \mathrm{dom}(q)$ which demonstrate that $p \cup q$ is not a condition.

Case 1. $s = t$ and $p(s) \neq q(t)$. Since $|q| < \beta$ it follows $|t|_T < \beta$ and so $s \in \mathrm{dom}(\hat{p})$.

Case 2. $s = t^\frown m$ for some m and $p(s) = q(t) = 1$. But then $|s|_T < |t|_T < \beta$ and so again $s \in \mathrm{dom}(\hat{p})$.

Case 3. $t = s^\frown m$ for some m and $p(s) = q(t) = 1$. Since $|t|_T < \beta$ either $|s|_T \leqq \beta$ and so $s \in \mathrm{dom}(\hat{p})$ or $|s|_T = \lambda$ a limit ordinal $> \beta$ in which case $t \in F$ so there exists $n < \omega$ such that $r(t^\frown n) = 1$ and so $t^\frown n \in \mathrm{dom}(\hat{p})$ and so $\hat{p}$ and q are incompatible. In all three cases $\hat{p}$ and q are incompatible. $\square$

The next lemma asserts the fact that statements of small rank should be forced by conditions of small rank. M is the ground model of ZFC and $\boldsymbol{P}$ is any partial order with a notion of rank.

LEMMA 2. *Let $B(r)$ be any $\boldsymbol{\Sigma}^0_\beta$ predicate with parameter in M, $1 \leqq \beta$, $\Vdash_P "\tau \in 2^\omega"$, $|\tau| = 0$, and $p \in \boldsymbol{P}$ such that $p \Vdash "B(\tau)"$. Then $\exists \hat{p} \in \boldsymbol{P}$, $|\hat{p}| < \beta$, p and $\hat{p}$ are compatible and $\hat{p} \Vdash "B(\tau)"$.*

Proof. The proof is by induction β.

Case 1. $\beta = 1$. Then $p \Vdash "\exists n R(\tau \restriction n, x \restriction n)"$ where R is primitive recursive and $x \in M \cap 2^\omega$. Find q extending p and $s \in 2^n$ for some n such that $q \Vdash "\tau \restriction n = \check{s}"$ and $R(s, x \restriction n)$ holds. By the definition of $|\tau| = 0$, $\exists \hat{p}$ compatible with q (and hence with p) such that $|\hat{p}| = 0$ and $\hat{p} \Vdash "\tau \restriction n = \check{s}"$. Thus $\hat{p} \Vdash "\exists n R(\tau \restriction n, x \restriction n)"$.

Case 2. β a limit ordinal. Then $p \Vdash "\exists n B_n(\tau)"$ where each $B_n(r)$ is a $\boldsymbol{\Sigma}^0_{\beta_n}$ predicate for some $\beta_n < \beta$. Let p_0 extend p such that $\exists n_0 < \omega$ $p_0 \Vdash "B_{n_0}(\tau)"$. By induction $\exists \hat{p}$ compatible with p_0 (and hence with p) such that $|\hat{p}| < \beta_{n_0} < \beta$ and $\hat{p} \Vdash "B_{n_0}(\tau)"$ (and hence $\hat{p} \Vdash "\exists n B_n(\tau)"$).

Case 3. $\beta = \gamma + 1$ and $\gamma > 0$. As in Case 2 we may as well assume $p \Vdash "B(\tau)"$ where $B(r)$ is a $\boldsymbol{\Pi}^0_\gamma$ predicate. By Lemma 1, $\exists \hat{p} \in \boldsymbol{P}$, $\hat{p}$ and p compatible, $|\hat{p}| \leqq \gamma$, and $\forall q \in \boldsymbol{P}$ if $|q| < \gamma$ and q and $\hat{p}$ are compatible, then q and p are compatible. Then $\hat{p} \Vdash "B(\tau)"$. Otherwise $\exists r$ extending $\hat{p}$, $r \Vdash "\neg B(\tau)"$. Since $\neg B(r)$ is a $\boldsymbol{\Sigma}^0_\gamma$ predicate, by induction $\exists \hat{r} \in \boldsymbol{P}$, $|\hat{r}| < \gamma$, $\hat{r}$ and r compatible, and $\hat{r} \Vdash "\neg B(\tau)"$. But $\hat{r}$ and p are incompatible (since $p \Vdash "B(\tau)"$) and so by choice of

