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LOCALLY COMPACT GROUPS ACTING ON TREES

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Following Serre's original description of groups having the fixed point property for actions on trees, Bass has introduced the notion of a group of type FA'. Groups of type FA' can not be nontrivial free products with amalgamation. We show that a locally compact (hausdorff) topological group with a compact set of connected components is of type FA'. Furthermore, any locally compact group which is a nontrivial free product with amalgamation has an open amalgamated subgroup.

1. A group G is called an amalgam if it is a free product with amalgamation of subgroups A and B along C , i.e., $G = A *_C B$, so that $C \neq A$, $C \neq B$.

If a group G acts without inversions on a tree so that it has a fixed vertex we say G has property FA on X . Serre has introduced the notion of a group of type FA. We say that G is of type FA if G has property FA whenever it acts on a tree. The following theorem characterizes G group theoretically.

THEOREM 1 (Serre). *A group G is of type FA if and only if it satisfies the following conditions:*

- (1) G has no infinite cyclic quotient.
- (2) G is not an amalgam.
- (3) G is not the union of any sequence

$$G_0 \subsetneq G_1 \subsetneq G_2 \cdots \subsetneq G_n \subsetneq \cdots$$

of its proper subgroups. □

This theorem was originally formulated by Serre for countable groups [6, Theorem 15; 2, Theorem 3.2]. Bass has introduced the notion of a group of type FA'. In order to formulate this we introduce the ends of a tree X . Consider the collection \mathcal{L} of half-lines of X : $L \in \mathcal{L}$ is isometric to the standard half-line

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 1 & 2 & 3 & 4 & & \end{array}$$

The ends of X is the set of equivalence classes \mathcal{E} of \mathcal{L} under the equivalence relation \sim :

$$L \sim M \text{ iff } L \cap M \text{ is a half-line.}$$

Notice that if $e, f \in \mathcal{E}$, $e \neq f$, we can choose representatives $L \in e$, $M \in f$ so that $L \cup M$ is a doubly infinite line of X denoted (e, f) . If G acts as a group of isometries on X then it also acts on the set \mathcal{L} of half-lines of X and the set \mathcal{E} of ends of X . If $L \in \mathcal{L}$, $g \in G$ we say L is neutral, repulsing or attracting for g if $gL = L$ (i.e., pointwise fixed), $gL \supsetneq L$, or $gL \subsetneq L$ respectively. If L contains a half-line $L'(L - L'$ is finite) which is neutral, repulsing or attracting for g then we say L is almost neutral, repulsing or attracting for g . An end $e \in \mathcal{E}$ is neutral, repulsing or attracting for $g \in G$ if it possesses a representative half-line which is so for g . Denote the ends which are fixed by G ($ge = e, \forall g \in G$) by \mathcal{E}^G .

We can now formulate the property FA'.

THEOREM 2. *Suppose G acts without inversion on the tree X . The following conditions are equivalent.*

- (i) *Each element of G has a fixed vertex.*
- (ii) *Each finitely generated subgroup of G has a fixed vertex.*
- (iii) *There is either a fixed vertex for G or a neutral fixed end.*

Proof. The implication (i) \Rightarrow (ii) is proved by Serre [6, Corollary 3 to Proposition 26]. The implication (i) \Rightarrow (iii) is proved by Tits [8, Corollary 3.4]. The implications (ii) \Rightarrow (i) and (iii) \Rightarrow (i) are obvious. \square

If G satisfies the equivalent conditions of Theorem 2 for a given action without inversions on a tree X then we shall say G has property FA' on X . This property has been further analyzed by Bass [2, Propositions 1.6, 3.7]. In case G has property FA' on X and has no fixed vertex then there is a half-line L with vertices (v_n) , $n \geq 0$, so that $G_{v_n} \subset G_{v_{n+1}}$, $n \geq 0$, and

$$G = \bigcup_{n \geq 0} G_{v_n}.$$

We say that G is of type FA' if G has property FA' whenever it acts on a tree.

THEOREM 3 (Bass). *A group G is of type FA' if and only if it satisfies the following conditions:*

- (1) *G has no infinite cyclic quotient.*
- (2) *G is not an amalgam.*

\square

One obtains information about homomorphisms from a group G of type FA or FA' to amalgams using the next propositions.

