ON MEASURABLE PROJECTIONS IN BANACH SPACES

Elias Saab

ON MEASURABLE PROJECTIONS IN BANACH SPACES

Elias Saab

Let E be a Banach space that is complemented in its bidual by a projection $P: E^{**} \to E$. It is shown that E has the Radon Nikodym property if and only if for every Radon probability measure λ on the unit ball K of E^{**} such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every weak* Borel subset A of K, the projection P is λ-Lusin measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula for λ on K.

J. J. Uhl Jr. asked the following question: Let E be a Banach space which is complemented in its bidual by a projection $P: E^{**} \to E$ which is weak* to norm universally Lusin measurable. Does E have the Radom-Nikodym property?

In [4] we showed that if E is the dual of a Banach space Y and if P is the natural projection from $E^{**} = Y^{***}$ to $Y^* = E$ then the above condition is necessary and sufficient for E to have the Radon-Nikodym property.

In [4] we also showed that for any Banach space E, if P is weak* to weak Baire-1 function then E has the Radon-Nikodym property.

Recently G. Edgar showed using an idea of Talagrand and Weizsäcker that the projection

$$L_1[0, 1]^{**} \longrightarrow L_1[0, 1]$$

is weak* to weak universally-Lusin measurable. This shows that Uhl's question does not have a positive answer in general, however if one examines the results of [4] he can see that if P is Baire-1, it is universally Lusin-measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula. It turns out that a Banach space E has the Radon-Nikodym property if and only if for every Radon probability measure λ on the unit ball K of E^{**} such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every ω^*-Borel subset A of K the projection P is λ-Lusin measurable and for every x^* in E^* the map x^*P satisfies the barycentric formula for λ on K.

Let us fix some terminology and conventions. All topological spaces in this paper will be completely regular. The set of all Radon probability measures on a topological space (X, τ) will be denoted by $M_+^1(X, \tau)$.

Definition 1. Let (X, τ_1) and (Y, τ_2) be two topological spaces and let

$$f: X \longrightarrow Y \quad \text{and} \quad \mu \in M_1(X, \tau_1)$$

the map f is said μ-Lusin measurable if for every compact set K in X and for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset K$ such that $\mu(K \backslash K_\varepsilon) < \varepsilon$ and the restriction $f \,|\, K_\varepsilon$ of f to K_ε is continuous.

If f is μ-Lusin measurable, the image of μ denoted by $f(\mu)$ and defined by $f(\mu)(A) = \mu(f^{-1}(A))$ for every Borel subset A of (Y, τ_2) belongs to $M_+^1(Y, \tau_2)$.

DEFINITION 2. Let E be a Banach space and let (T, Σ, λ) be a probability space. A function $f: T \to E$, is Bochner integrable if there exists a sequence (f_n) of simple functions such that

(i) $\lim_n \| f(t) - f_n(t) \| = 0$ for λ-almost all $t \in T$ and

(ii) $\lim_n \int_T \| f(t) - f_n(t) \| \, d\lambda = 0$.

If f is Bochner integrable we denote by

$$\text{Bochner} - \int_A f d\lambda = \lim_n \int_A f_n d\lambda$$

for every A in Σ.

DEFINITION 3. A Banach space E is said to have the Radon-Nikodym property if for every probability space (T, Σ, λ) and every vector measure $m: \Sigma \to E$ such that $\| m(A) \| \leq \lambda(A)$ for every A in Σ, there exists $f: T \to E$ Bochner integrable such that

$$m(A) = \text{Bochner} - \int_A f d\lambda \quad \text{for}$$

every A in Σ.

For more about the Radon-Nikodym property see [1].

If (X, τ) is a topological space, Σ the Borel subset of (X, τ) and $\lambda \in M_+^1(X, \tau)$ and $f: X \to (E, \| \; \|)$ which is λ-Lusin measurable and bounded then f is Bochner integrable.

If C is a w^*-compact convex subset of the dual E^* of a Banach space E and $f: (X, \tau) \to (C, \sigma(E^*, E))$ then f is said to be w^*-integrable with respect to $\lambda \in M_+^1(X, \tau)$ if

(i) For every $x \in E$ the map $t \to x(f(t))$ is λ-integrable.