$\hat{p}$, $\hat{r}$ and $\hat{p}$ are incompatible a contradiction. $\square$

Next we describe almost disjoint forcing (similar to the way it is done in [2]). Given $X = \{x_\alpha : \alpha < \omega_1\} \subseteq 2^\omega$ distinct and $\langle Y_\alpha : \alpha < \omega_1 \rangle = Y$ where each $Y_\alpha \subseteq \omega^{<\omega}$, we want to force a sequence of G_δ sets $\langle G_s : s \in \omega^{<\omega} \rangle$ such that $\forall s \forall \alpha (x_\alpha \in G_s \leftrightarrow s \in Y_\alpha)$. Let $\boldsymbol{B}$ be the family of all clopen subsets of 2^ω. Define $\boldsymbol{P}(X, Y)$ as follows:

it is the set of all r such that

(a) r is a finite subset of $\omega^{<\omega} \times \omega \times (\boldsymbol{B} \cup X)$;

(b) if $\langle s, n, B \rangle, \langle s, n, x_\alpha \rangle \in r$ then $x_\alpha \notin B$;

(c) if $\langle s, n, x_\alpha \rangle \in r$ then $s \notin Y_\alpha$.

As usual r extends p, $(r \geqq p)$ iff $r \supseteq p$. It is well known that $\boldsymbol{P}(X, Y)$ satisfies the c.c.c. and also for any G which is $\boldsymbol{P}(X, Y)$-generic if we define $G_s = \bigcap_n \bigcup \{B : \{\langle s, n, B \rangle\} \in G\}$ then $\forall s \forall \alpha (x_\alpha \in G_s \leftrightarrow s \in Y_\alpha)$.

1. Forcing a Souslin set. We now describe how to force Souslin sets. Let M be our ground model of ZFC. Working in M let F^* be some standard fixed bijection between $\omega^{<\omega}$ and ω, and define $F: 2^\omega \to 2^{(\omega^{<\omega})}$ by $F(x)(s) = x(F^*(s))$. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be a fixed subset of 2^ω such that for all $\alpha < \omega_1$, $F(x_\alpha)$ is the characteristic function of a normal α-tree T_α. Let

$$\boldsymbol{P}_0 = \sum_{\alpha < \omega_1} \boldsymbol{P}(T_\alpha) ,$$

note that $\boldsymbol{P}_0$ has c.c.c. since it is equivalent to adding ω_1 Cohen reals. Note that any G which is $\boldsymbol{P}(T_\alpha)$-generic over M determines (and is determined by) a map $G_\alpha : T_\alpha \to 2$. $G_\alpha \upharpoonright T_\alpha^*$ in fact determines G_α by the rule $G_\alpha(s) = 1$ iff $\forall n\, G_\alpha(s^\frown n) = 0$. Given G^0 $\boldsymbol{P}_0$-generic over the ground model M, let $G^0 = \langle G_\alpha : \alpha < \omega_1 \rangle$ and let $y_\alpha = \{s \in T_\alpha^* : G_\alpha(s) = 0\}$. Let $\boldsymbol{P}_1 = \boldsymbol{P}(X, Y)$ where $Y = \langle y_\alpha : \alpha < \omega_1 \rangle$. (So $\boldsymbol{P}_1 \in M[G^0]$.) Let $\boldsymbol{P} = \boldsymbol{P}_0 {}^* \boldsymbol{P}_1$.

Working in $M[G]$ for G $\boldsymbol{P}$-generic over M (so $G = (\langle G_\alpha : \alpha < \omega_1 \rangle, \langle G^s : s \in \omega^{<\omega} \rangle)$) let:

$$A = \{x_\alpha \in X : G_\alpha(\phi) = 1\} .$$

To see that A is analytic in X we will define $\hat{A}$ a $\boldsymbol{\Sigma}_1^1$ set such that $\hat{A} \cap X = A$. Define $x \in \hat{A}$ iff $\exists T \subseteq \omega^{<\omega}$, $\exists p: \omega^{<\omega} \to 2$, $\exists T^* \subseteq \omega^{<\omega}$ such that

(a) $F(x)$ is the characteristic function of T;

(b) T is a tree;

(c) $T^* = \{s \in T : \exists n\ s^\frown n \notin T\} = \{s \in T : \forall n\ s^\frown n \notin T\}$;

(d) $\forall s \in T^*\ p(s) = 1$ iff $x \in G_s$;

(e) $\forall s \in T - T^*\ p(s) = 1$ iff $\forall n\ p(s^\frown n) = 0$;

(f) $p(\phi) = 1$.

