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REPRESENTATIONS OF DIVISION ALGEBRAS OVER LOCAL FIELDS. II

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Let D be a division algebra of degree p^4 over a non-Archimedean locally compact field K (i.e., index $(D/K)=p^2$). We show how to construct the irreducible unitary representations of the multiplicative group of D .

1. Let K be a non-Archimedean locally compact field, with residue field k having $q = p^r$ elements (p is a prime), and let D be a locally compact division algebra with center K . Then $[D:K]$ is a perfect square— m^2 , say; m is called the index of D . It is a question of some interest to determine the irreducible unitary representations of D^\times . In [1], [2], and [3], the representations were determined and their characters computed for the case where m is prime to p (the “tamely ramified” case). As was pointed out in [4], the same procedure serves to compute the representations (though not the characters) in the case $m = p$.

In this paper we compute the irreducible representations of D^\times in the case where $m = p^2$. The procedure is like that of the previous cases, but there are added technical difficulties. The method as given here does work in somewhat greater generality; see § 5 for a brief discussion.

Before describing the method, we need some notation. Let K_1 be a maximal unramified extension of K contained in D , so that $[K_1:K] = m$; let d be the residue class field of K_1 (and of D). It is possible to pick a prime element π of D such that conjugation by π generates $\text{Gal}(K_1/K)$. (For these and other facts used here, see, e.g., Chapter 1 of [8].) Then π^m is a prime element of K . We may also choose a set $k_1 \subset K_1$ of coset representatives of d such that $0 \in k_1$ and the other elements of k_1 are the $(q^m - 1)$ th roots of unity in K_1 . Then every element of D has a unique expression of the form

$$y = \sum_{j=0}^{\infty} \alpha_j \pi^j, \quad j_0 \in \mathbb{Z},$$

where each $\alpha_j \in k_1$. Set

$$D_1 = \left\{ y = \sum_{j=0}^{\infty} \alpha_j \pi^j : \alpha_0 \neq 0 \right\}$$

and

$$G = \{ y \in D_1 : \alpha_0 = 1 \}.$$

Then

$$D^{\times} = D_1 \ltimes \{\pi^j: j \in \mathbb{Z}\}$$

and

$$D_1 = G \ltimes k'_1, \quad k'_1 = k_1 \setminus \{0\}.$$

As noted in [1], the major problem in finding irreducible representations of D^{\times} lies in finding those of G .

Let $\mathfrak{O} = \mathfrak{O}_D = \{y = \sum_{j=0}^{\infty} \alpha_j \pi^j\}$ be the ring of integers of D , and let $P = \pi \mathfrak{O} = \mathfrak{O} \pi$ be the maximal ideal of \mathfrak{O} . Then for each $j \geq 1$, $G_j = 1 + P^j$ is a subgroup of G (and $G = G_1$). Any irreducible representation ρ of the compact group G must be trivial on some G_j ; let n be the smallest integer (≥ 1) such that $\rho|_{G_{n+1}}$ is trivial. (In the terminology of [3], $n+1$ is the conductor of ρ .) Let $n_0 = [(n+1)/2]$, where $[\]$ denotes the greatest integer function. Then $G_{n_0}/G_n \cong P^{n_0}/P^{n+1}$ is Abelian, and therefore $\rho|_{G_{n_0}}$ is a direct sum of 1-dimensional representations. Let χ be one of these, and regard χ as a representation of P^{n_0} . We may describe χ as follows: let ψ be a character of K^+ which is trivial on $\mathfrak{O}_K = \mathfrak{O}_D \cap K$, but not on $\pi^{-m} \mathfrak{O}_K$. Then any character of D^+ is of the form

$$\chi_x(y) = \psi \circ \tau(xy),$$

where $x \in D$ and $\tau: D \rightarrow K$ is the reduced trace. If we identify D with D^{\wedge} via $x \mapsto \chi_x$, then $\mathfrak{O}_D^{\wedge} = P^{-m+1}$ and $(P^{n+1})^{\perp} = P^{-m-n}$, $(P^{n_0})^{\wedge} = D/P^{-m-n_0+1}$. The character χ is an element of $(P^{n_0})^{\wedge}$ that annihilates P^{n+1} , and is thus given by an element of P^{-m-n}/P^{-m-n_0+1} . That is, $\chi = \chi_x$, where $x \in P^{-m-n}$ (and x is determined mod P^{-m-n_0+1}). Conversely, ρ appears in $\text{Ind}_{G_{n_0} \rightarrow G}(\chi_x)$, and we therefore determine all irreducibles ρ containing χ by finding all irreducible components of $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$. Replacing χ_x by $\chi_{x'}$, where x and x' are conjugate in G , does not change the induced representation; if x and x' are conjugate in D , the induced representation may be changed, but in an unimportant way so far as representations of D are concerned.

The construction of ρ in the tamely ramified case proceeds as follows: one first chooses x to generate a field of degree as small as possible. Then one constructs a subgroup $H = H_x$ of G which contains G_{n_0} (roughly speaking, H consists of elements of G commuting with x mod an appropriate power of P), extends χ_x to an irreducible representation $\chi_{x,a}$ of H_x , and induces up to G ; the induced representation is irreducible, and one obtains all the irreducibles in $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$ by varying the $\chi_{x,a}$. The procedure in our case is similar, but in some cases the group H_x depends on the extended representation $\chi_{x,a}$.

The wildly ramified case seems to be harder than the tamely

ramified one for two related reasons. Write $x = \sum_{j=-m-n}^{\infty} \alpha_j \pi^j$, $x_1 = \alpha_{-m-n} \pi^{-m-n}$. In the tamely ramified case, we may assume (by taking a conjugate of x) that the elements $\alpha_j \pi^j$ all commute. Furthermore, an important part in the analysis is played by elements x_i which are conjugate to $x \bmod P^{-m-n+i}$ and which generate fields of degrees as small as possible. In the tamely ramified case, these elements can be assumed to lie in the division algebra D_{x_1} of elements commuting with x_1 , and therefore many proofs can proceed inductively. In the wildly ramified case, it is quite possible that $1 < [K(x_1): K] < [K(x): K]$ although $K(x)$ contains no proper extension of K . As a result, we need some technical lemmas to show that D_{x_1} has elements "close" (in D) to those of $K(x)$. When these technical difficulties do not arise, the proofs are easier; we discuss these cases in § 2.

For the rest of the paper, we set $m = p^2$ (though for typographical convenience we continue to use m). Let $K_0 \subseteq K_1$ be an unramified extension of K of order p .

In what follows, σ denotes the element of $\text{Gal}(K_1/K)$ generated by conjugation by π ; σ is generally written exponentially. We also use σ to denote the corresponding element of $\text{Gal}(d/k)$. We shall often denote an element of k_1 and the corresponding element of d by the same letter; for example, we shall write $\text{Tr}_{d/k} \alpha$ for $\alpha \in k_1$. This convention should cause no confusion; in case $\text{Char } K = p$, it is justified by the fact that $d \cong k_1$. Similarly, we write k, k_0 for $K \cap k_1, K_0 \cap k_1$, and regard elements of, e.g., k as either in K or in the residue class field. For $0, 1 \neq x \in D$, D_x is the division algebra of elements commuting with x ; however, D_0 will be the division algebra of elements commuting with K_0 . Note that if $[K(x): K] = m_0$, then the index of D_x over $K(x)$ is $(m/m_0)^2$. Then reduced trace (norm) is denoted by $\tau(\nu)$. Finally, we shall write " ρ contains χ " for "the restriction to $(1 + P^{n_0})$ of the irreducible representation ρ of G contains χ ."

2. The results of this section are, generally speaking, either specifically contained in those of [1], [2], or [3], or easily deduced from them.

We continue with the notation of the first section. Let $\chi = \chi_x$, $x = \sum_{j=-n-m}^{\infty} \alpha_j \pi^j$, and set $x_1 = \alpha_{-m-n} \pi^{-m-n}$. We assume throughout this section and the next two that n is odd.

LEMMA 1. *If $y = \sum_{j=-m-n}^{\infty} \beta_j \pi^j$, with $\beta_{-m-n} = \alpha_{-m-n}$, then $e(K(y)) \geq e(K(x_1))$ and $f(K(y)) \geq f(K(x_1))$, where e = ramification index and f = residue class degree.*

Proof. As the extra effort involved is small, we give a proof

valid for all m . Write α for α_{-m-n} . Since $\pi^{-m} \in K$, $K(x_1) = K(\alpha\pi^{-n}) = K(x_0)$, say. Let $m = st$, where s is the greatest common divisor of m and n ; $x_0^t \in K_1$. Let $K' = K(x_0^t)$. Then $f(K(x_1)) = [K': K]$ and $e(K(x_1)) = t$.

Suppose that $am - bn = s$. Then $\pi^{m(a+b)}y^b \in P^s$, so that $K(y)$ certainly has an element y_0 such that $\pi^{-m}y_0^t$ is a unit, and hence $e(K(y))$ is a multiple of t . Furthermore, $y^t \equiv x_1^t \pmod{P^{-(m+n)+1}}$; now Hensel's lemma shows that $K(y)$ contains a field isomorphic to K' . Thus $f(K(y)) \geq [K': K]$, as claimed.

We now proceed to look at the representations of G . The analysis splits into four basic cases, depending on the nature of x_1 :

- (1) $x_1 \in K$;
- (2) $[K(x_1): K] = m$;
- (3) $[K(x_1): K] = p$ and $K(x_1)$ is unramified over K ;
- (4) $[K(x_1): K] = p$ and $K(x_1)$ is totally ramified over K .

