ON ULTIMATELY NONEXPANSIVE SEMIGROUPS

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A semigroup G of continuous selfmappings of a metric space (X, d) is called ultimately nonexpansive if for every u, v in X and $\alpha > 0$, there is an f in G such that for all g in G, $d(fg(u), fg(v)) \leq (1+\alpha)d(u, v)$. It is shown that if G is an ultimately nonexpansive commutative semigroup of self-mappings, then G has a fixed point when any one of the following conditions is satisfied: (1) X is a reflexive Banach space and each orbit under G is precompact; (2) X is a finite dimensional Banach space and there is a point in X with a bounded orbit; (3) X is a reflexive, locally uniformly convex Banach space having a point with a precompact orbit.

1. **Introduction.** Let G be a family of selfmappings of a metric space (X, d) and suppose that G is a semigroup under composition. For any $x \in X$, $G(x) = \{g(x): g \in G\}$ is called the orbit of x under G and a point $z \in X$ such that $G(z) = \{z\}$ is called a fixed point of G. Fixed point properties of semigroups having various contractivity properties have been investigated by several writers (cf. [1], [3], [5], [6], [7], [8], [9], [10]). In [5], for example, an asymptotically nonexpansive semigroup G (see definition given below), satisfying certain conditions, was shown to have a fixed point if X is a finite dimensional Euclidean space and, for some $x \in X$, co $G(x)$ does not contain any affine flat of positive dimension. Similarly, in [8] a semigroup of selfmappings of a closed, bounded convex set in a uniformly convex Banach space—called eventually nonexpansive—was shown to have a fixed point provided certain additional hypotheses were satisfied.

In this paper we consider semigroups of mappings related to those of [8] but much more general; (e.g., members of the semigroup need not be lipschitzian).

DEFINITION. A semigroup G of continuous selfmappings of a metric space (X, d) is called ultimately nonexpansive if for every pair of points $u, v \in X$ and for every $\alpha > 0$, there is an $f \in G$ such that for all $g \in G$,

$$(1) \qquad d(fg(u), fg(v)) \leq (1 + \alpha)d(u, v) .$$

It is called asymptotically nonexpansive (cf. [5]) if (1) above is satisfied with $\alpha = 0$.

It is the purpose of this paper to show that some of the fixed point theory for semigroups of nonexpansive mappings carries over to this much larger class of mappings. In particular this is the case

when the orbits under G and the space X satisfy additional hypotheses; e.g., that some orbits be precompact and that X be locally uniformly convex and reflexive. Thus if X has the above property and at least one orbit is precompact then the existence of a fixed point follows.

2. **Ultimately nonexpansive semigroups in general metric spaces.** Certain properties of ultimately nonexpansive semigroups, some of them similar to those discussed in [5], and partially relying on the ideas and results of [5], are needed for the proof of our main results. We begin by reproducing the definition of "G-closure point" as given in [5].

DEFINITION. The G-closure of X, where (X, d) is a metric space, consists of all points x in X such that for some $z \in X$, any $\varepsilon > 0$ and any $f \in G$ there is a $g \in G$ such that

$$(2) \qquad\qquad d(fg(z), x) < \varepsilon .$$

The G-closure of X is denoted by X^G and it is a generalization of a similar notion for a single mapping as given in [2].

PROPOSITION 1. *Let G be an ultimately nonexpansive commutative semigroup of selfmappings of a metric space (X, d),*
  (a) *if x is a G-closure point, then z can be replaced by x in* (2) *(and any reference to z can be deleted from the definition of X^G);*
  (b) *if* cl $G(x)$ *is compact then it contains a G-closure point.*

Proof. (a) Let $z \in X$ be such that for $\varepsilon > 0$ and $f \in G$ there is a $g_\varepsilon \in G$ such that $d(fg_\varepsilon(z), x) < \varepsilon/2(1 + \varepsilon)$. Since G is ultimately nonexpansive there is an $h_\varepsilon \in G$ (corresponding to the points $g_\varepsilon f(z)$, $x \in X$ and ε) such that for all $k \in G$,

$$(3) \qquad d(kh_\varepsilon g_\varepsilon f(z), kh_\varepsilon(x)) \leqq (1 + \varepsilon)d(g_\varepsilon f(z), x) < \frac{\varepsilon}{2} .$$