(a) thru (f) are easily seen to be a Borel predicate of x, T, T^*, and p, and hence $\hat{A}$ is $\boldsymbol{\Sigma}_1^1$.

In order to show A is a new Souslin set we first want to extend our notion of rank to $\boldsymbol{P}$. Let $\boldsymbol{Q} = \{r \mid r$ satisfies (a) and (b) in the definition of $\boldsymbol{P}(X, Y)\}$ (thus $\boldsymbol{Q} \in M$). Then

$$\{(p, q)\colon p \in \boldsymbol{P}_0,\ q \in \boldsymbol{Q}\ , \quad \text{and} \quad p \Vdash {}''\check{q} \in \boldsymbol{P}(X, Y)''\}$$

ordered by $(\hat{p}, \hat{q}) \geqq (p, q)$ iff $\hat{p} \geqq p$ and $\hat{q} \geqq q$, is clearly dense in $\boldsymbol{P}$, so for simplicity assume it is $\boldsymbol{P}$. Let us unravel $p \Vdash {}''\check{q} \in \boldsymbol{P}(X, Y)''$. This means that whenever $\langle s, n, x_\alpha\rangle \in q'$ then $p \Vdash {}''s \notin Y_\alpha''$. But $p \Vdash {}''s \notin Y_\alpha''$ iff $s \notin T_\alpha^*$ or ($s \in T_\alpha^*$, $s \in \operatorname{dom}(p_\alpha)$, and $p_\alpha(s) = 1$). The fact which we note is that if $p, p' \in \boldsymbol{P}_0$ and $\forall \alpha < \omega_1\ p_\alpha \upharpoonright T_\alpha^* = p'_\alpha \upharpoonright T_\alpha^*$, then $\forall r \in \boldsymbol{Q}\ \langle p, r\rangle \in \boldsymbol{P}$ iff $\langle p', r\rangle \in \boldsymbol{P}$.

For any $\alpha < \omega_1$, we define the following rank function on $\boldsymbol{P}$:

$$|(p, q)|_\alpha = \max\{|s|_{T_\gamma}\colon \gamma > \alpha \quad \text{and} \quad s \in \operatorname{dom}(p_\gamma)\}\ .$$

Note that the rank depends only on the part of the condition in $\boldsymbol{P}_0$. To see that it is a rank function, let (p, q) be any condition and $\beta \geqq 1$. For each $\gamma > \alpha$ by Lemma 1 $\exists \hat{p}_\gamma \in \boldsymbol{P}(T_\gamma)$ such that $\hat{p}_\gamma \upharpoonright T_\gamma^* = p \upharpoonright T_\gamma^*$, $\hat{p}_\gamma$ and p_γ are compatible, $|\hat{p}_\gamma| \leqq \beta$, and $\forall q \in \boldsymbol{P}(T_\gamma)$ if $|q| < \beta$ and $\hat{p}_\gamma$ and q are compatible, then p_γ and q are compatible. Let $\hat{p} \in \boldsymbol{P}_0$ be defined by:

$$\hat{p}_\gamma = \begin{cases} p_\gamma & \text{if} \quad \gamma \leqq \alpha \\ \hat{p}_\gamma & \text{if} \quad \gamma > \alpha\ . \end{cases}$$

By what we have already remarked

$$(\hat{p}, q) \in \boldsymbol{P},\ |(\hat{p}, q)|_\alpha \leqq \beta,\ (p, q) \text{ and } (\hat{p}, q) \text{ are compatible}\ ,$$
$$\forall (p', q') \in \boldsymbol{P} \text{ if } |(p', q')|_\alpha < \beta \text{ and}$$

(p', q') is compatible with $(\hat{p}, q)$, then (p', q') is compatible with (p, q).