We deal with the first three cases in this section. Case (4), which splits into three subcases, is the subject of the next two sections.

Case (1) is handled by reducing it to the other cases; see [1], [3], or [4].

THEOREM 1. *If $K(x_1) = K$, then every irreducible representation ρ containing χ_x is of the form $\rho = \rho_1 \otimes (\chi_0 \circ \nu)$, where ν is the reduced norm map (from D to K), χ_0 is a 1-dimensional representation of K^* , and ρ_1 is a representation of G with conductor $< n$.*

Proof. If $x_1 \in K$ and $1 + y \in G_n$, then

$$\chi_x(1 + y) = \psi \circ \tau_{D/K}(xy) = \psi(x\tau_{D/K}y).$$

But $\nu(1 + y) \equiv 1 + \tau_{D/K}(y) \pmod{G_{n+1}}$, so that $\chi_x(1 + y)$ depends only on $\nu(y)$. Hence there is a character χ_0 of K^* such that $\chi = \chi_0 \circ \nu$ on G_n . A simple counting argument (given in [1]) shows that $\text{Ind}_{G_{n-1}-G}(\chi|_{G_{n-1}})$ consists of multiples of representations $\rho_1 \otimes (\chi_0 \circ \nu)$, where ρ_1 has conductor $< n$. This proves the theorem.

Case (2) is also easy to deal with; the analysis follows that in § 4 of [4] (or that of [1], Theorem 2.3).

LEMMA 2. *If $[K(x_1): K] = m$ and $y \in K(x)$, $y = \sum_{j=r}^{\infty} \beta_j \pi^j$, then $\beta_r \pi^r \in K(x_1)$; conversely, if $\beta_r \pi^r \in K(x_1)$, then $\exists y \in K(x)$ with $y = \sum_{j=r}^{\infty} \beta_j \pi^j$.*

Proof. Since $xy = yx$, the lowest order terms of x and y must commute—that is, if $y_1 = \beta_r \pi^r$, then $y_1 x_1 = x_1 y_1$. Thus $y_1 \in D_{x_1}$. Since $K(x_1)$ is its own commutator in D , $y \in K(x_1)$. For the converse, fix r

and let $h' = \{\beta \in k_i: \exists y \in K(x) \text{ with } y = \sum_{j=r}^{\infty} \beta_j \pi^j \text{ and } \beta_r = \beta\}$, $k'' = \{\beta \in k_i: \beta \pi^r \in K(x_i)\}$. Then k' and k'' have the same cardinality (Lemma 1), and $k' \subseteq k''$, from the first part of this lemma. Hence $k' = k''$.

Since we assume that n is odd, we have $n_0 = (n + 1)/2$. Let $H_x = G_{n_0} \cap (G \cap K(x))$; we write H for H_x when no confusion will result.

LEMMA 3. $[H, H] \subseteq G_{n_0}$, and $\chi \equiv 1$ on $[H, H]$.

Proof. Because $K(x)$ is commutative and G_{n_0} is normal, $[H, H] \subseteq G_{n_0}$. For the rest, we need to show that if $w \in G_{n_0}$ and $y \in G \cap K(x)$, then $\chi(ywy^{-1}) = \chi(w)$. Let $w = 1 + w'$; then

$$\begin{aligned} \chi(ywy^{-1}) &= \chi(1 + yw'y^{-1}) \\ &= \psi \circ \tau(yw'y^{-1}x) = \psi \circ \tau(w'y^{-1}xy) \\ &= \psi \circ \tau(w'x) = \chi(w), \end{aligned}$$

since x and y commute.

There are $[H: G_n]$ different extensions of χ_x to H . Label them $\chi_{x;a}$, where a runs through the integers from 1 to $[H: G_{n_0}]$.

THEOREM 2. (a) $\rho_{x;a} = \text{Ind}_{H \rightarrow G} \chi_{x;a}$ is irreducible.

(b) $\rho_{x;a} \not\cong \rho_{x;b}$ unless $a = b$.

(c) The $\rho_{x;a}$ exhaust the irreducible representations of G containing χ .

Proof. (a) It suffices, from Theorem 6 of [7], to show that if $y \notin H$, then there is some $w \in H$ such that $ywy^{-1} \in H$ and $\chi_{x;a}(w) \neq \chi_{x;a}(ywy^{-1})$. Let $y = 1 + \beta_r \pi^r + \dots$; we may assume, since $y \notin H$ and we may multiply y by any element of H , that $r < n_0$ and $\beta_r \pi^r$ does not commute with x_1 (see Lemma 2). Let $w = 1 + w'$, with $w' \in P^{n-r-2}$. Then

$$\begin{aligned} \chi_{x;a}(ywy^{-1}) &= \psi \circ \tau(yw'y^{-1}x) \\ &= (\psi \circ \tau(w'x))(\psi \circ \tau(w'\beta_r \pi^r x - \beta_r \pi^r w'x)), \end{aligned}$$

as one sees by expanding $y^{-1} = [1 + (\beta_r \pi^r + \dots)]^{-1}$ in a series. Therefore $\chi_{x;a}(ywy^{-1}) = \chi_{x;a}(w)$ for all such w only if

$$\begin{aligned} 1 &= \psi \circ \tau(w'\beta_r \pi^r x - \beta_r \pi^r w'x) = \psi \circ \tau(w'(\beta_r \pi^r x - x\beta_r \pi^r)) \\ &= \psi \circ \tau(w'(\beta_r \pi^r x_1 - x_1 \beta_r \pi^r)), \end{aligned}$$

since, e.g., $w'\beta_r \pi^r(x - x_1) \in P^{m-1}$, and so $\psi \circ \tau(w'\beta_r \pi^r(x - x_1)) = 1$.

But since w' is any element of P^{n-r-1} , this implies that $\beta_r \pi^r$ and x_1 commute, a contradiction.

(b) If $\rho_{x;a} \cong \rho_{x;b}$ with $a \neq b$, then Theorem 7 of [7] implies that $\exists y \notin H$ such that for all $w \in H \cap yHy^{-1}$, $\chi_{x;a}(w) = \chi_{x;b}(ywy^{-1})$. We saw in (a) that this is impossible.

(c) The dimension of $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$ is $[G:G_{n_0}]$ and the dimension of each $\rho_{x;a} [G:H]$. As there are $[H:G_{n_0}]$ possible $\chi_{x;a}$'s, and as every irreducible ρ containing χ_x occurs in $\text{Ind}_{G_{n_0} \rightarrow G_n} \chi_x$ (by Frobenius reciprocity), the result follows from counting.

Next, we deal with Case (3). We continue with the notation as above. Let $H = G_{n_0}(G \cap D_0)$, where D_0 is the algebra defined in § 1.

THEOREM 3. *Suppose that $K(x_1)$ is unramified, with $[K(x_1):K] = p$. Then $D_{x_1} = D_0$. Let ρ_0 be any representation of $G \cap D_0$ such that $\rho_0|_{G_{n-1} \cap D_0}$ is a multiple of $\chi_x|_{G_{n-1} \cap D_0}$. Extend ρ_0 to a representation of letting $\rho_0(1 + \alpha\pi^r) = I$ if $p \nmid r$, $r \geq n_0$, and $\alpha \in k_0$. Then $\rho = \text{Ind}_{H \rightarrow G} \rho_0$ is irreducible, and $\rho|_{G_{n-1}}$ is a multiple of $\chi_x|_{G_{n-1}}$. If ρ'_0 is another representation of $D_0 \cap G$ satisfying the above condition and ρ' is the induced representation of G , then $\rho \cong \rho' \Leftrightarrow \rho_0 \cong \rho'_0$. Every irreducible representation of G which agrees with χ on G_{n-1} is obtained in this way.*

Proof. Since $K(x_1)$ is unramified, we have $p^2|_{-n-m}$, and $K(x_1) = K(\alpha_{-m-n}) = K_1$; thus $K(x_1) = K_0$, and $D_{x_1} = D_0$. Now the proof is the same as that of Theorem 2.3 of [1], except that, since $D_0 = \{y = \sum_{j=r}^{\infty} \beta_j \pi^j: \beta_j = 0 \text{ unless } p \nmid j\}$, one can simplify the notation and some of the arguments.

REMARK. The above theorem does not directly identify the representations ρ containing χ_x . The following observations let one make this identification. For definiteness, assume that $[K(x):K] = p^2$. From Lemma 1, $K(x)$ contains a field isomorphic with $K(x_1)$; the Skolem-Noether theorem (see, e.g., p. 166 of [8]) lets us assume (by conjugating x) that $K(x_1) \subseteq K(x)$. Then χ_x defines a character on $G_{n_0} \cap D_0$ which agrees with the norm map on $G_{n-1} \cap D_0$. As described in § IV of [4], we may use an appropriate extension of χ_x on a subgroup of $G \cap D_0$ to induce ρ_0 on $G \cap D_0$, and ρ_0 determines ρ . It is not hard to use this method to give an explicit description of the subgroup H_x of G such that ρ is induced by χ extended to H_x . In the next section, we shall see more complicated examples along the same lines.

We should perhaps add a remark on the generality of these results. Theorem 2 applies to general m (not just $m = p^2$); it thus permits the construction of many representations of G . So does

Theorem 1, but that theorem reduces the problem only trivially. Theorem 3 permits a reduction of the problem whenever $[K(x_1):K]$ is unramified, but one must find a general procedure for finding the irreducible representations of $(D_{x_1})^*$ to reap maximum benefit from this procedure.