On the other hand, there is a mapping $t \in G$ (corresponding to $h_\varepsilon g_\varepsilon ff \in G$) such that

$$(4) \qquad\qquad d(fth_\varepsilon g_\varepsilon f(z), x) < \frac{\varepsilon}{2} .$$

From (3) and (4) above we conclude that

$$d(fth_\varepsilon(x), x) \leqq d(fth_\varepsilon(x), fth_\varepsilon g_\varepsilon f(z)) + d(fth_\varepsilon g_\varepsilon f(z), x) < \varepsilon$$

proving the assertion.

(b) Suppose not. Then for every $u \in \operatorname{cl} G(x)$ there is an $\varepsilon_u > 0$ and a $g_u \in G$ such that $d(g_u f(x), u) \geqq \varepsilon_u$ for all $f \in G$. The open balls $B(u, \varepsilon_u)$ form an open cover of $\operatorname{cl} G(x)$. Hence there is a finite subcover of such balls with centers at $u_1, u_2 \cdots, u_n$, say. Corresponding each u_i, $i = 1, \cdots, n$, there is to a $g_{u_i} \in G$. Now $g_{u_1} \cdot g_{u_2} \cdot \cdots \cdot g_{u_n}(x)$ is in some $B(u_i, \varepsilon_{u_i})$ contradicting the defining property of g_{u_i}; hence the result.

PROPOSITION 2. *Let G and (X, d) be as in Proposition 1. Then,*

(a) *if $x \in X^G$ then the restriction $G|_{G(x)}$ of G to $G(x)$ is a semigroup of isometries;*

(b) *if $\operatorname{cl} G(x)$ is compact then $G|_{\operatorname{cl} G(x)}$ is a semigroup of isometries (of $\operatorname{cl} G(x)$ onto itself).*

Proof. (a) Since each member of G maps $G(x)$ into itself it is clear that $G|_{G(x)}$ is a semigroup again. Since the conclusion concerning isometries was obtained in [5] for asymptotically nonexpansive semigroups it suffices to show that $G|_{G(x)}$ is such a semigroup. Suppose $G|_{G(x)}$ fails to be asymptotically nonexpansive. Then, clearly, members $f_1(x)$, $f_2(x)$ of $G(x)$ must exist such that for all $g \in G$ there is a $t \in G$ with

$$\delta = d(tgf_1(x), tgf_2(x)) - d(f_1(x), f_2(x)) > 0 .$$

Let $\varepsilon = \delta(1 + d(f_1(x), f_2(x)))^{-1}$. From the definition of an ultimately nonexpansive semigroup it follows that corresponding to ε and $f_1(x)$, $f_2(x)$ there is an $h \in G$ such that for all $f \in G$,

$$d(fhf_1(x), fhf_2(x)) \leqq (1 + \varepsilon)d(f_1(x), f_2(x)) .$$

Hence,

$$d(tghf_1(x), tghf_2(x)) \leqq (1 + \varepsilon)d(f_1(x), f_2(x)) .$$

Since $x \in X^G$ and each member in G is continuous there is a $g_0 \in G$ such that

$$\max \{d(tghg_0 f_i(x), tgf_i(x)): i = 1, 2\} < \frac{\varepsilon}{2} .$$

It follows that

$$\begin{aligned}
d(tgf_1(x), tgf_2(x)) \leqq\ & d(tgf_1(x), tghg_0 f_1(x)) + d(tghg_0 f_1(x), tghg_0 f_2(x)) \\
& + d(tghg_0 f_2(x), tgf_2(x)) \\
<\ & \frac{\varepsilon}{2} + (1 + \varepsilon)d(f_1(x), f_2(x)) + \frac{\varepsilon}{2} \\
=\ & d(f_1(x), f_2(x)) + \varepsilon(1 + d(f_1(x), f_2(x))) \\
=\ & d(tgf_1(x), tgf_2(x))
\end{aligned}$$

implying that $\delta = 0$.

(b)   This follows from (a) (and the fact that an isometry on a compact set is onto).