Let G be $\boldsymbol{P}$-generic over M, and let A be the generic Souslin subset of X determined by G. We first show that $M[G] \models {}''A$ is not Borel in X''. Suppose on the contrary that $\exists \tau$, $wB(v, \omega) a \boldsymbol{\Sigma}_\beta^0$ predicate with parameters in M, and $r \in \boldsymbol{P}$ such that

$$r \Vdash {}''\forall x \in X(x \in A \text{ iff } B(\tau, x))''\ .$$

By c.c.c. we can find $\alpha < \omega_1$ such that $|\tau|_\alpha = 0$, $|r|_\alpha = 0$, and $\beta < \alpha$. Let γ be any countable ordinal greater than $\alpha + \omega$. Extend $r = (p, q)$ by adding $p_\gamma(\phi) = 1$ to p, and call the result r_1. By this addition, $r_1 \Vdash {}''x \in A''$, so $r_1 \Vdash {}''B(\tau, x_\gamma)''$, so there exists r_2 compatible with r_1 such that $|r_2|_\alpha < \beta$ and $r_2 \Vdash {}''B(\tau, x_\gamma)''$. But since $\gamma > \alpha + \omega$ and $|r_2|_\alpha < \beta < \alpha$, it follows that $\exists r_3 \geqq r_2$ such that $p_\gamma^3(\phi) = 0$ and thus $\Vdash$ "$x_\gamma \notin A$". This is a contradiction since r_3 and r_1 are compatible (since r_2 and r_1 are compatible).

Now let us prove something a little stronger. Let $M \models$ "$H \subseteq P(X)$, $|H| \leqq \omega$", then, we claim $M[G] \models$ "$A \notin B(H)$ (the σ-algebra generated by H)".

Work in M. Let $H = \{A_n: n < \omega\}$ and define $K: X \to 2^\omega$ by $K(x)(n) = 1$ iff $x \in A_n$. Let Y be the range of K, then K has the property that it maps the σ-algebra generated by H into the Borel subsets of Y.

For any $C \in B(H)^{M[G]} \exists B$ Borel subset of Y, and $p \in \boldsymbol{P}$ such that

$$p \Vdash \text{"}\forall x \in X(x \in C \quad \text{iff} \quad K(x) \in B)\text{"} .$$

The preceding proof now goes through. Finally we are ready to state the theorem.

THEOREM 3. *It is consistent with* ZFC *that there does not exist* $H \subseteq P(2^\omega)$ *countable such that every analytic set is in the* σ*-algebra generated by* H.

Proof. Let M, X, and $\boldsymbol{P}$ be as above. Working in M let $\{\boldsymbol{P}_\alpha: \alpha < \omega_2^M\}$ be a set of isomorphic copies of $\boldsymbol{P}$. Force with $\Sigma\{\boldsymbol{P}_\alpha: \alpha < \omega_2^M\}$. Let $\langle G_\alpha. \alpha < \omega_2^M\rangle$ be generic over M. If $M[G_\alpha: \alpha < \omega_2^M] \models$ "$H \subseteq P(2^\omega)$, $|H| \leqq \omega$" then by c.c.c. $\exists\alpha_0 < \omega_2^M$ such that $\{B \cap X: B \in H\} \in M[G_\alpha: \alpha \neq \alpha_0]$. Let $M[G_\alpha: \alpha \neq \alpha_0]$ be the new ground model and $\hat{A}$ the analytic set created by $\boldsymbol{P}_{\alpha_0}$. Note that although $\boldsymbol{P}_{\alpha_0}$ is not the same as adding Cohen reals, because of its finite nature it is the same partial order whether defined in M or any extension of M (e.g., $M[G_\alpha: \alpha \neq \alpha_0]$). We have already noted that $\hat{A} \cap X$ is not in the σ-algebra generated by $\{B \cap X: B \in H\}$ and therefore $\hat{A}$ is not in the σ-algebra generated by H. $\square$

2. Making subsets generic Souslin sets. Let Σ be the set of countable successor ordinals greater than two. As in §1 let $X^* = \{x_\alpha: \alpha \in \Sigma\} \subseteq 2^\omega$ and $F: 2^\omega \to 2^{(\omega^{<\omega})}$ be the map such that $\forall\alpha \in \Sigma$, $F(x_\alpha)$ is a normal α-tree T_α. For $i = 0$ or 1 and $T \subseteq \omega^{<\omega}$ define:

$$\boldsymbol{P}^i(T) = \{p \in \boldsymbol{P}(T): \exists\hat{p} \text{ an extension of } p,\ \hat{p}(\phi) = i\} .$$

It is easy to check that for any G which is $\boldsymbol{P}^i(T)$-generic over M, $G(\phi) = i$. Given $Z \subseteq \Sigma$ define $\boldsymbol{P}(Z)$ a suborder of $\boldsymbol{P}$ by $(p, q) \in \boldsymbol{P}(Z)$ iff $(p, q) \in \boldsymbol{P}$ and $\forall\alpha \in \Sigma$

(a) if $\alpha \in Z$ then $p_\alpha \in \boldsymbol{P}^0(T_\alpha)$;

(b) if $\alpha \notin Z$ then $p_\alpha \in \boldsymbol{P}^1(T_\alpha)$.