3. We have reduced the problem to Case (4): $K(x_1)$ is a totally ramified extension of degree p . Thus $x_1^p \in K$. Because the map $\alpha \mapsto \alpha^p$ is an automorphism of k , one can see easily that there is an element $\beta\pi^{-m-n}$ with $\beta \in k_1 \cap K$ and $x_1^p = (\beta\pi^{-m-n})^p$. By conjugating with an element of D (or, equivalently, by replacing π with another prime element $\gamma\pi$, $\gamma \in k_1'$), we may assume that $x_1 = \beta\pi^{-m-n}$, or that $\alpha_{-m-n} \in K \cap k_1$. We keep to this assumption throughout this section and the next.

We divide Case 4 into three subcases, and we deal with the easiest one first.

LEMMA 4. *Let x generate a field of degree p ; let $y = \sum_{j=r}^{\infty} \beta_j \pi^j$ commute with x . Then $\beta_r \pi^r$ commutes with x_1 . Conversely, if $\beta_r \pi^r$ commutes with x_1 , then there is an element $y = \sum_{j=r}^{\infty} \beta_j \pi^j$ which commutes with x .*

Proof. This has the same proof as in Lemma 2, once one notes that the division algebras D_x and D_{x_1} are both p^3 -dimensional vector spaces over K , and that both K_x , K_{x_1} are totally ramified of degree p over K .

THEOREM 4. *Let x, x_1 generate fields of degree p . Then:*

- (1) $\chi_x|_{D_x \cap G_{n_0}}$ factors through the reduced norm $\nu_{D_x/K(x)}$;
- (2) if $y \in D_x \cap G$ and $z \in G_{n_0}$, then $\chi_x(yzy^{-1}) = \chi_x(y)$.

Thus one can extend χ_x to a 1-dimensional representation of $G_{n_0}(G \cap D_1)$, also to be called χ_x , which factors through the reduced norm on $G \cap D_1$. Let ξ be any irreducible representation of $G \cap D_1$ which is trivial on $G_{n_0} \cap D_1$. Then $\xi \otimes \chi_x$ is a representation of $G_{n_0}(G \cap D_1)$ which is equivalent to a multiple of χ_x on G_{n_0} . Let $\rho_{\xi, x} = \text{Ind}_{G_{n_0}(G \cap D_1) \rightarrow G} \xi \otimes \chi_x$. Then:

- (3) $\rho_{\xi, x}$ is an irreducible representation of G ;
- (4) $\rho_{\xi_1, x} \cong \rho_{\xi_2, x} \iff \xi_1 \cong \xi_2$;
- (5) every irreducible representation of G whose restriction to G_{n_0} contains χ_x is a $\rho_{\xi, x}$.

Proof. (1) If $y = 1 + y' \in D_1 \cap G_{n_0}$, then let xy' generate a field K' over K . We have (assuming that $[K':K] = p^2$)

$$\begin{aligned} \chi_x(y) &= \psi \circ \tau_{D/K}(xy') = \psi(\text{Tr}_{K'/K}(xy)) = \psi(\text{Tr}_{K_1/K}(\text{Tr}_{K'/K}(xy))) \\ &= \psi(\text{Tr}_{K_1/K}(x \text{Tr}_{K'/K_1} y)) = \psi(\text{Tr}_{K_1/K}(x \tau_{D_1/K_1} y')). \end{aligned}$$

This result also holds if $y' \in K_1$. But $\nu_{D_1/K_1}(y) \equiv 1 + \tau_{D_1/K_1} y' \pmod{P^n}$, and this shows that $\chi_x(y)$ depends only on $\nu_{D_1/K_1}(y)$ for $y \in D_1 \cap G_{n_0}$.

(2) If $z = 1 + z' \in G \cap D_1$, then (for y as above)

$$\chi_x(yzy^{-1}) = \psi \circ \tau(yz'y^{-1}x) = \psi \circ \tau(z'y^{-1}xy) = \psi \circ \tau(z'x) = \chi_x(z).$$

(3)–(5) The proof is just like that of Theorem 1, except that the counting argument is different. It is still routine; we omit details.

We are thus in the situation where $[K(x):K] = p^2$ and $K(x_1)$ is a totally ramified extension of degree p . This is the difficult case. We shall divide it into two subcases, depending on whether or not $K(x)$ is totally ramified. While the approach is similar in the subcases, and some lemmas could be combined, it seems clearer to divide the work.

If $K(x)$ contained a totally ramified extension (over K) of degree p , we could proceed just as in the tamely ramified case. In general, no such extension exists, and we must work harder.

For each r with $1 \leq r \leq n_0$, let x_r be an element of D such that

$$(1) \quad x_r \equiv x \pmod{P^{-m-n+r}};$$

$$(2) \quad [K(x_r):K] \text{ is as small as possible, subject to (1).}$$

Note that this notation is consistent with the definition of x_1 .

There is a number s such that $[K(x_s):K] = p$ and $[K(x_{s+1}):K] = p^2$. The general strategy of the proof is to show that for many purposes, we can use $K(x_s)$ as a substitute for a subfield of $K(x)$ of degree p over K .

We begin by investigating some properties of x_s .

LEMMA 5. *Let $y = \sum_{j=r}^{\infty} B_j \pi^j \in D$ generate a field of ramification index p over K . Then there is a conjugate (under G) of y , $\sum_{j=r}^{\infty} \beta_j \pi^j$, with $\beta_j = 0$ unless $p \mid j$.*

Proof. D_x certainly contains an unramified extension K'_0 of $K(y)$ with $[K'_0:K(y)] = p$; by the Skolem-Noether theorem, we may assume (by conjugating in D) that $K'_0 = K(y, \alpha)$, where $K(\alpha) = K_0$. But under this assumption, y commutes with α , and this implies directly that the β_j are 0 unless $p \mid j$. Since conjugating by an element of k'_1 or by a power of π gives an automorphism of K_0 , we may equally well assume that the conjugation is in G .

LEMMA 6. *Let $y = \sum_{j=r}^{\infty} \beta_j \pi^j$ generate a totally ramified extension of order p ; let $y_1 = \beta_r \pi^r$. Assume that $\beta_j = 0$ unless $p \mid j$ and that $[K(y_1):K] = p$, $\beta_r \in k$. Let $w = \beta' \pi^t$, $t > r$, be such that if $y' \equiv y + w \pmod{P^{t+1}}$, then $[K(y'):K] = p^2$, and assume further that $K(y + w)$ is*

not totally ramified over K . (Therefore $f(K(y+w)/K) = p$.) Then:

- (a) $p \nmid t$ and $\text{Tr}_{d/k_0} \beta' \notin k$;
- (b) if $y' \equiv y + w \pmod{P^{t+1}}$, then $f(K(y')/K) = p$.

Proof. Note that $K_0 \subseteq D_y$. Since multiplying by an element of K does not affect the hypotheses, we may assume that $0 < r < p^2$. Let F be the minimal polynomial for y :

$$F(X) = \sum_{j=0}^p a_j X^j, \quad a_p = 1 \quad \text{and} \quad a_j \in \mathfrak{O}_K, \quad \text{all } j.$$

By considering the valuation (plus the fact that $F(y) = 0$), one sees that $a_0 \in P^{pr}/P^{pr+1}$ and that the terms $a_j y^j$ ($1 \leq j \leq p-1$) are in $P^{p(r+1)}$. (Note that $p \mid r$.)

Now consider $F(y+w)$; again, valuation considerations show that

$$F(y+w) \equiv x^{p-1} (\text{Tr}_{d/k_0} \beta') \pi^t \pmod{P^{(p-1)r+t+1}}.$$

If $p \nmid t$ and $\text{Tr}_{d/k_0} \beta' \neq 0$, then this last calculation shows that $F(y+w)$ is an element which generates an ideal P^{t_0} with $(t_0, p) = 1$; it follows that $K(y+w)$ is totally ramified. On the other hand, if $\text{Tr}_{d/k_0} \beta' = 0$, then β' is of the form $\gamma^{\sigma'} - \gamma$ for some $\gamma \in d_0$ (where $p \mid r$ and $p^2 \nmid r$), and

$$(1 - \gamma \pi^{t-r}) y (1 - \gamma \pi^{t-r})^{-1} \equiv y + w \pmod{P^{t+1}}.$$

Thus there is an element $y' \equiv y + w \pmod{P^{t+1}}$ with $[K(y'):K] = p$, a contradiction. It follows that $p \mid t$.

We know from general structure theory (and the fact that $K_0 K(y)$ is a maximal unramified field in D_y) that $\exists u \in D_y$ such that $u^p = y$ and conjugation by u induces an automorphism generating $\text{Gal}(K_0/K)$. Then $u = \sum_{j=s}^{\infty} \delta_j \pi^j$, where $ps = r$, $\delta_s \in k_1 \cap K$, $\delta_s^p = \beta_r$, and $\delta_j = 0$ unless $j \equiv s \pmod{p}$. (This last follows because otherwise conjugating by u will not be an automorphism of K_0 .) Because $\beta_r \in k$ and taking p th powers is an automorphism of k , $\delta_s \in k$. Let $u' = u + \gamma_0 \pi^{s+(t-r)}$; note that $p \mid t - r$. Then

$$(u')^p \equiv y + \delta_s (\gamma_0 + \gamma_0^{\sigma} + \cdots + \gamma_0^{(\sigma^{p-1})^s}) \pi^t \pmod{P^{t+1}}.$$

So if β' can be written as $\beta_s (\gamma_0 + \gamma_0^{\sigma} + \cdots + \gamma_0^{(\sigma^{p-1})^s})$, then there exists $y' \equiv y + w \pmod{P^{t+1}}$ such that y' is a p th power in D ; then, of course, $[K(y'):K] < p^2$, and we again have a contradiction. Thus β' is not of that form.