PROPOSITION 1 (Serre [6, Proposition 21]). *If G is a group of*

type FA and $\varphi: G \rightarrow A^*B$ is a homomorphism to an amalgam then $\varphi(G)$ is contained in a conjugate of A or B . \square

PROPOSITION 2. *If G is a group of type FA' and $\varphi: G \rightarrow A^*B$ is a homomorphism then $\varphi(G)$ is contained in a conjugate of A or B .*

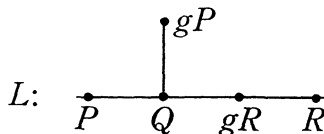
Proof. First notice that a homomorphic image of type FA' is also of type FA'. Thus $\varphi(G)$ acts without inversions on the tree X for A^*B . Using condition iii) of Theorem 2, $\varphi(G)$ has a fixed point and consequently $\varphi(G)$ is contained in a conjugate of A or B , or there is a neutral fixed end for $\varphi(G)$. However, the edge stabilizers for this fixed end are trivial since A^*B has no amalgamation; this is impossible and consequently $\varphi(G)$ has a fixed point. \square

2. If H is a normal subgroup of type FA of a group G and G/H is of type FA'(FA) then G is of type FA'(FA). To see this notice that if G acts on a tree X and K is a finitely generated subgroup of G then $L = K \cap H$ has a fixed tree X^L and thus the finitely generated subgroup $KH/H \cong K/L$ of G/H acts on X^L with a fixed vertex which is then fixed by K . Also, if G contains a subgroup of finite index H of type FA' then G is also of type FA'. Indeed, if K is a finitely generated subgroup of G then $L = K \cap H$ is a finitely generated subgroup of H ; without loss of generality we may assume H is normal and thus the finite group K/L has a fixed point for its action on X^L .

Based on some remarks of Tits [7; §2.3] we shall show that every extension of groups of type FA' is again of type FA'. For this we shall need some further comments on ends. We suppose that a group G is acting without inversions on a tree X .

PROPOSITION 3. *Let $e \in \mathcal{E}^G$. Any half-line $L \in e$ is almost neutral, repulsing or attracting for $g \in G$.*

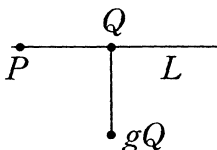
Proof. Given $g \in G$ and $L \in e$. Let P be the initial vertex of L . If $gP \in L$ then L is neutral or attracting for g . If $gP \notin L$ there are two possibilities: (1) The geodesic from P to gP meets L only at P or (2) The geodesic from P to gP meets L at a vertex $Q \neq P$. In the first case L is repulsing for g . In the second case let L' be the half-line contained in L starting at Q .



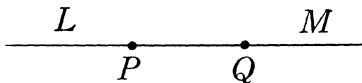
If $R \in L'$ then gR belongs to the geodesic from gP to Q for only finitely many R . Thus there is a half-line $L'' \subset L'$ so that $gL'' \subset L'$. Choose $R \in L''$ so that Q belongs to the geodesic from P to R and gQ belongs to the geodesic from gP to gR . If gQ belongs to the geodesic from gP to Q then L' is repulsing or neutral for g ; otherwise, L' is attracting for g . It is now easy to see that if one half-line $L \in e$ is almost neutral, almost repulsing or almost attracting for g then so is every half-line in e ; viz. if g has a fixed point on L in e then it must be almost neutral for g . \square

PROPOSITION 4. *If G has a neutral fixed end e then either $\mathcal{E}^G = \{e\}$ or there is a doubly infinite line of fixed points for G .*

Proof. Suppose f is a repulsing or attracting end for $g \in G$. Let $P \in e$ so that $gP = P$ and choose $L \in f$ on which g is repulsing or attracting starting at Q .



This is impossible since the length of the geodesic from P to Q is different from that of gP to gQ . Thus any other fixed end f for G must be a neutral fixed end. Choose representative $L \in e, M \in f$ so that $L \cup M$ is a double infinite line.