PROPOSITION 3.   *Let X and G be as in Proposition 1.   Then,*

(a)   *if $u \in \operatorname{cl} G(x)$ with $x \in X^G$, and $v \in X$, $v \neq u$, there is a neighborhood U of u such that for any positive integer n there exists an $f \in G$ with the property that for all $g \in G$ and $u' \in U$,*

$$d(fg(u'), fg(v)) \leqq \left(1 + \frac{1}{n}\right) d(u', v) ;$$

(b)   *if $\operatorname{cl} G(x)$ is compact, with $x \in X^G$, and $v \notin \operatorname{cl} G(x)$, then for any positive integer n, there is a $f \in G$ such that for all $g \in G$ and $u \in \operatorname{cl} G(x)$, $d(fg(u), fg(v)) \leqq (1 + 1/n)d(u, v)$.*

Proof.   (a)   Let $f \in G$ be such that for all $g \in G$, $d(fg(u), fg(v)) \leqq (1 + 1/4n)d(u, v)$. Let $U = \{u' \in \operatorname{cl} G(x): d(u', u) < (1/4n)d(u', v)\}$. Then,

$$d(fg(u'), fg(v)) \leqq d(fg(u'), fg(u)) + d(fg(u), fg(v))$$

$$\leqq d(u', u) + \left(1 + \frac{1}{4n}\right)d(u, v)$$

$$< \frac{1}{4n}d(u', v) + \left(1 + \frac{1}{4n}\right)(d(u', u) + d(u', v))$$

$$= \left(\frac{1}{4n} + \left(1 + \frac{1}{4n}\right)^2\right)d(u', v) \leqq \left(1 + \frac{1}{n}\right)d(u', v) .$$

(b)   Let $\{U_i | i = 1, 2, \cdots, k\}$ be a finite cover of $\operatorname{cl} G(X)$ with the property that corresponding to each U_i and for any positive integer n, there is an $f_i \in G$ such that for all $g \in G$, $u_i \in U_i$.

$$d(f_i g(u_i), f_i g(v)) \leqq \left(1 + \frac{1}{n}\right)d(u_i, v) .$$

(Such a cover exists by (a) above and the compactness of $\operatorname{cl} G(x)$.) Let $f = f_1 \circ f_2 \circ \cdots \circ f_k$ then f is as desired. Indeed, if $u \in \operatorname{cl} G(x)$ then $u \in U_i$ for some $i \in 1, 2, \cdots, k$. Let $\hat{f}_i$ denote the product of $k - 1$ members of $\{f_1, \cdots, f_n\}$ obtained by deleting f_i. Then

$$d(fg(u), fg(v)) = d(f_i \hat{f}_i g(u_i), f_i \hat{f}_i g(v)) \leqq \left(1 + \frac{1}{n}\right)d(u_i, v)$$

$$= \left(1 + \frac{1}{n}\right)d(u, v) .$$

3.   Centers of orbits.   In this section A will denote a nonempty closed convex subset of a Banach space X.   Throughout, $G(x)$ will

denote a precompact orbit under an ultimately nonexpansive semi-group of selfmappings of X.

In the proofs of the main results we shall make use of certain properties of Chebyshev centers of precompact orbits $G(x)$.

By a Chebyshev center C_A of $G(x)$ with respect to A we understand the subset of A consisting of points u such that the radius $R(u)$ of the smallest closed ball about u containing $G(x)$ is minimal.
Thus, if

$$R(u) = \sup\{\|u - g(x)\| : g \in G\}$$

and

$$R_A = \inf\{R(u) : u \in A\}$$

then

$$C_A = \{u \in A : R(u) = R_A\} \, .$$

The set C_A is obviously also the intersection of closed balls about points of $G(x)$, each of radius R_A. Hence C_A is closed convex and bounded; it is nonempty if X is reflexive.