Let $\sigma' = \sigma^s$; let $T\gamma = \gamma + \gamma^{\sigma'} + \cdots + \gamma^{(\sigma')^{p-1}}$, $\gamma \in k_1$. Note that T is k -linear and $\delta_s \in k$, so that $\beta' \notin \text{Range}(T)$. Thus we need to show that $\text{Range } T = \{\delta \in d: \text{Tr}_{d/k_0} \delta \in k\}$. Call this last described set V . Then V is a vector space over k of dimension $p^2 - p + 1$, since

$\text{Ker}(Tr_{d/k_0})$ is a $(p-1)$ -dimensional k_0 -space. On the other hand, since σ' generates $\text{Gal}(d/k)$, $T\delta \in k \Leftrightarrow \sigma'(T\delta) = T\delta \Leftrightarrow \delta = \delta^{(\sigma')^p} \Leftrightarrow \delta \in k_0$, and on k_0 , $T = Tr_{k_0/k}$. Hence

$$\text{Ker } T = \text{Ker}(Tr_{k_0/k}),$$

a $(p-1)$ -dimensional k -subspace of k_0 . Thus $\text{Range } T$ and V have the same dimension. Since, however, $Tr_{d/k_0} \circ T = Tr_{d/k}$, we have $\gamma \in \text{Range } T \Rightarrow Tr_{d/k_0} \gamma \in k$, or $\text{Range } T \subseteq V$. This proves (a).

For (b), note that if $y' \equiv y + w \pmod{\pi^t}$, then $F(y') \equiv x^{p-1}(Tr_{d/k_0}\beta)\pi^t \pmod{P^{(p-1)r+t+1}}$; thus $\alpha(y')^a F(y')(\pi^{p^2})^b = Tr_{d/k_0}\beta \pmod{P}$ for appropriate $\alpha \in k_0$ and $a, b \in Z$. By Hensel's lemma, $K(y')$ contains an unramified extension of order p .

Now we return to the problem of representations. We let $\chi = \chi_x$, where $[K(x):K] = p^2$, $[K(x_i):K] = p$, $K(x_i)$ is totally ramified over K , and $K(x)$ is not totally ramified. We choose x_s as in the discussion before Lemma 5. Then x satisfies the hypotheses of Lemma 5, and we may thus assume that $x = \sum_{j=m-n}^{\infty} \alpha_j \pi^j$, with $\alpha_j = 0$ when $p \nmid j$. Moreover, x_s and $w = x - x_s$ satisfy the hypotheses of Lemma 6. Note that $p \mid s$.

We need to show that certain elements of D_{x_s} are "close" to elements of $K(x)$, so that we will be able to use $K(x_s)$ as if it were a subfield of $K(x)$. The next lemma gives what is needed.

LEMMA 7. *We can find elements t' , t and u such that:*

- (a) *t' is a prime element of $K(x)$, and u is a prime element of D_{x_s} ;*
- (b) *u commutes with $t' \pmod{P^{p+s+1}}$; that is, $ut' - t'u \in P^{p+s+1}$;*
- (c) *$u^p = t$, and t is a prime element of $K(x_s)$ with $t' \equiv t \pmod{P^{p+s}}$.*

Proof. Recall that $x = \alpha_{jp} \pi^{jp} + \text{higher order terms}$, where $jp = -m - n$. Choose h so that $jh \equiv 1 \pmod{p}$; then we may find $a \in K$ such that $t' = ax^h$ generates P^p . In fact, we may assume that $t' = \pi^p + \text{higher order terms}$. Let $t = ax_x^h$. Then $t' \equiv t \pmod{P^{p+s}}$. Since t generates a field of degree p over K , we can find a p th root u of t in D with $u = \sum_{j=1}^{\infty} \beta_j \pi^j$, $0 \neq \beta_1 \in k$, as was done in Lemma 6. Then, of course, u commutes with $t' \pmod{P^{p+s+1}}$, since $u \in P$ and $t' - t \in P^{p+s}$.

Now assume for simplicity that s is even; let $s = 2s_0$. We define

$$H_x = G_{n_0}(D_{x_s} \cap G_{n_0-s_0})(G \cap K(x)).$$

Note that coset representatives for $D_{x_s} \cap (1 + P^j)/D_{x_s} \cap (1 + P^{j+1})$ are

given by elements $1 + \gamma u^r$, $\gamma \in k_0$ and u as in Lemma 7, and that coset representatives for $K(x) \cap (1 + P^{jp})/K(x) \cap (1 + P^{j^{p+1}})$ are given by elements $1 + \gamma(t')^j$, $\gamma \in d_0$. (If $p \nmid r$, $K(x) \cap (1 + P^r) = K(x) \cap (1 + P^{r+1})$.)

The next lemma is the crucial one for extending χ to H_x .

LEMMA 8. (a) If $n_0 - s_0 < j < n_0$, then

$$[H_x, H_x] \cap (1 + P^j) / [H_x, H_x] \cap (1 + P^{j+1}) \cong \begin{cases} (1 + P^j)/(1 + P^{j+1}) & \text{if } p \nmid j \text{ and there is a multiple} \\ & \text{of } p \text{ between } n_0 - s_0 \text{ and } j; \\ \{1\} & \text{otherwise;} \end{cases}$$

(b) If $z \in [H_x, H_x] \cap (1 + P^{n_0})$, then $\chi_x(z) = 1$.

REMARK. Part (b) of the lemma says that χ_x has an extension to H_x . Any such extension must be trivial on $[H_x, H_x]$, of course; part (a) therefore enables us to count the extensions of χ_x .

Proof. We use the notation of Lemma 7.

(a) Modulo G_{n_0} , the commutators generating $[H_x, H_x]$ are of the form

$$(f, g)' = (1 + f(t'))(1 + g(u))(1 + f(t'))^{-1}(1 + g(u))^{-1},$$

where f is a polynomial over K_0 with integral coefficients and no constant term, while g is a polynomial with integral coefficients in K_0 and all terms of degree $\geq n_0 - s_0$. (The coefficients of $g(u)$ need not commute with u .) Substitute

$$(1 + f(t'))^{-1} = 1 - f(t') + f(t')^2 + \cdots,$$

$$(1 + g(u))^{-1} = 1 - g(u) + g(u)^2 + \cdots,$$

in the formula for $(f, g)'$. Since $g(u)^2 \in P^{n_0}$ and we are calculating modulo P^{n_0} , we can drop all powers of $g(u)$ beyond the first, setting

$$\begin{aligned} (f, g)' &\equiv 1 + \sum_{i=0}^{n_0} (-1)^i [f(t')^{i+1}g(u) - f(t')g(u)f(t')^i] \\ &\equiv 1 + \sum_{i=0}^{n_0} (-1)^i [f(t)^{i+1}g(u) - f(t)g(u)f(t)^i]. \end{aligned}$$

This last congruence holds because $f(t) \equiv f(t') \pmod{P^{2+s}}$ and $g(u) \in P^{n_0-s_0}$.

Any term in $g(u)$ whose degree is a multiple of p can be written as a power of t , since $t = u^p$; that term commutes with f . Let g_0 be the polynomial obtained from g by omitting all terms whose degree is a multiple of p . Then

$$(f, g)' \equiv 1 + \sum_{i=0}^{n_0} (-1)^i [f(u^p)^{i+1}g_0(u) - f(u^p)g(u)f(u^p)^i],$$

and every term in this expression is of degree prime to p and greater than $n_0 - s_0$. To complete the proof of (a), note that if $n_0 - s_0 \leq j_0 p < j_0 p + j_1 = j < n_0$, then the elements

$$(1 + (\beta t')^{j_0})(1 + \gamma u^{j_1})(1 + (\beta t')^{j_0})^{-1}(1 + (\gamma u^{j_1}))^{-1},$$

where β, γ vary over k_0 , give the coset representatives.

(b) This is more complicated; the idea is to replace computations of χ_x with computations of χ_{x_s} . Define $(f, g)'$ as in (a), and set

$$(f, g)'_1 = \sum_{i=0}^{\infty} (-1)^i [f(t')^{i+1} g(u) - f(t') g(u) f(t')^i].$$

If $w' = \prod_{j=1}^h (f_j, g_j)'$, let $w'_1 = \sum_{j=1}^h (f_j, g_j)'_1$, $w'_2 = w'(1 + w'_1)^{-1}$. One can think of w'_1 as containing all the terms in w' that are linear in some g_j . Similarly, we define

$$(f, g) = (1 + f(t))(1 + g(u))(1 + f(t))^{-1} + (1 + g(u))^{-1},$$

$$(f, g)_1 = \sum_{i=0}^{\infty} (-1)^i [f(t)^{i+1} g(u) - f(t) g(u) f(t)^i];$$

note that we have replaced t' with t . If w' is as above, let

$$w = \prod_{j=1}^h (f_j, g_j), \quad w_1 = \sum_{j=1}^h (f_j, g_j)_1, \quad w_2 = w(1 + w_1)^{-1}.$$

Now:

(1) $w \in G_{n_0}$. For $w' \in G_{n_0}$, and (as we saw in the proof of part (a)) replacing t' by t in the commutators changes terms only by elements of P^{n_0} .