Thus for each $g \in G$ there exists $P \in L, Q \in M$ fixed by g ; hence the doubly infinite $L \cup M$ is fixed identically for all $g \in G$. \square

From the above remarks we see that every half-line in $e \in \mathcal{E}^G$ is one of the mutually exclusive alternatives for a given $g \in G$. We can then define $v_e: G \rightarrow \mathbb{Z}$ for a fixed end e as follows

$$v_e(g) = \begin{cases} 0 & \text{if } e \text{ is neutral for } g \\ \min_{L \in e, L \text{ attracting for } g} |L - (L \cap gL)| & \text{if } e \text{ is attracting for } g \\ -\min_{L \in e, L \text{ repulsing for } g} |gL - (L \cap gL)| & \text{if } e \text{ is repulsing for } g. \end{cases}$$

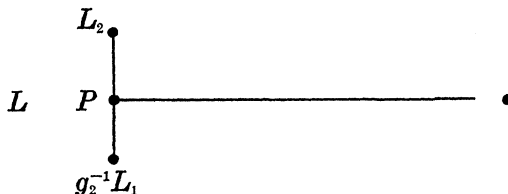
THEOREM 4. *For each fixed end $e \in \mathcal{E}^G$ there is a canonical*

homomorphism

$$v_e: G \longrightarrow \mathbb{Z}$$

with the property that $v_e(g) = 0$ if and only if e is neutral for g and $L^g \neq \emptyset$ for all $L \in e$.

Proof. To see that v_e is a homomorphism let $g_1, g_2 \in G$ achieve their v_e value on L_1, L_2 respectively and let $g_1 g_2$ achieve its v_e value on L . Consider the half-line $g_2^{-1} L_1 \cap L_2 \cap L$ starting at P .



Now $P \in L_2$ and thus $g_2 P \in L_1$ so P has moved $v_e(g_2)$ under the action of g_2 . However $g_1(g_2 P) \in L_1$ so that $g_2 P$ has moved $v_e(g_1)$ under the action of g_1 . Thus P has moved $v_e(g_1) + v_e(g_2)$ under the action of $g_1 g_2$; however $P \in L$ so P moves $v_e(g_1 g_2)$ under the action of $g_1 g_2$, so that

$$v_e(g_1 g_2) = v_e(g_1) + v_e(g_2).$$

If $v_e(g) = 0$ then e is neutral for g and $L_g \neq \emptyset$ for some $L \in e$. Moreover, since g fixes identically a half-line $L' \subset L$ then it must have a fixed point on every half-line in e . Conversely if g has a fixed point on some $L \in e$ then L is almost neutral for g and e is neutral for g ; thus $v_e(g) = 0$. \square

COROLLARY 1. *If for a given action of G on a tree X a normal subgroup H has a unique neutral fixed end e then either e is a neutral fixed end for G or there is a nontrivial homomorphism $v: G/H \rightarrow \mathbb{Z}$.*

Proof. It is easy to see that e is a fixed end for G ; viz. suppose $ge = f \in \mathcal{E}$, then for $h \in H$

$$e = g^{-1} h g e = g^{-1} h f.$$

Thus $f = ge = hf$ and by uniqueness $f = e$. Thus from the theorem above $v_e: G \rightarrow \mathbb{Z}$ factors through a homomorphism $v: G/H \rightarrow \mathbb{Z}$ since e is neutral for H . \square

COROLLARY 2. *If G has a normal subgroup H so that H and*

G/H have property FA' then G has property FA'.

Proof. Let G act on a tree X . If H has a fixed point then there is an action of G/H on X^H . Since G/H has property FA' we can find a fixed point for $g \in G$ by finding a fixed point for gH on X^H . If H has no fixed points on X then it has a neutral fixed end and thus since G/H has no homomorphism to Z this neutral fixed end for H is also a neutral fixed end for G . \square

COROLLARY 3. *Suppose that G acts without inversions on a tree X . If G is generated by a set S with $X^s \neq \emptyset$ for all $s \in S$ then either G has no fixed end or a fixed end is neutral.*

Proof. If G has a fixed end e then any $s \in S$ has a fixed point lying on some half-line $L \in e$; thus $v_s(s) = 0$. It follows immediately from the theorem then that v_e is trivial and consequently that e is neutral for G . \square

A nonempty collection of subgroups $\mathcal{N} = \{N_\alpha \mid \alpha \in \mathcal{A}\}$ of a group G is called a normal filtering family if

- (1) given $\alpha, \beta \in \mathcal{A}$, $\exists \gamma \in \mathcal{A}$ so that $N_\gamma \subset N_\alpha \cap N_\beta$ and
- (2) given $\alpha \in \mathcal{A}$, $g \in G$, $\exists \beta \in \mathcal{A}$ so that $N_\beta \subset gN_\alpha g^{-1}$.