(ii) For every $A \in \Sigma$ there exists $x_A^* \in C$ such that $x(x_A^*) = \int_A x(f(t)) d\lambda$ for every $x \in E$. The element x_A^* will be denoted by

$$x_A^* = \omega^* - \int_A f d\lambda \, .$$

Let $\mu \in M^1_+(C, \sigma(E^*, E))$ it is easy to see that the identity map $I: (C, \sigma(E^*, E)) \to (C, \sigma(E^*, E))$ is μ weak*-integrable. An affine function $h: (C, \sigma(E^*, E)) \to R$ which is μ-Lusin measurable is said to satisfy the barycentric formula for μ on C if for every w^*-Borel subset A of C

$$h\left(w^* - \int_A Id\mu\right) = \int_A h \cdot Id\mu .$$

If $\lambda \in M_1(X, \tau)$ we denote by $\operatorname{supp} \lambda$ the support of λ.

LEMMA 4. *Let (X, τ) be a topological space and $\lambda \in M^1_+(X, \tau)$. Let C be a w^*-compact convex subset of the dual E^* of a Banach space E and f and ϕ*

$$f, \phi: (X, \tau) \longrightarrow (C, \sigma(E^*, E))$$

two λ-Lusin measurable maps such that for every Borel subset A in (X, τ),

$$\omega^* - \int_A fd\lambda = \omega^* - \int_A \phi d\lambda .$$

Then $f = \phi$ λ-almost everywhere.

Proof. Let K be a compact set in (X, τ) such that $\phi|K$ and $f|K$ are continuous from $(K, \tau) \to (C, \sigma(E^*, E))$ then we claim that $f = \phi$ λ-almost everywhere on K. Let $\mu = \lambda|K$, it is enough to show that

$$\phi|\operatorname{supp} \mu = f|\operatorname{supp} \mu$$

if not there exists $t_0 \in \operatorname{supp} \mu$ such that $\phi(t_0) \neq f(t_0)$. Let $x \in E$ such that $x(\phi(t_0) - f(t_0)) = 1$, the scalar map $t \to \psi(t) = x(\phi(t) - f(t))$ is continuous on K, therefore there exists a neighborhood V of t_0 in K such that

$$t \in V \Longrightarrow \psi(t) \geq \frac{1}{2} .$$

Observe that $t_0 \in \operatorname{supp} \mu \Rightarrow \mu(V) > 0$ and hence

$$\int_V \psi(t)d\lambda = \int_V \psi(t)d\mu \geq \frac{1}{2}\mu(V) > 0$$

on the other hand we have $\omega^* - \int_V fd\lambda = \omega^* - \int_V \phi d\lambda$ which in turn implies that $\int_V x(f(t))d\lambda = \int_V x\phi(t)d\lambda$ there fore $\int_V \psi(t)d\lambda = 0$ a contradiction that finishes the proof of the claim. To finish the proof choose for every $n \geq 1$ a compact K_n such that

(i) $f \,|\, K_n$ and $\phi \,|\, K_n$ are both continuous on K_n.

(ii) $\lambda(X \backslash K_n) \leqq 1/n$.

(iii) $K_n = H_n \cup N_n$ where $f \,|\, H_n = \phi \,|\, H_n$ and $\lambda(N_n) = 0$

Let $K = \bigcup_{n=1}^{\infty} H_n$, $M = X \backslash \bigcup_{n=1}^{\infty} K_n$ and $M = \bigcup_{n=1}^{\infty} N_n$ then $X = K \cup M \cup N$ where $\lambda(M \cup N) = 0$ and $f = \phi$ on K.

From now on, E will be a Banach space complemented in its second dual E^{**} by a projection $P \colon E^{**} \to E$ and K will denote the closed unit Ball of E^{**}.