(2) $w'_2 \equiv w_2 \pmod{G_{n+1}}$. For $w'_2 - 1$ is a sum of terms involving products of two or more $g_j(u)$ and one or more $f_i(t')$, and $w_2 - 1$ can be written as a similar sum involving the $g_j(u)$ and $f_i(t)$. Since $g_{j_1}(u)g_{j_2}(u) \in P^{n-s+1}$ and $t' - t \in P^s$, we have $w'_2 - w_2 \in P^{n+1}$.

(3) $w'_2 \in G_{n-s+1}$. This follows from the calculation in (2), since $g_{j_1}(u)g_{j_2}(u) \in G_{n-s+1}$.

(4) $\chi_x(w'_2) = \chi_{x_s}(w'_2) = \chi_{x_s}(w_2)$, from (2) and (3), since χ_x and χ_{x_s} agree on G_{n-s+1} .

(5) $\chi_x(1 + w'_1) = 1$. For

$$\chi_x(1 + w'_1) = \chi(\tau_{D/K}(w'_1 x)),$$

and, because x and t' commute,

$$\begin{aligned} \tau(w'_1 x) &= \tau \left(\sum_{j=1}^h \sum_{i=0}^{\infty} (-1)^i (f_j(t')^{i+1} g_j(u) x - f_j(t') g_j(u) f_j(t')^i x) \right) \\ &= \tau \left(\sum_{j=1}^h \sum_{i=0}^{\infty} (-1)^i (g_j(u) x f_j(t')^{i+1} - g(u) f_j(t')^i x f_j(t')) \right) \\ &= 0. \end{aligned}$$

(6) $\chi_{x_s}(1 + w_1) = 1$, by a calculation like that in (5), since x_s and t commute.

(7) $\chi_x(w') = \chi_{x_s}(w)$, from (4), (5) and (6).

(8) Thus we need only compute $\chi_s(w)$. But w is a commutator in $D_{x_s} \cap (1 + P^{n_0})$, and therefore $w = 1 + w_0$, with $\tau_{D_{x_s}/K(x_s)} w_0 \equiv 0 \pmod{P^{n+1}}$. Hence $\tau(w_0 x_s) \equiv 0 \pmod{P^{n+1}}$, and hence $\chi_{x_s}(w_0) = 1$.

To finish the proof of (b), we need only mention that all commutators in $H_x \cap G_{n_0}$ are products of the commutators just considered with commutators of the form $z = z_1 z_2 z_1^{-1} z_2^{-1}$, $z_1, z_2 \in D_{x_s} \cap G_{n_0 - s_0}$. Then $z \in G_{n-s+1}$, and so $\chi_x(z) = \chi_{x_s}(z) = 1$ for the same reasons as in (4) and (8) above.

We label the different extensions of χ_x to H_x with integers; a typical one is $\chi_{x,a}$. Let pj_1 be the smallest multiple of p that is greater than $n_0 - s_0$. We say that $\chi_{x,a}$ and $\chi_{x,b}$ are equivalent if they agree on $K(x) \cap H_x$. (Thus they may differ on elements $1 + \eta w^r$, $n_0 - s_0 \leq r < pj_1$.)

It will be convenient in what follows to have a reference additive character $\psi^{(s)}$ defined on $K(x_s)$ which corresponds to ψ on K . (Recall: ψ is an additive character. See §1.) We let $\psi^{(s)}$ be any character on $K(x_s)$, trivial on P^{-m+p} , whose restriction to K is ψ . Note that if ψ_k is the character defined on k by $\psi_k(\gamma) = \psi(\gamma\pi^{-m})$, then $\psi^{(s)}(\gamma t^{-p}) = \psi_k(\gamma)$, $\forall \gamma \in k$.

THEOREM 5. (a) If $\chi_{x;a}$ is any extension of χ_x to H_x , then $\text{Ind}_{H_x \rightarrow G} \chi_{x;a} = \rho_{x;a}$ is irreducible.

(b) Inequivalent $\chi_{x;a}$ give rise to inequivalent $\rho_{x;a}$, and equivalent $\chi_{x;a}$ give rise to equivalent $\rho_{x;a}$.

(c) Any irreducible representation of G whose restriction to G_{n_0} contains χ_x is equivalent to some $\rho_{x;a}$.

Proof. (a) We need to show that if $z = 1 + \sum_{j=r}^{\infty} \gamma_j \pi^j$, $r \geq 1$, is such that $\chi_{x;a}(zyz^{-1}) = \chi_{x;a}(z)$ whenever y, zyz^{-1} are both in H_x , then $z \in H_x$. Suppose otherwise. We may then assume that $zz' \notin G_{r+1}$ whenever $z' \in H_x$.

Let $y = 1 + y_0$, $y_0 \in P^{n_0}$. Then $\chi_{x;a}(y) = \chi_x(y)$ and $\chi_{x;a}(zyz^{-1}) = \chi_x(zyz^{-1}) = \chi_{x^{-1}xz}(y)$, so that $z^{-1}xz \equiv z \pmod{P^{-n_0-m}}$. In particular, $\gamma_r \pi^r$ commutes with x_1 . From Lemma 4 (applied to x_s), there is an element $z_0 = 1 + \gamma_r \pi^r + \text{higher order terms}$ with $z \in D_{x_s}$. As $zz_0^{-1} \in G_{r+1}$, we may not have $z_0 \in H_x$. Therefore $r < n_0 - s_0$. Moreover, if $p|r$, then there is an element $q' = 1 + \gamma_r \pi^r + \text{higher order terms}$ with $z' \in K(x)$. We conclude that $p \nmid r$.

Thus far, we have dealt only with χ_s ; now we analyze $\chi_{x;a}$ more carefully. On $D_{x_s} \cap G_{n_0}$, χ_{x_s} factors through $\nu_{D_{x_s}/K(x_s)}$. The proof of

this fact parallels that of Theorem 1. Define the character ψ_0 on $K(x_s)$ by $\psi_0 = \psi \circ Tr_{K(x_s)/K}$. If $1 + y \in D_{x_s} \cap G_{n_0}$, then

$$\nu_{D_{x_s}/K(x_s)}(1 + y) \equiv 1 + \tau_{D_{x_s}/K(x_s)} y \pmod{P^{n+1}},$$

and so $\chi_{x_s}(1 + y) = \psi_0(x_s \tau_{D_{x_s}/K(x_s)} y)$ depends only on $\nu(y)$.

Let ω be a 1-dimensional representation of $D_{x_s} \cap G$ which agrees with χ_{x_s} on $D_{x_s} \cap G_{n_0}$. Then $\chi_x \omega^{-1}$ is trivial on $D_{x_s} \cap G_{n-s}$. On $D_{x_s} \cap G_{n_0-s_0}$, we may write

$$\chi_{x;a} \omega^{-1}(1 + y) = \psi^{(s)} \tau_{D_{x_s}/K(x_s)}(y y_0),$$

where

$$y_0 = \sum_{j=-n+s-m}^{\infty} \beta_j \pi^j \in D_{x_s}.$$

Let $y_1 = \beta_{-n+s-m} u^{-n+s-m}$, and write $\delta = \beta_{-n+s-m}$; let $x - x_s = \gamma \pi^{-n+s-m} +$ higher order terms.

Since $x - x_s$ and y_1 both describe the behavior of $\chi_{x;a} \cdot \omega^{-1}$ on G_{n-s} , they should be related. We now describe the relationship. If $y = 1 + \beta u^{n-s} \in D_{x_s}$, then

$$\begin{aligned} \chi_{x;a} \cdot \omega^{-1}(y) &= \chi_{x-x_s}(y) = \psi \circ \tau_{D/K}((y-1)(x-x_s)) \\ &= \psi \circ \tau_{D/K}(\beta \gamma^{\sigma^{n-s}} \pi^{-m}) = \psi_k \circ Tr_{d/k}(\beta \gamma^{\sigma^{n-s}}) \\ &= \psi_k \circ Tr_{d/k}(\gamma \beta^{\sigma^{s-n}}) = \psi_k \circ Tr_{d_0/k} \beta^{\sigma^{s-n}} (Tr_{d/d_0} \gamma), \end{aligned}$$

since $\beta \in d_0$. Also,

$$\begin{aligned} \chi_{x;a} \cdot \omega^{-1}(y) &= \psi^{(s)} \circ \tau_{D_{x_s}/K(x_s)}((y-1)y_0) \\ &= \psi^{(s)} \circ \tau_{D_{x_s}/K(x)}(\beta \delta^{\sigma^{n-s}} \pi^{-m}) \\ &= \psi_k \circ Tr_{d_0/k}(\beta \delta^{\sigma^{n-s}}) = \psi_k \circ Tr_{d_0/k}(\beta^{\sigma^{s-n}} \delta). \end{aligned}$$

Thus $\delta = Tr_{d/d_0} \gamma$. By Lemma 6, $\delta \notin k_0$. Therefore y_1 generates an unramified extension of order p over $K(x_s)$.

We use this fact to restrict the element z further. On $H_x \cap D_{x_s}$, conjugating by z commutes with ω . Therefore it commutes with $\chi_{x;a} \cdot \omega^{-1}$. Let $y \in D_{x_s} \cap G_{n-s-r}$. Then

$$\chi_{x;a} \cdot \omega^{-1}(y) = \chi_{x;a} \cdot \omega^{-1}(zyz^{-1}),$$

and a calculation like that at the start of the proof shows that

$$z^{-1} y_0 z \equiv y_0 \pmod{P^{-n_0-s_0-m}}.$$

In particular, y_1 and $\gamma_r \pi^r$ commute. But the residue class degree of y_1 is divisible by p ; it follows that $p \mid r$. This contradicts our earlier demonstration that $p \nmid r$, and part (a) follows.