LEMMA 1. *If $G(x)$ is precompact, with $G(x) \neq \{x\}$, and $x \in X^G$, then* cl $G(x)$ *and C_A are disjoint for any closed convex subset A of X.*

Proof. Suppose not and let $u \in (\mathrm{cl}\, G(x)) \cap C_A$. Then $R(u) = R_A = \mathrm{diam}\, G(x)$ and since members of G act as isometries on cl $G(x)$ it readily follows that cl $G(x) \subset C_A$. This, in turn, implies that

$$R(w) = R_A = \mathrm{diam}\, G(x)$$

for all w in $\overline{\mathrm{co}}\, G(x)$, the closed convex hull of $G(x)$; against the known property of compact convex sets K in a Banach space to contain a nondiametral point (i.e., a point z such that $\sup\{\|z - w\| : w \in K\} < \mathrm{diam}\, K$).

LEMMA 2. *Let C_A be the Chebyshev center of a precompact orbit $G(x)$, where $x \in X^G$, with respect to a closed convex set $A \subset X$. Suppose that $G(c)$ is containd in A for some $c \in C_A$ and $\{f_n : n = 1, 2, \cdots\}$ has the property that*

$$\|f_n g(c) - f_n g(u)\| \leqq \left(1 + \frac{1}{n}\right)\|c - u\|$$

for all $u \in \mathrm{cl}\, G(x)$ and some $g \in G$. If $f_n g(c) \to \bar{c}$ then $G(\bar{c}) \subset C_A$.

Proof. Since $\{f_n g(c)\} \subset G(c) \subset A$ it suffices to show that for any $h, k \in G$ and any $\varepsilon > 0$,

(5) $$\|k(\bar{c}) - h(x)\| < R_A + \varepsilon \, .$$

By continuity of k there is a positive integer n such that

$$\|k(\bar{c}) - f_n g k(c)\| < \frac{\varepsilon}{2} \ ;$$

and since h is an isometry on $\mathrm{cl}\, G(x)$ there is a $g_n \in G$ such that

$$\|f_n g g_n k h(x) - h(x)\| < \frac{\varepsilon}{2} \ .$$

Hence

$$\|k(\bar{c}) - h(x)\| \leqq \|k(\bar{c}) - f_n g k(c)\| + \|f_n g k(c) - f_n g g_n k h(x)\|$$
$$+ \|f_n g g_n k h(x) - h(x)\|$$
$$< \frac{\varepsilon}{2} + \left(1 + \frac{1}{n}\right)\|c - g_n h(x)\| + \frac{\varepsilon}{2}$$
$$\leqq \left(1 + \frac{1}{n}\right)R_A + \varepsilon \ .$$

This being true for all $n = 1, 2, \cdots$, (5) follows.

LEMMA 3. *Let C_A be the Chebyshev center of precompact orbit $G(x)$, where $x \in X^G$, with respect to a closed convex set $A \subset X$ and suppose that $c \in C_A$. If $G(c)$ is a precompact subset of A then there is a $\bar{c} \in \mathrm{cl}\, G(c)$ such that $G(\bar{c}) \subset C_A$.*

Proof. By Lemma 1, $c \notin \mathrm{cl}\, G(x)$. Hence, by Proposition 3(b), a sequence $\{f_n g(c)\}$ can be defined as in the preceding lemma. Since it is contained in $G(c)$, it contains a subsequence which converges to some $\bar{c} \in \mathrm{cl}\, G(x)$. Thus, after a suitable relabelling of the indices if required, the preceding lemma applies and the conclusion follows.

LEMMA 4. *Let X be a strictly convex Banach space and suppose that, for some $x \in X$, $G(x)$ is precompact. Then the Chebyshev center C of $G(x)$ with respect to X consists of one point at most.*

Proof. Suppose that $u, v \in C$ and let $w \in \mathrm{cl}\, G(x)$ be such that

$$\left\|\frac{u + v}{2} - w\right\| = R_x \ .$$

Then $2R_x = \|u + v - 2w\| \leqq \|u - w\| + \|v - w\| \leqq 2R_x$. Hence $\|u - w\| = \|v - w\| = R_x$. By strict convexity there is a λ such that $u - w = \lambda(v - w)$. If $R_x = 0$ there is nothing to prove (for then $G(x) = \{x\} = C$). Otherwise, we have $|\lambda| = 1$; and, again by $R_x \neq 0$, we must have $\lambda = 1$. So $u = v$, proving the assertion.