THEOREM 5.   *The Banach space E has the Radon-Nikodym property if and only if for every $\lambda \in M_{+}^{1}(K, \sigma(E^{**}, E^{*}))$ such that $\omega^{*} - \int_{A} x^{**} d\lambda \in E$ for every w^{*}-Borel subset A of K, the projection P is weak* to norm λ-Lusin measurable and for every x^{*} in E^{*} the map $x^{*}P$ satisfies the barycentric formula for λ on K.*

Proof.   Let $\lambda \in M_{+}^{1}(K, \sigma(E^{**}, E^{*}))$ such that

$$m(A) = \omega^{*} - \int_{A} x^{**} d\lambda \qquad \text{belongs}$$

to E for every ω^{*}-Borel subset A of K. It is easy to see that

$$\| m(A) \| \leqq \lambda(A) \qquad \text{for every}$$

ω^{*}-Borel subset A of K and therefore m is a σ-additive E-valued vector measure. If E has the Radon-Nikodym property one can find

$$f \colon K \longrightarrow (E, \| \quad \|)$$

λ-Bochner integrable such that for every w^{*}-Borel subset A of K we have

$$m(A) = \text{Bochner} - \int_{A} f d\lambda = \omega^{*} - \int_{A} x^{**} d\lambda \,.$$

Apply Lemma 4 to conclude that $f(x^{**}) = x^{**}$ λ-almost everywhere and use the fact that f is λ-Lusin measurable from $K \to (E, \| \quad \|)$ to write $K = \bigcup_{n=1}^{\infty} K_n \cup N$ where (K_n) is a sequence of disjoint norm compact subset of E and $\lambda(N) = 0$. This shows that the identity

$$I \colon (K, \sigma(E^{**}, E^{*})) \longrightarrow (K, \| \quad \|)$$

is λ-Lusin measurable and therefore P is λ-Lusin measurable. Let x^{*} in E^{*}, we have to show that

$$x^{*}P\!\left(\omega^{*} - \int_{A} x^{**} d\lambda\right) = \int_{A} x^{*}P(x^{**}) d\lambda \,.$$

To this end observe that

$$x^* P\left(\omega^* - \int_A x^{**} d\lambda\right) = x^*\left(\omega^* - \int_A x^{**} d\lambda\right) = x^*(m(A))$$

$$= x^*\left(\sum_{n=1}^{\infty} m(K_n \cap A)\right) = \sum_{n=1}^{\infty} x^*(m(K_n \cap A))$$

$$= \sum_{n=1}^{\infty} \int_{K_n \cap A} x^*(x^{**}) d\lambda = \sum_{n=1}^{\infty} \int_{K_n \cap A} x^* P(x^{**}) d\lambda$$

$$= \int_A x^* P(x^{**}) d\lambda .$$

Conversely, let λ be in $M_+^1(K, \sigma(E^{**}, E^*))$ such that for every weak* Borel subset A of K we have

$$m(A) = \omega^* - \int_A x^{**} d\lambda \in E .$$

Let $x^* \in E^*$, then

$$x^*(m(A)) = x^* P(m(A)) = \int_A x^* P(x^{**}) d\lambda = \int_A x^*(x^{**}) d\lambda .$$

Therefore $\omega^* - \int_A Id\lambda = \omega^* - \int_A Pd\lambda$ where I is the identity map on K. Now apply Lemma 4 to deduce that K can be written

$$K = \bigcup_{n=1}^{\infty} K_n \cup N$$

where each K_n is w^*-compact on which $I = P$ and $\lambda(N) = 0$. This implies that for every $n \geq 1$, K_n is norm compact and is contained in E and hence $I: (K, \sigma(E^{**}, E^*)) \to (K, \| \quad \|)$ is λ-Lusin measurable. To prove now that E has the Radon-Nikodym property, let Σ be σ-algebra of all Lebesgue measurable subsets of $[0, 1]$ and let μ be the Lebesgues measure on $[0, 1]$. Consider a vector measure m: $\Sigma \to E$ such that $\| m(A) \| \leq \mu(A)$ for every $A \in \Sigma$. By [5], there exists a map $f: [0, 1] \to K$ such that

(i) For every ω^*-Borel subset B of K, $f^{-1}(B)$ belongs to Σ.

(ii) The image measure $f(\mu)$ belongs to $M_+^1(K, \sigma(E^{**}, E^*))$.