(b) If $\rho_{x;a}$ and $\rho_{x;b}$ are equivalent, then $\exists z: \chi_{x;a}(y) = \chi_{x;b}(zyz^{-1})$ whenever y and zyz^{-1} are both in H_x . In particular, this holds for all $y \in G_{n_0}$. Write z as in part (a). The analysis given there shows that

$z \in D_{x_s} \cap G_{n_0-s}$, that $p \nmid r$, and that conjugation by z changes $\chi_{x;a}$ on elements of the form $1 + \beta u^{n-s-r}$. We know that $p \nmid n-s-r$. If $n-s-r > pj_0$ (where pj_0 is the smallest multiple of p greater than n_0-s_0), then $\chi_{x;a}$ and $\chi_{x;b}$ must agree, from (a) of Lemma 8. Hence $n-s-r < pj_0$. It is now easy to check that z normalizes H_x and that $\chi_{x;a}$ and $\chi_{x;b}$ are equivalent.

That equivalent $\chi_{x;a}$ produce equivalent $\rho_{x;a}$ can be proved by picking z as in the above paragraph. It is also implied by (c); see below.

(c) This is a matter of counting. Observe first that if $z \in G$ fixes χ_x , then (modulo H_x) $z \in D_{x_s} \cap G_{n_0-s}$. (This follows from the analysis in part (a); the only difference is that in the second part of the analysis, we are concerned only with elements $y \in D_{x_s} \cap G_{n_0}$.) Conversely, a straightforward calculation shows that any such z does fix χ_x . Hence the multiplicity of χ_x in $\rho_{x;a}|G_{n_0}$ is $[D_{x_s} \cap G_{n_0-s} : D_{x_s} \cap G_{n_0-s_0}]/[K(x) \cap G_{n_0-s} : K(x) \cap G_{n_0-s_0}]$. By Frobenius reciprocity, this is also the multiplicity of $\rho_{x;a}$ in $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$. We also know that there are $[G \cap K(x) : G_{n_0} \cap K(x)]$ equivalence classes of $\chi_{x;a}$'s and that

$$\begin{aligned} \dim \rho_{x;a} &= [G : H_x] \\ &= [G/G_{n_0} : (G \cap K(x))(G_{n_0-s_0} \cap D_{x_s}) / (G_{n_0} \cap K(x))(G_{n_0} \cap D_{x_s})]. \end{aligned}$$

Since $(G \cap K(x)) \cap (G_{n_0-s_0} \cap D_{x_s}) = G_{n_0-s_0} \cap K(x)$ and $(G_{n_0} \cap K(x)) \cap (G_{n_0} \cap D_{x_s}) = G_{n_0} \cap K(x)$, we have

$$\begin{aligned} \dim \rho_{x;a} &= [G/G_{n_0} : G \cap K(x)/G_{n_0} \cap K(x)] \\ &\quad \times [G_{n_0-s_0} \cap D_{x_s}/G_{n_0} \cap D_{x_s} : G_{n_0-s_0} \cap K(x)/G_{n_0} \cap K(x)]^{-1}. \end{aligned}$$

Thus the different $\chi_{x;a}$ account for a subspace in $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$ of dimension

$$\begin{aligned} &[G/G_{n_0}][G \cap K(x)/G_{n_0} \cap K(x)]^{-1}[G_{n_0-s_0} \cap D_{x_s}/G_{n_0} \cap D_{x_s}]^{-1} \\ &\quad \times [G_{n_0-s_0} \cap K(x)/G_{n_0} \cap K(x)][G \cap K(x)/G_{n_0} \cap K(x)] \\ &\quad \times [D_{x_s} \cap G_{n_0-s}/D_{x_s} \cap G_{n_0-s_0}][K(x) \cap G_{n_0-s} : K(x) \cap G_{n_0-s_0}]^{-1}. \end{aligned}$$

But $[G_{n_0-s_0} \cap K(x)/G_{n_0} \cap K(x)]/[G_{n_0-s} \cap K(x)/G_{n_0-s_0} \cap K(x)] = 1$ and $[D_{x_s} \cap G_{n_0-s_0}/D_{x_s} \cap G_{n_s}]/[D_{x_s} \cap G_{n_0-s}/D_{x_s} \cap G_{n_0-s_0}] = 1$. (Note that $p \neq 2$, since $p|n$ and n is odd; also, $p|s$. Hence $p|s_0$, and the number of multiples of p between n_0-s_0 and n_0 equals the number between n_0-s and n_0-s_0 . From this, the first relation follows; the second is even easier to verify.) That is, the $\chi_{x;a}$ account for a subspace in $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$ of dimension $[G : G_{n_0}] = \dim \text{Ind}_{G_{n_0} \rightarrow G} \chi_x$.

That proves (c). Also, as we have accounted for all the representations in $\text{Ind}_{G_{n_0} \rightarrow G} \chi_x$, it must be the case that $\chi_{x;a}$ equivalent to $\chi_{x;b}$ implies $\rho_{x;a} \cong \rho_{x;b}$.

4. We still need to deal with the subcase in which $[K(x_1) : K] = p$,

$[K(x):K] = p^2$, and $K(x)$ is totally ramified. This situation is similar to the last one we dealt with, but there are complications.

We define x_s as in the previous section. We begin with three lemmas similar to Lemmas 6, 7 and 8.

LEMMA 9. *Let y be as in Lemma 6. Let $w = \beta\pi^t$, $t > r$, be such that if $y' \equiv y + w \pmod{P^{t+1}}$, then $[K(y'):K] = p^2$, and suppose that $K(y + w)$ is totally ramified over K . Then:*

- (a) $p \nmid t$ and $\text{Tr}_{d/k_0}\beta \neq 0$;
- (b) if $y' \equiv y + w \pmod{P^{t+1}}$, then $K(y')$ is totally ramified over K ;
- (c) by perhaps changing the choice of prime element, one can also arrange that $\text{Tr}_{d/k_0}\beta \in k$ (without upsetting the other normalizations already made).

Proof. The proofs of (a) and (b) are like that of Lemma 6, with easy modifications. For (c), notice first that if $\delta \in k_0$, then $(\delta\pi)^p = \delta'\pi^p$, $\delta' \in k$. Thus using $\pi' = \delta^{-1}\pi$ as prime element does not affect our assumptions about y . On the other hand, this change means that $w = \varepsilon\beta(\pi')^t$, where $\varepsilon = \delta\delta^\sigma \cdots (\delta^\sigma)^{t-1}$; of course, $\text{Tr}_{d/k_0}\varepsilon\beta = \varepsilon\text{Tr}_{d/k_0}\beta$, and $N_{k_0/k}\varepsilon = (N\delta)^t$. Because d^* is cyclic, we can modify $\text{Tr}_{d/k_0}\beta$ by any element of k_0 whose norm in k is a t th power. Since $N_{k_0/k}$ is onto (as a map from k to itself), we have $N_{k_0/k}(\text{Tr}_{d/k_0}\beta) = N_{k_0/k}\beta_0$, $\beta_0 \in k_0$. That is, we can modify $\text{Tr}_{d/k_0}\beta$ to be $(\beta_0(\text{Tr}_{d/k_0}\beta)^{-1})\text{Tr}_{d/k_0}\beta = \beta_0 \in k$, since $N(\beta_0)\text{Tr}_{d/k_0}\beta^{-1}$ is a p th power.

LEMMA 10. *We can find elements u' and u such that:*

- (a) u' is a prime element of $K(x)$ and u is a prime element of D_{x_s} ;
- (b) $u' \equiv u \pmod{P^s}$.

Proof. Let u' be any prime element of $K(x)$. Then u' commutes with $x_s \pmod{P^{-n-m+s}}$, and Lemma 4 plus an easy induction shows that there is a prime element u of D_{x_s} with $u \equiv x' \pmod{P^s}$.

Assume that s is even, $s = 2s_0$. Define $H_x = G_{n_0}(G_{n_0-s_0} \cap D_{x_s})(G \cap K(x))$.

LEMMA 11. (a) *If $n_0 - s_0 < j < n$, then $[H_x, H_x] \cap G_j/[H_x, H_x] \cap G_{j+1}$ has as coset representatives the elements $1 + \alpha u^p$, $\text{Tr}_{k_0/k}\alpha = 0$. When $j = n_0 - s_0$, the quotient is trivial.*

- (b) *If $z \in [H_x, H_x] \cap G_{n_0}$, then $\chi_x(z) = 1$.*

Proof. This is almost the same as the proof of Lemma 8; the modifications are easy.

From now on, the analogy with the procedure in § 3 is not nearly so close. It is true that we could now extend χ_x to a group H_x (as in Theorem 5) and induce to G , and that the result would be an irreducible representation of G . The trouble is that we would not get enough representations of G in this way. The way out of this difficulty is to vary the group H_x . The details, as we shall see, are notationally unpleasant.