As before for G $\boldsymbol{P}(Z)$-generic over M, in $M[G]$, $\{x_\alpha: \alpha \in Z\}$ is analytic in X^*. The reason for Σ will be evident in the proof of Lemma 5.

THEOREM 4. *There exist a generic extension N of M such that $N \models$ "Every subset of X^* is analytic in X^* but some subset of X^* is not Borel in X^*".*

Proof. N will be obtained by iterating with finite support $\boldsymbol{P}(Z)$. Since each $\boldsymbol{P}(Z)$ is a relatively simple suborder of $\boldsymbol{P}$ we can give the following simpler definition. We assume $M \models "2^{\omega_1} = \omega_2"$. Let $\boldsymbol{Q} = \sum_{\alpha<\omega_2} \boldsymbol{P}_\alpha$ as in §1 and for $p \in \boldsymbol{Q}$ define $\operatorname{supp}(p) = \{\alpha < \omega_2 \colon p(\alpha) \neq 0\}$. Let A_α for $\alpha < \omega_2$ list with ω_2 repetitions all maps $A \colon \omega_1 \to [\boldsymbol{Q}]^{\leqq\omega}$. Inductively define $\boldsymbol{Q}_\alpha \subseteqq \boldsymbol{Q}$ for $\alpha < \omega_2$. For $\alpha = 0$ let $\boldsymbol{Q}_\alpha = \{p \in \boldsymbol{Q} \colon \operatorname{supp}(p) = \{0\}\}$ (i.e., $\boldsymbol{Q}_0 = \boldsymbol{P}$). For all $\alpha \boldsymbol{Q}_\alpha \subseteqq \{p \in \boldsymbol{Q} \colon \operatorname{supp}(p) \subseteqq \alpha\}$. For α a limit ordinal let $\boldsymbol{Q}_\alpha = \cup\{\boldsymbol{Q}_\beta \colon \beta < \alpha\}$. For $\alpha + 1$ let G_α be $\boldsymbol{Q}_\alpha$-generic over M and let $Z_\alpha = \{\beta \in \Sigma \colon A_\alpha(\beta) \cap G_\alpha \neq \phi\}$. Then

$$\boldsymbol{Q}_{\alpha+1} = \{p \in \boldsymbol{Q} \mid p \restriction \alpha \in \boldsymbol{Q}_\alpha,\ p \restriction \alpha \Vdash_{\boldsymbol{Q}_\alpha} "p(\alpha) \in \boldsymbol{P}(Z_\alpha)",\\ \text{and} \quad \operatorname{supp}(p) \subseteqq \alpha + 1\}\,.$$

(Of course by $p \restriction \alpha$ here we mean that condition in $\boldsymbol{Q}$ whose restriction to α is the same as p's and whose support is contained in α.)

Thus if G_{ω_2} is $\boldsymbol{Q}_{\omega_2}$ generic over M then $M[G_{\omega_2}] \models$ "Every subset of X^* is analytic in X^*". Work in M. Given $\alpha < \omega_1$ recall the definition $|p|_\alpha$ for $p \in \boldsymbol{P}$ given in §1. Given $K \subseteqq \omega_2$ and $\alpha < \omega_1$ define a map $F \colon \boldsymbol{Q}_{\omega_2} \to \alpha \cup \{\infty\}$ by $F(p) = \max\{|p(\delta)|_\alpha \colon \delta \in K\}$ if $\operatorname{supp}(p) \subseteqq K$ and the max is less than α, and otherwise let $F(p) = \infty$. Denote $F(p)$ by $|p|(K, \alpha)$. For suitably chosen K and α we will show $|p|(K, \alpha)$ is a rank function. Given $\Gamma \subseteqq \boldsymbol{Q}_{\omega_2}$ and θ a sentence we say Γ decides θ iff $\forall p \in \boldsymbol{Q}_{\omega_2} \exists q \in \Gamma$ p and q are compatible, and $q \Vdash "\theta"$ or $q \Vdash "\neg\theta"$.