(iii) For every $A \in \Sigma$

$$m(A) = \omega^* - \int_A f d\mu .$$

It follows easily that for any w^*-Borel subset B of K

$$\omega^* - \int_A x^{**} df(\mu) \in E .$$

Therefore $I: (K, \sigma(E^{**}, E^*)) \to (K, \| \quad \|)$ is $f(\mu)$-Lusin measurable by what we did above. Consequently K can be written $K = \bigcup_{n=1}^{\infty} K_n \cup N$ where $f(\mu)(N) = \mu(f^{-1}(N)) = 0$ and K_n is norm compact subset of

E^{**}. It follows that If: $[0, 1] \to (K, \| \ \|)$ is μ-almost separably valued. Also note that if 0 is an open set in $(K, \| \ \|)$ then $f^{-1}(0) \in \Sigma$. This shows that the map

$$f = \mathrm{I}f : [0, 1] \to (K, \| \quad \|)$$

is μ-Lusin measurable and therefore Bochner integrable and hence

$$m(A) = \omega^* - \int_A f d\mu = \mathrm{Bochner} - \int_A f d\mu$$

for every $A \in \Sigma$. This shows that f takes its values μ-almost everywhere in E, therefore E has the Radon-Nikodym property.

The proof of the above theorem implies the following corollary.

COROLLARY 6. *For any Banach space E the following two conditions are equivalent:*

(i) *The space E the Radon-Nikodym property.*

(ii) *For every $\lambda \in M^1_+(K, \sigma(E^{**}, E^*))$ such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every w^*-Borel subset A of K, the identity*

$$(K, \sigma(E^{**}, E^*)) \to (K, \| \quad \|)$$

is λ-Lusin measurable.

If E is completed in E^{**} by a projection $P: E^{**} \to E$ then (i) *and* (ii) *are equivalent to*

(iii) *For every $\lambda \in M^1_+(K, \sigma(E^{**}, E^*))$ such that $\omega^* - \int_A x^{**} d\lambda \in E$ for every ω^*-Borel subset A of K, the projection P is λ-Lusin measurable and for every $x^* \in E^*$, the map $x^* P$ satisfies the barycentric formula for λ on K.*

COROLLARY 7 [4]. *If E is complemented in E^{**} by a weak* to weak Baire-1 projection P, then E has the Radon-Nikodym property.*

Proof. If P is Baire-1, it is λ-Lusin-measurable for any $\lambda \in M^1_+(K, \sigma(E^{**}, E^*))$ and for every $x^* \in E^*$, the map $x^* P$ is Baire-1 and therefore satisfies the barycentric formula for λ on K.

In [4] it was shown that if $P: (E^{**}, \sigma(E^{**}, E^*)) \to (E, \sigma(E^*, E))$ is Baire-1, then E is a weakly compactly generated Banach space. Using this fact we can now give the following:

Example of a Banach space having the Radon-Nikodym property and complemented in its bidual by a nonweak to weak Baire-1 projection.*

Let R be the Banach space constructed by Rosenthal in [2], this

space bas the following properties:

(1) It is a dual space, therefore it is complemented in R^{**}.

(2) It is a closed subspace of a weakly compactly generated Banach space, therefore it has the Radon-Nikodym property [3].

(3) It is not weakly compactly generated so $P: R^{**} \to R$ is not Baire-1.

For more examples related to this paper see [4].

REFERENCES

1. J. Diestel and J. J. Uhl Jr., *Vector measures*, Mathematical Survey No. 15, American Mathematical Society, Providence, 1977.
2. H. P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Comp. Math., Groningen, (28), (1974), 83-111.
3. E. Saab, *A characterization of w*-compact convex sets having the Radon-Nikodym property*, Bull. Soc. Math., 2 ème serie, **104** (1980), 79-88.
4. ————, *Universally Lusin-measurable and Baire-1 projections*, Proc. Amer. Math. Soc., to appear, **78** (1980), 514-518..
5. H. Weizsäker, *Strong measurability, lifting and the Choquet Edgar theorem*, Lecture notes no 645, 209-218.

THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B. C.
V6T 1Y4 CANADA

Current address: The University of Missouri
                   Department of Mathematics
                   Columbia, MO 65211

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

Pacific Journal of Mathematics
Vol. 97, No. 2   February, 1981