We begin by analyzing the characters χ_x and χ_{x_s} more closely. Define the additive character $\psi^{(s)}$ on K_{x_s} as was done in § 3, and let ω be a 1-dimensional representation of $D_{x_s} \cap G$ which agrees with χ_{x_s} on $D_{x_s} \cap G_{n_0}$. We may extend ω to $G_{n_0}(D_{x_s} \cap G)$ by letting it equal χ_{x_s} on G_{n_0} . Then ω is trivial on all commutators of $D_{x_s} \cap G_{n_0-s_0}$, and $\chi_x \omega^{-1}$ is trivial on G_{n-s} . Hence if ϕ is any 1-dimensional representation of $D_{x_s} \cap G_{n_0-s_0}$ which agrees with $\chi_x \omega^{-1}$ on $D_{x_s} \cap G_{n_0}$, then we can extend χ_x to $G_{n_0}(D_{x_s} \cap G_{n_0-s_0})$ by letting it equal $\phi \omega$ on $D_x \cap G_{n_0-s_0}$. There is an element $w \in D_{x_s}$ such that

$$\phi(1+z) = \psi^{(s)} \circ \tau_{D_{x_s}/K(x_s)}(zw), \quad z \in D_{x_s} \cap P^{n_0-s_0}.$$

We write $\phi = \chi_w^{(s)}$. In fact,

$$w = \sum_{j=-n-m+s}^{\infty} \beta_j u^j,$$

and, if $x - x_s = \alpha' \pi^{-n-m+s} + \text{higher order terms}$, then a calculation like that in part (a) of Theorem 5 shows that

$$\beta_{-n-m+s} = \text{Tr}_{k/k_0}(\alpha').$$

From Lemma 9, $\beta_{-n-m+s} \in k$.

LEMMA 12. (a) *There are q^{s_0} orbits of the $\chi_w^{(s)}$ under the action of $D_{x_s} \cap G_{n_0-s}$.*

(b) *If $3s_0 \geq n_0$, then $w = \sum_{j=-n-m+s}^{\infty} \beta_j u^j$, with $\beta_j \in k$ for $j \leq -n_0 - m$*

(c) *If $3s_0 < n_0$, let t be the smallest integer with $2t \geq n_0 - s_0$. Then $\beta_j \in K \cap k_1$ for $j \leq -n_0 \neq s_0 - m - t$.*

Proof. (a) The element $\beta_{-n+s-m} u^{-m-n+s}$ generates a totally ramified field of degree p over $K(x_s)$. It follows (as in Lemma 1) that the elements of $D_{x_s} \cap G_{n_0-s}/D_{x_s} \cap G_{n_0-s_0}$ commuting with χ'_w are (congruent mod $P^{n_0-s_0}$ to) elements of $K(x_s, w)$, and there are q^{s_0} such elements. The orbit calculation follows easily.

(b) Let u, u' be as in Lemma 10. If $y \in D_{x_s} \cap G_{n_0-s_0}$ and $v' \in K(x) \cap G_{s_0}$, then $\chi_x(v' y v'^{-1} y^{-1}) = 1$, from Lemma 11. But $\chi_{x_s}(v' y v'^{-1} y^{-1}) = 1$ as well. For if $y = 1 + y_0$, $v' = 1 + v_0$, then

$$\chi_{x_s}(v'yv'^{-1}y^{-1}) = \psi \circ \tau_{D/K} x_s \left(\sum_{j=0}^{\infty} (-1)^j (v_0 y_0 v_0^j - y_0 v_0^{j+1}) \right).$$

But

$$\tau_{D/K}(x_s v_0 y_0 - x_s y_0 v_0) = \tau_{D/K}(x_s v_0 y_0 - y_0 x_s v_0) = 0,$$

while the other terms in the parentheses are in P^{n-s} , so that we can replace x_s by x . (It is here that we need the estimate of the size of s .) As v_0 and x commute, a similar calculation shows that the remaining terms have trace 0.

Thus $\chi'_w(v'yv'^{-1}y^{-1}) = 1$. Recall that v' is given by a polynomial in u' ; let v be the corresponding polynomial in u . Then

$$\chi'_w(vy v^{-1}y^{-1}) = 1,$$

since $vy v^{-1}y^{-1}$ and $v'yv'^{-1}y^{-1}$ agree modulo G^{n-s} . Since χ'_w is trivial on these commutators, we must have $w \in K(u) \bmod P^{-m-n_0}$.

The proof of (c) is essentially the same as that of (b).

Note. Parts (b) and (c) of Lemma 12 may seem paradoxical. After all, we may vary $x - x_s$ freely on P^{-m-n+s} ; how can we restrict the β_j in any way? The answer is that the element $u \in D_{x_s}$ depends on x ; varying x means a new choice of u , and hence a different expression for w .

Define $\chi_w^{(s)}(y) = \psi^{(s)} \circ \tau_{D_{x_s}/K(x_s)}(y - 1)$, $y \in G_{n_0-s_0}$, if $w \in D_{x_s} \cap P^{-n-m+s}$, and let $H_{x,0} = G_{n_0}(G_{n_0-s_0} \cap D_{x_s})(G \cap K(x))$. We know that there are extensions of χ_x to $H_{x,0}$. Let $\chi_{x,0}$ be one, and let $\chi_{x,0}\omega^{-1}|D_{x_s} \cap G_{n_0-s_0} = \chi_{w_1}^{(x)}$. Write $w_1 = w_{1;0} + w_{1;1}$, where $w_{1;0} \in K(x_s)$ and $w_{1;1} \in P^{-m-n_0+1}$ or $P^{-m-n_0-s_0-t+1}$, depending on whether (b) or (c) of Lemma 12 applies. Choose one element from each of the orbits in Lemma 12(a); a typical element is $w = w_1 + w_b$, $w_b \in P^{-m-n+s} \cap D_{x_s}$. For the w_1 -orbit, we may, and do, take $w_b = 0$.

Suppose that $v' \in K(x) \cap G$, and $y \in D_{x_s} \cap G_{n_0-s_0}$. In investigating whether the induced representations we shall construct are irreducible, we need to know when yv' commutes with $\chi_{x;0} \cdot \chi_{w_b}^{(s)}$ on $D_{x_s} \cap G_{n_0-s_0}$. Write $z(\chi)$ for the conjugate action of z on χ . Since v' commutes with $\chi_{x;0}$, we have

$$\begin{aligned} (yv')(\chi_{x;0} \cdot \chi_{w_b}^{(s)})(\chi_{x;0} \cdot \chi_{w_b}^{(s)})^{-1} &= y(\chi_{x;0} \cdot (yv')(\chi_{w_b}^{(s)})) \cdot (\chi_{x;0} \cdot \chi_{w_b}^{(s)})^{-1} \\ &= y(\chi_{x;0} \cdot \omega^{-1})(yv')(\chi_{w_b}^{(s)})(\chi_{x;0} \cdot \omega^{-1} \cdot \chi_{w_b}^{(s)})^{-1} \\ &= y(\chi_{w_1}^{(s)})\chi_{(yv'w_b v'^{-1}y^{-1})} \cdot (\chi_{w_1}^{(s)} \cdot \chi_{w_b}^{(s)})^{-1} \\ &= \chi_{(yw_{1;0}y^{-1})}^{(s)}\chi_{(yw_{1;1}y^{-1})}^{(s)}\chi_{yw'w_b v'^{-1}y^{-1}}^{(s)}\chi_{w_1-w_b}^{(s)}. \end{aligned}$$

It is easy to check that y commutes with $\chi_{w_{1;1}}^{(s)}$; for instance, if Lemma 12(b) applies, then

$$yw_{1;1}y^{-1} - w_{1;1} \in P^{-m-n_0+1+n_0-s_0} = P^{-m-s_0+1};$$

and thus $\chi_{yw_{1;1}y^{-1}-w_{1;1}}^{(s)}$ is trivial on $D_{x_s} \cap G_{n_0-s_0}$. Similarly, $\chi_{v'w_{1;0}v'^{-1}}^{(s)} = \chi_{w_{1;0}}^{(s)}$. We can therefore write the last expression as

$$\chi_{yv'w_{1;0}v'^{-1}y^{-1}}^{(s)} \chi_{(yv'w_bv'^{-1}y^{-1})}^{(s)} \chi_{-w_{1;0}}^{(s)} \chi_{-w_b}^{(s)}.$$

Thus yv' commutes with $\chi_{x;0} \cdot \chi_{w_b}^{(s)}$ iff yv' commutes with $w_{1;0} + w_b$ (modulo a sufficiently high power of ρ).

LEMMA 13. *Let $u_1 = (w_{1;0} + w_b)^{r_1} \pi^{-mr_2}$ be a prime element in $K(w_{1;0} + w_b, x_s)$, so that $u_1 = \sum_{j=1}^{\infty} \varepsilon_j u^j$, with $\varepsilon_j \in k$ for $1 \leq j \leq n_0 - s$ and $\varepsilon_j \in k_0$ generally. Let $u'_1 = \sum_{j=1}^{\infty} \varepsilon_j (u')^j$. Then elements in $G \cap K(u'_1)$ commute with the representation on $G_{n_0}(G_{n_0-s_0} \cap D_{x_s})$ given by χ_s on G_{n_0} and extended (as indicated above) by $\chi_w^{(s)}$, $w = w_{1;0} + w_{1;1} + w_b$.*

Let $H_{x;b} = G_{n_0}(G_{n_0-s_0} \cap D_{x_s})(G \cap K(u'_1))$. The 1-dimensional representations described above have exactly $q^{n_0-s_0-1}$ extensions to $H_{x;b}$.

Proof. We checked above that the elements in $G \cap K(u'_1)$ commute with the given representation; they also commute with other. Therefore the representation extends. The last statement holds because $[H_{x;b}; G_{n_0}(G_{n_0-s_0} \cap D_{x_s})] = q^{n_0-s_0-1}$.

We are now (finally!) ready to describe the irreducible representations of G containing χ_x .