LEMMA 5. *Suppose that $\forall \delta \in K \forall \beta < \alpha \{p \in \boldsymbol{Q}_\delta \colon |p|(K, \alpha) = 0\}$ decides $"\beta \in Z_\delta"$. Then $|p|(K, \alpha)$ is a rank function.*

Proof. We must show that given $p \in \boldsymbol{Q}_{\omega_2}$ and $1 \leqq \beta \leqq \alpha$ there exists $\hat{p} \in \boldsymbol{Q}_{\omega_2}$ compatible with p, $|\hat{p}|(K, \alpha) \leqq \beta$, and $\forall q \in \boldsymbol{Q}_{\omega_2}$ if $|q|(K, \alpha) < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.

Recall that in the proof that $|\ \ |_\alpha$ is a rank function on $\boldsymbol{P}$ we obtained for each $p \in \boldsymbol{P}$ a $\hat{p} \in \boldsymbol{P}$ such that:

(a) $|\hat{p}|_\alpha \leqq \beta$;

(b) $\hat{p}$ and p are compatible;

(c) $\forall q \in \boldsymbol{P}$ if $|q|_\alpha < \beta$ and q and $\hat{p}$ are compatible, then q and p are compatible;

(d) $\forall\gamma < \alpha$, $\hat{p}(\gamma) = p(\gamma)$.

Given $p \in \boldsymbol{Q}_{\omega_2}$ define $\hat{p}$ by letting $\forall\delta \notin K$, $\hat{p}(\delta) = 0$ and $\forall\delta \in K$, $\hat{p}(\delta)$ is the condition in $\boldsymbol{P}$ obtained above for $p(\delta)$. We show that $\hat{p} \in \boldsymbol{Q}_{\omega_2}$. Suppose not and let δ be the least such that $\hat{p} \upharpoonright \delta$ does not force "$\hat{p}(\delta) \in \boldsymbol{P}(Z_\delta)$". Clearly $\delta \in K$. Let $\hat{p}(\delta) = (p', q)$. Then there must be some $\gamma \in \Sigma$ such that $p'_\gamma \notin \boldsymbol{P}^0(T_\gamma)$ or $p'_\gamma \notin \boldsymbol{P}^1(T_\gamma)$, and $\hat{p} \upharpoonright \delta$ does not force "$\gamma \notin Z''_\delta$" respectively "$\gamma \in Z''_\delta$". If $p'_\gamma \notin \boldsymbol{P}^0(T_\gamma)$ then $\phi \in \mathrm{dom}(p'_\gamma)$ and $p'_\gamma(\phi) = 1$. If $p'_\gamma \notin \boldsymbol{P}^1(T_\gamma)$ then either $\phi \in \mathrm{dom}(p'_\gamma)$ and $p'_\gamma(\phi) = 0$ or $\exists n < \omega$, $\langle n\rangle \in \mathrm{dom}(p'_\gamma)$ and $p'_\gamma(\langle n\rangle) = 1$. Since $\gamma \in \Sigma$ it is a successor ordinal. Since $|p(\delta)|_\alpha \leqq \beta < \alpha$ and $|\langle n\rangle|_{T_\gamma} \geqq \gamma - 1$ it must be that $\gamma < \alpha$. By the properties of K and α, $\exists q \in \boldsymbol{Q}_\delta$, $|q|(K, \alpha) = 0$, $q \Vdash$ "$\gamma \notin Z''_\delta$" (respectively "$\gamma \in Z''_\delta$"), and q is compatible with $\hat{p} \upharpoonright \delta$. But since q is compatible with $\hat{p} \upharpoonright \delta$, it is compatible with $p \upharpoonright \delta$. This is a contradiction, since by (d) $q \Vdash$ "$\hat{p}(\delta) \notin \boldsymbol{P}(Z_\delta)$". $\square$

If A is the analytic subset of X^* which is created at the first step, then A is not Borel in X^* in the model $M[G_{\omega_2}]$. To see this suppose not and $\exists p \in \boldsymbol{Q}_{\omega_2}$

$$p \Vdash \text{"}\forall x \in X^*(x \in A \text{ iff } x \in B_\tau)\text{"}$$

where B_τ is a $\boldsymbol{\Sigma}^0_\beta$ set with parameter $\tau \in 2^\omega$. Using the c.c.c. of $\boldsymbol{Q}_{\omega_2}$ it is easy to obtain $K \subseteqq \omega_2$ countable, $0 \in K$, and $\alpha < \omega_1$ with $\beta < \alpha$, such that $|p|(K, \alpha) = 0$, $|\tau|(K, \alpha) = 0$, and K and α satisfy the requirements set down in Lemma 5. As in §1 this leads to a contradiction.