(These are the conditions that guarantee G is a topological group with \mathcal{N} as a fundamental system of open subgroups.)

PROPOSITION 5. *Suppose that G acts without inversions on a tree X such that $\mathcal{E}^G = \emptyset$. If \mathcal{N} is a normal filtering family of subgroups of G having property FA' on X then some $N \in \mathcal{N}$ has a fixed point.*

Proof. Suppose by way of contradiction that no $N \in \mathcal{N}$ has a fixed point; it follows then from the FA' property that each $N_\alpha \in \mathcal{N}$ has a unique neutral fixed end e_α . Given $N_\alpha, N_\beta \in \mathcal{N}$, choose $N_\gamma \subset N_\alpha \cap N_\beta$; we have then

$$\{e_\alpha\} = \mathcal{E}^{N_\alpha} = \mathcal{E}^{N_\gamma} = \mathcal{E}^{N_\beta} = \{e_\beta\}.$$

Thus there is a common neutral fixed end e for \mathcal{N} . Given $N_\alpha \in \mathcal{N}$, $g \in G$, choose $N_\beta \subset gN_\alpha g^{-1}$; it follows that

$$\{e\} = \mathcal{E}^{N_\beta} = \mathcal{E}^{gN_\alpha g^{-1}} = \{ge\}$$

and thus e is a fixed end for G . \square

Since an amalgam has elements which have no fixed points on

the tree corresponding to the amalgamation it follows from Corollary 3 and Theorem 2 that there can be no fixed end for this action. Similarly, for an HNN extension $A_C^*(C \neq A)$ acting on its corresponding tree there can be no fixed end. To see this, we may choose without loss of generality a representative half-line for this end with initial vertex and edge having stabilizers A and C respectively; then for $g \in A$ it follows from Theorem 4 that g is neutral on this half-line and thus $g \in C$, whence $C = A$. We shall use these remarks together with Proposition 5 to derive some important consequences for topological groups. Also, this proposition will provide useful information if the family consists of a single normal subgroup.

As a further remark on extensions of groups having property FA' we have the following result.

THEOREM 5. *If H and K are subgroups of G having property FA' and $G = HK$ then G has property FA'.*

Proof. Let $g \in G$ be written as $g = hk$, $h \in H$, $k \in K$; express now $kh = h'k'$, $h' \in H$, $k' \in K$. Thus we have

$$(h^{-1}h')k'(h^{-1}k^{-1}h) = 1.$$

By results of Serre [6, Corollary 1 to Proposition 26] we can find a common fixed point $P \in X$ of the automorphisms $h^{-1}h'$, k' , $h^{-1}k^{-1}h$ for an action of G on the tree X if each has a fixed point; this is so from the FA' hypothesis for H and K . Consequently, we have the properties:

$$hP = h'P, k'P = P, k(hP) = hP.$$

Let $X^k, X^{k'}$ be the trees of fixed points of k and k' ; $X^k \cap X^{k'} \neq \emptyset$ since K has property FA' (condition (ii)). Since $P \in X^{k'}$, $hP \in X^k$, it follows that the midpoint Q of the geodesic from P to hP is fixed by h [6, Corollary 2 to Proposition 23] and also by h' since $hP = h'P$; thus $Q \in X^k$ or $Q \in X^{k'}$. If $Q \in X^k$ then $hkQ = Q$. If $Q \in X^{k'}$ then $h'k'Q = Q$; but $hk = h(kh)h^{-1} = h(h'k')h^{-1}$ so $hk(Q) = hk(hQ) = h(kh)h^{-1}(hQ) = hh'k'Q = hQ = Q$. Hence g has a fixed point for its action on X . \square