THEOREM 6. *Let $\chi_{x;b;a}$ be one of the 1-dimensional representations of $H_{x;b}$ described in Lemma 13. (Recall that one chooses exactly one $w = w_1 + w_b$ from each of the orbits in Lemma 12(a).) Let $\rho_{x;b;a} = \text{Ind}_{H_{x;b} \rightarrow G} \chi_{x;b;a}$. Then:*

- (a) $\rho_{x;b;a}$ is irreducible;
- (b) the $\rho_{x;b;a}$ are mutually disjoint;
- (c) the $\rho_{x;b;a}$ exhaust the irreducible representations of G containing χ_x .

Proof. For (a), let $H'_{x;w} = G_{n_0}(G_{n_0-s} \cap D_{x_s})(G \cap K(u'_1))$, $J_x = G_{n_0-s} D_{x_s}$. The elements of G which fix $\chi_x|_{G_{n_0}}$ are all in $H'_{x;w}$, and $H'_{x;w}$ normalizes $H_{x;w}$. Moreover, the coset representatives for $H'_{x;w}/H_{x;w}$ can all be chosen from among coset representatives of $J_x/J_x \cap G_{n_0-s_0}$. We need only show, therefore, that any element of J_x (modulo $J_x \cap G_{n_0-s_0}$) which commutes with $\chi_{x;w;a}$ lies in $H_{x;w}$. From Lemma 11, there are exactly $q^{n_0-s_0}$ such elements, and elements of $H_{x;w}$ account for them. Thus $\rho_{x;w;a}$ is irreducible.

The proof of (b) is nearly identical; we need to show that no element of $H'_{x;w}$ can transform one $\chi_{x;b;a}$ into a $\chi_{x;b';a'}$, and this is evident because w, w' are in different orbits. Finally, (c) is the usual counting argument.

Theorems 1-6 solve the problem of finding the irreducible repre-

sentations of G , except for some minor details to be dealt with in the next section.

5. We still need to remove the assumptions on the parity of n and s . As the procedure is the same as in [1] and [4], we omit proofs. (Some of the counting arguments above must be modified; this, too, is not hard.)

If n is even, but $n - s$ is odd, let $n_0 = 1 + (n/2)$, and let $(H_0)_x$ or $(H_0)_{x;w}$ be the group described in § 2 or 3 (so that if, e.g., $K(x_1) = K(x) = p^2$, then $(H_0)_x = G_{n_0}(K(x) \cap G)$). We shall write $(H_0)_x$ in what follows, but the remarks apply equally to $(H_0)_{x;w}$. Consider the bilinear form

$$B_x: (\alpha, \beta) \longmapsto \chi_x(1 + \alpha\pi^{n_0-1})(1 + \beta\pi^{n_0-1})(1 + \alpha\pi^{n_0-1})^{-1}(1 + \beta\pi^{n_0-1})^{-1}$$

taking $d \times d$ to the unit circle. It is antisymmetric, and the radical is $\{\beta: K(x_1) \text{ contains an element } \sum_{j=n_0-1}^{\infty} \beta_j \pi^j \text{ with } \beta_{n_0-1} = \beta\}$. Therefore the elements $1 + \beta\pi^{n_0-1}$ with $\beta \in \text{Rad}(B_x)$ are already contained in $(H_0)_x$. Let h be a maximal isotropic subspace of B_x ; we let H_x be the group generated by $(H_0)_x$ and the elements $(1 + \beta\pi^{n_0-1})$ with $\beta \in h$. Two extensions of χ_x to H_x are equivalent if they are equivalent on $(H_0)_x$. The representations $\rho_{x;a} = \text{Ind}_{H_x \rightarrow G} \chi_{x;a}$ are irreducible and exhaust the representations of G whose restrictions to G_{n_0} contain χ_x , and inequivalent $\chi_{x;a}$ give rise to inequivalent $\rho_{x;a}$.

If $n - s$ is even, the procedure is similar; let $n_0 - s_0 = (n - s)/2 + 1$ and consider a maximal isotropic subspace h' of

$$B'_x: (\alpha, \beta) \longmapsto \chi_{x-x_s}((1 + \alpha u^{n_0-s_0-1})(1 + \beta u^{n_0-s_0-1}) \\ \times (1 + \alpha u^{n_0-s_0-1})^{-1}(1 + \beta u^{n_0-s_0-1})^{-1}) ;$$

add the elements $(1 + \alpha u^{n_0-s_0-1})$, $\alpha \in h'$, to $(H_0)_x$, and call extensions of χ to $(H_0)_x$ equivalent if they agree on H_0 .

There is another description of this procedure, in which if, e.g., n is odd, one tensors $\chi_{x;a}$ on $(H_0)_x$ with a Heisenberg-like representation of G_{n_0-1} ; see [4].

We also need to extend representations from G to D^x . The method is generally like that in [1]; here, for example, is a description for the case $[K(x_1): K] = p^2$. We may assume that the maximal unramified extension in $K(x)$ is contained in K_1 . Let x' be a prime element of $K(x)$, and let

$$J_x = \{(x')^j: j \in \mathbb{Z}\} \cdot \{k_1 \cap K^x(x)\} .$$

Then conjugation by elements of J_x fixes $\chi_{x;a}$ on H_x . Tensor the $\chi_{x;a}$ with representations of J_x and induce to find the irreducible representations of D^x .

The case considered in §4 is somewhat different; we merely sketch the argument, since the result is not quite satisfactory. We use the notation of §4. Let $\rho_{x;b;a}$ be induced from $\chi_{x;b;a}$ on $H_{x;b}$, and let F be the trace character of $\chi_{x;b;a}$. We note that u'_1 normalizes $H_{x;b}$. Let $y \in G$. If no G -conjugate of y lies in $H_{x;b}$, then no G -conjugate of $u'_1 y (u'_1)^{-1}$ lies in $H_{x;b}$; hence $F(y) = F(u'_1 y u'^{-1}_1) = 0$. On the other hand, it is not hard to see that if $y \in H_{x;b}$, then there is a prime element $v = \sum_{j=1}^{\infty} \eta_j \pi^j$, $\eta_1 = 1$, such that $F(vy v^{-1}) = F(y)$. (For instance, if $y \in G_{n_0}$, then $\chi_{(u')^{-1} x u'} = \chi_x$, and so $F(u'_1 y (u'_1)^{-1}) = F(y)$.) Since $vu' \in G$, we conclude that, e.g., conjugation by u'_1 fixes F . Thus one can extend $\rho_{x;b;a}$ to J_x . Now one proceeds as in [1]. It would, of course, be preferable to have a concrete description to J_x .

The procedure for constructing representations described in this paper also applies to division algebras D whose index m is the product of two primes. (If neither prime is p , then, of course, we are in the situation of [1] or [3].) There is some difference in the consideration of cases because the behavior of $Tr_{k_0/k}$ depends on whether or not $[k_0:k]$ is prime to p . The procedure also applies to some representations in the case where m is a power of p ; for instance, if $m = p^r$ and $[K(x):K(x)] = p^{r-1}$, one can find the representations of G containing χ_x by methods like those of §§3 and 4.

The general case is murkier. The problems with extending the procedures of §§3 and 4 are twofold: the approximation lemma analogous to Lemmas 7 and 10 becomes more difficult to prove, and one needs to deal with certain finite-dimensional representations of H_x which are hard to describe. I hope to return to this topic in future papers.

Two other matters deserve mention. In [6], Langlands predicted a correspondence between representations of D and certain representations of the Weil group of K . This correspondence has been verified in the tamely ramified case (see [5]). It is of considerable interest to know whether the correspondence holds more generally.

Finally, there is the question of finding the characters of the representations. In the tamely ramified case, the computations are found in [4]. The wildly ramified case seems much harder.

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Natália Bebiano, On the evaluation of permanents	1
David Borwein and Bruce Brigham Watson, Tauberian theorems between the logarithmic and Abel-type summability methods	11
Leo George Chouinard, II, Hermite semigroup rings	25
Kun-Jen Chung, Remarks on nonlinear contractions	41
Lawrence Jay Corwin, Representations of division algebras over local fields. II	49
Mahlon M. Day, Left thick to left lumpy—a guided tour	71
M. Edelstein and Mo Tak Kiang, On ultimately nonexpansive semigroups	93
Mary Rodriguez Embry, Semigroups of quasinormal operators	103
William Goldman and Morris William Hirsch, Polynomial forms on affine manifolds	115
S. Janakiraman and T. Soundararajan, Totally bounded group topologies and closed subgroups	123
John Rowlay Martin, Lex Gerard Oversteegen and Edward D. Tymchatyn, Fixed point set of products and cones	133
Jan van Mill, A homogeneous Eberlein compact space which is not metrizable	141
Steven Paul Plotnick, Embedding homology 3-spheres in S^5	147
Norbert Riedel, Classification of the C^* -algebras associated with minimal rotations	153
Benedict Seifert, Combinatorial and geometric properties of weight systems of irreducible finite-dimensional representations of simple split Lie algebras over fields of 0 characteristic	163
James E. Simpson, Dilations on locally convex spaces	185
Paolo M. Soardi, Schauder bases and fixed points of nonexpansive mappings	193
Yoshio Tanaka, Point-countable k -systems and products of k -spaces	199
Fausto A. Toranzos, The points of local nonconvexity of starshaped sets	209
Lorenzo Traldi, The determinantal ideals of link modules. I	215
P. C. Trombi, Invariant harmonic analysis on split rank one groups with applications	223
Shinji Yamashita, Nonnormal Blaschke quotients	247