3. Abstract Souslin sets.

Recall that $R = \{A \times B: A, B \subseteqq 2^\omega\}$, $B(R)$ is the σ-algebra generated by R, and $S(R)$ the family of sets which are gotten by applying the Souslin operation to sets in $B(R)$.

THEOREM 6. *It is consistent with* ZFC *that* $S(R) = P(2^\omega \times 2^\omega) \neq B(R)$.

The model used will be a minor modification of the one obtained in §2.

LEMMA 7. *Suppose* $X \subseteqq 2^\omega$, $|X| = |2^\omega|$, *and every subset of* X *of cardinality less than* $|2^\omega|$ *is analytic in* X. *Then* $S(R) = P(2^\omega \times 2^\omega)$.

Proof. Let $\kappa = |2^\omega|$ and $X = \{x_\alpha: \alpha < \kappa\}$. Since $S(R)$ is closed under finite union, it is enough to show that any $Y \subseteqq \kappa^2$ with the property that $\langle\alpha, \beta\rangle \in Y \to \alpha \leqq \beta$, is in $S(R)$. For each β let $X_\beta = \{x_\alpha: \langle\alpha, \beta\rangle \in Y\}$. For each β and $s \in \omega^{<\omega}$ let C^β_s be a closed subset of X such that $X_\beta = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} C^\beta_{f\upharpoonright n}$.

For each $s \in \omega^{<\omega}$ define $B_s = \{\langle\alpha, \beta\rangle: x_\alpha \in C_s^\beta\}$. Since $Y = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} B_{f\restriction n}$ it is enough to check that each $B_s \in B(R)$. Fix $s \in \omega^{<\omega}$ and let $\{D_n: n < \omega\}$ be an open basis for X. For each β define $y_\beta(n) = 1$ iff $D_n \cap C_s^\beta = \phi$. It follows that $\alpha \in C_s^\beta$ iff $\forall n$ (if $y_\beta(n) = 1$ then $\alpha \notin D_n$). Letting $E_n = (D_n \times X) \cup (D_n \times \{\beta: y_\beta(n) = 0\})$ we have that $B_s = \bigcap_{n<\omega} E_n$. $\square$

LEMMA 8. *Suppose $F: X \to Y$ is $1 - 1$ and $\forall U$ open in Y $F^{-1}(U)$ is Borel in X. If every subset of Y is analytic in Y then every subset of X is analytic in X.*

Proof. Given $A \subseteq X$ let $B = F''A$. Then there are Borel subsets of Y, B_s for $s \in \omega^{<\omega}$ such that $B = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} B_{f\restriction n}$. Let $A_s = F^{-1}(B_s)$, then A_s is Borel in X and $A = \bigcup_{f\in\omega^\omega} \bigcap_{n<\omega} A_{f\restriction n}$. $\square$

We now prove Theorem 2. Let M, the ground model of ZFC in §2, be a model of $MA + 2^\omega = \omega_2$. We first show that for G_{ω_2} $\boldsymbol{Q}_{\omega_2}$-generic over M, $M[G_{\omega_2}]$ models that $S(R) = P(2^\omega \times 2^\omega)$. Working in M for any $Z, W \subseteq 2^\omega$ with $|Z| = |W| = \omega_1$, if $F: Z \to W$ is any $1 - 1$ map then by Silver's lemma (see [6]) for every U open in W, $F^{-1}(U)$ is Borel in Z. F still has this property in any extension of M since W is second countable and M contains an open basis for W. Working in M there exists $X \subseteq 2^\omega$ such that $|X| = \omega_2$ and $\forall Y \subseteq X$ if $|Y| \leqq \omega_1$ then Y is Borel in X (a generalized Luzin set is such an example, see [9]). We claim that in $M[G_{\omega_2}]$ every subset of X of size $\leqq \omega_1$ is analytic in X and thus by Lemma 7, $S(R) = P(2^\omega \times 2^\omega)$. Working in $M[G_{\omega_2}]$ for any $Z \subseteq X$ if $|Z| \leqq \omega_1$ then $\exists Y \in M Z \subseteq Y$ and $|Y| \leqq \omega_1$. Letting $F: Y \to X^*$ be any $1 - 1$ map in M we have by Lemma 8 that every subset of Y is analytic in Y, and since Y is Borel in X, Z is analytic in X.