3. We now derive consequences for topological groups from the results of the previous sections.

THEOREM 6. *If G is a connected locally compact topological group then G is of type FA'.*

Proof. As a first step we decompose G as

$$G = LCR$$

where L is a semisimple (connected) Lie subgroup, C is a compact connected semisimple subgroup and R is the radical of G (maximal solvable connected closed normal subgroup) and CR is a closed normal subgroup [5, Theorem 1]. [One uses the solution of Hilbert's fifth problem to see the equivalence of connected locally compact and Iwasawa's notion of (L) group [3].] Now using [4, Lemma 3.12] we decompose the group L as $L = HM$ where H is a connected solvable Lie group and M is either the maximal compact subgroup K of L or $M = K \times V$ where V is a vector group. It suffices then using Theorem 5 to verify that H, M, C, R are of type FA'. Compact groups are of type FA' [1]; also any vector group being divisible and abelian is FA'. It remains to show that a connected solvable group S is of type FA'. Using Iwasawa's decomposition of a locally compact connected group as

$$G = H_1 H_2 \cdots H_r K$$

where K is maximal compact and $H_i \cong \mathbf{R}$ $1 \leq i \leq r$ [4; Theorem 13], we see that G has no nontrivial homomorphisms to \mathbf{Z} . Now if S is an amalgam then using Bass' result for solvable groups [2, Theorem 6.1] we obtain a surjective homomorphism $S \xrightarrow{\varphi} \mathbf{Z}_2^* \mathbf{Z}_2$. However using the Iwasawa decomposition above for $S(=G)$ we obtain $\varphi|_{H_i}, \varphi|_K$ are trivial homomorphisms. To see this notice that H_i, K are of type FA' and hence by Proposition 2 each of the restrictions has image in a conjugate of one of the \mathbf{Z}_2 factors; the divisibility of H_i $1 \leq i \leq r$, K then forces each image to be trivial and thus also φ . \square

COROLLARY 1. *If G is a locally compact topological group with G/G_0 compact then G is of type FA'.*

Proof. This follows immediately from the theorem above, Corollary 2 to Theorem 4 and main result of [1].

COROLLARY 2. *Suppose G is a locally compact topological group. If G is an amalgam, $G = A_c^* B$, or an HNN extension, $G = A_c^*$, then $G_0 \subset C$.*

Proof. The connected component of the identity G_0 is of type FA'. Using Proposition 5 and the remarks following it we see that G_0 has a fixed point for its action on the tree corresponding to the amalgam or the HNN extension if $C \neq A$. Since G_0 is normal we see immediately that $G_0 \subset C$. In case the HNN extension has $C = A$ the corresponding tree has a fixed end, say e ; using Theorem 4 now,

each element of G_0 has a fixed point so $G_0 \subset \ker v_e = A$. \square

COROLLARY 3. *Suppose G is a locally compact topological group. If G is an amalgam, $G = A_c^* B$ or an HNN extension, $G = A_c^*$, then C is open in G .*

Proof. Without loss of generality we may replace G by G/G_0 using Corollary 2 above and assume then that G is a locally compact totally disconnected topological group. It is well known that G has a neighborhood basis of the identity given by compact open subgroups [Hewitt and Ross, *Abstract Harmonic Analysis*, p. 62]. Since compact groups are of type FA' this is a normal filtering family of type FA'; by dint of Proposition 5 then some compact open subgroup U has a fixed point. Without loss of generality we may assume $U \subset A$. In case G is an amalgam choose $g \in B - C$ so that $U \cap g U g^{-1} \subset A \cap g A g^{-1} \subset C$; hence C is open. If G is an HNN extension it is generated by A together with an element t (which generates the fundamental group of X/G [6, p. 62]); thus

$$U \cap t U t^{-1} \subset A \cap t A t^{-1} \subset C.$$

For the HNN extension to which Proposition 5 doesn't apply, viz. $G = A_c^*$ we notice as in the proof of Corollary 2 that there is a fixed end e and hence for any compact open subgroup U ,

$$U \subset \ker v_e \subset A$$

since U is of type FA'. \square

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