We next want to show that in $M[G_{\omega_2}]$, $P(2^\omega \times 2^\omega) \neq B(R)$. It is enough to show that in $M[G_{\omega_2}]$ there does not exist a countable $H \subseteq P(X^*)$ such that $B(H) = P(X^*)$. To see that this suffices let $\{X_\alpha: \alpha < \omega_2\} = P(X^*)$ and let $Y = \{(x, \alpha): x \in X_\alpha\} \subseteq X^* \times \omega_2$. If Y is in the σ-algebra generated by $\{A_n \times B_n: n < \omega\}$ then $B(\{A_n: n<\omega\}) = P(X^*)$. Just show by induction that $\forall K \in B(\{A_n \times B_n: n < \omega\}) \forall \beta < \omega_2$ $\{x \in X^*: (x, \beta) \in K\} \in B(\{A_n: n < \omega\})$.

By the technique of §1 and §2 we note that in M there is no countable $H \subseteq P(X^*)$ such that the generic Souslin set created at the first step is in $B(H)$. Note that for $Z = \phi$ and G $\boldsymbol{P}(Z)$-generic over M the set $A = \{x_\alpha \in X^*: G_\alpha(\langle 0\rangle) = 1\}$ is also a generic Souslin set over M. This is because the requirement that $G_\alpha(\phi) = 0$ puts no constraint on the value of $G_\alpha(\langle 0\rangle)$. $\square$

4. Remarks. (1) In the model used for Theorem 1 one can show that there does not exist any $H \subseteq P(2^\omega)$, $|H| < |2^\omega|$, such that every analytic subset of 2^ω is in $B(H)$. Note also that ω_2 can be replaced by any $\kappa > \omega_1$ of uncountable cofinality. Also in this model it is true that the universal Σ_1^1 subset of $2^\omega \times 2^\omega$ is not in the σ-algebra generated by the abstact rectangles.

(2) It is not hard to modify the technique of §2 to get it consistent with ZFC that $\exists X \subseteq 2^\omega |X| = \omega_2$ (or even $|X| = \aleph_{\omega_1}$) such that every subset of X is analytic in X but not every subset of X is Borel in X.

(3) X^* in §2 has Baire order ω_1 in $M[G_{\omega_2}]$.

(4) In [5] Kunen showed that if one adds ω_2 Cohen reals to a model of CH then $\{(\alpha, \beta): \alpha < \beta < \omega_2\}$ is not in the σ-algebra generated by $\{A \times B: A \subseteq \omega_2, B \subseteq \omega_2\}$. In the same model (actually CH is not necessary in ground model) there is a subset of $\omega_1 \times \omega_2$ not in the σ-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$. To prove this it is enough to find $F \subseteq P(\omega_1) |F| = \omega_2$ such that there does not exist $H \subseteq P(\omega_1)$ countable with $F \subseteq B(H)$. Let $\boldsymbol{P} = \{p \mid p: F \to 2$, for some $F \in [\omega_1]^{<\omega}\}$ and suppose G is $\boldsymbol{P}$-generic over M. Let

$$X = \{\alpha < \omega_1 \mid G(\alpha) = 1\}$$

and note that for any $H \subseteq P(\omega_1)$ countable and in M, $M[G] \Vdash "X \notin B(H)"$. This is because for any $Y \in B(H)$ $\exists t \in 2^\omega$ $Y \in M[t]$.

(5) In [12] Rothberger showed that $2^\omega = \omega_2 + 2^{\omega_1} = \aleph_{\omega_2}$ implies that not every subset of $\omega_1 \times \omega_2$ is in the σ-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$. To see this let G_α for $\alpha < \aleph_{\omega_2}$ list all countable subsets of $P(\omega_1)$. Since $|B(G_\alpha)| \leqq 2^\omega = \omega_2$ we can pick $K_\alpha \in P(\omega_1)$ for $\alpha < \omega_2$ such that $K_\alpha \notin \bigcup_{\beta < \omega_\alpha} B(G_\beta)$. It follows as in (4) that $\{(\beta, \alpha): \beta \in K_\alpha\}$ is not in the σ-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$.

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