4. Main results.

THEOREM 1. *Let X be a reflexive Banach space and G an ultimately nonexpansive commutative semigroup of selfmappings of X. If each orbit under G is precompact then G has a fixed point.*

Proof. Let $C_1 = X$ and x_1 be an arbitrary point in X^G. Suppose that for all ordinals $\mu < \lambda$, C_μ and x_μ are defined with C_μ closed convex and $G(x_\mu) \subset C_\mu$. If λ is not a limit ordinal, i.e., if an immediate predecessor $\lambda - 1$ exists, then $C_{\lambda-1}$ is defined and contains the precompact orbit $G(x_{\lambda-1})$ with $x_{\lambda-1} \in X^G$. We define C_λ as the Chebyshev center of $G(x_{\lambda-1})$ and choose $x_\lambda \in C_\lambda$ such that $G(x_\lambda) \subset C_\lambda$. This is possible by Lemma 3, Proposition 3(b) and Proposition 2(b).

If λ is a limit ordinal then

$$C_\lambda = \bigcap \{C_\mu \colon \mu < \lambda\} \ .$$

By reflexivity of X, $C_\lambda \neq \varnothing$. We proceed to show that an $x \in X^G$ exists such that $G(x) \subset C_\lambda$. Let $c \in C_\lambda$ be arbitrary. Then $c \in G_{\mu+1}$ for any $\mu < \lambda$; so c is in the Chebyshev center of $G(x_\mu)$ and Lemma 3 applies to the effect that a $\bar{c} \in \operatorname{cl} G(c)$ exists such that $G(\bar{c}) \subset G_{\mu+1}$. Now $\operatorname{cl} G(\bar{c})$ is also contained in $C_{\mu+1}$, and by compactness it contains a $\bar{c}_{\mu+1} \in X^G$ such that $G(\bar{c}_{\mu+1}) \subset \operatorname{cl} G(\bar{c}) \subset C_{\mu+1}$. Let $c_\lambda \in \bigcap \{\operatorname{cl} G(\bar{c}_\mu) \colon \mu < \lambda\} \subset C_\lambda$. Then $\operatorname{cl} G(c_\lambda) \subset C_\lambda$ and a $\bar{c}_\lambda \in \operatorname{cl} G(c_\lambda)$ exists such that $\bar{c}_\lambda \in X^G$. Choose $x_\lambda = \bar{c}_\lambda$. Thus a transfinite sequence of pairs $\{C_\lambda, x_\lambda\}$ arises. Since $C_{\lambda+1} \subset C_\lambda \backslash G(x_\lambda)$ whenever $G(x_\lambda)$ is not a singleton, a λ^* (with card $\lambda^* \leqq$ card 2^X) must exist such that

$$G(x_{\lambda*}) = \{x_{\lambda*}\} \ ,$$

so $x_{\lambda*}$ may serve as the desired fixed point.

THEOREM 2. *Let X be a finite-dimensional Banach space and G a commutative ultimately nonexpansive semigroup of selfmappings of X. If $G(y)$ is bounded for some $y \in X$ then G has a fixed point.*

Proof. Since $\operatorname{cl} G(y)$ is compact it contains an $x \in X^G$. Let C_1 be the Chebyshev center of $G(x)$ with respect to X and let $c_1 \in C_1$. As in Lemma 2, with A replaced by X, let $f_n \in G$, $n = 1, 2, \cdots$, be such that

$$\| f_n g(c_1) - f_n g(u) \| \leqq \left(1 + \frac{1}{n}\right) \| c_1 - u \|$$

for all $u \in \operatorname{cl} G(x)$ and some $g \in G$. (This is possible by Lemma 1 and Proposition 3(b).)

To apply Lemma 2 it suffices to show that a subsequence $\{f_{n_i} g(c_1)\}$

converges and to this end it is enough to check that $\{f_n g(c_1)\}$ is bounded. Now

$$\| f_n g(c_1) - f_m g(c_1) \|$$
$$\leq \| f_n g(c_1) - f_n gh(x) \| + \| f_n gh(x) - f_m gh(x) \| + \| f_m gh(x) - f_m g(c_1) \|$$
$$\leq \left(1 + \frac{1}{n}\right) \| c_1 - h(x) \| + \operatorname{diam} G(x) + \left(1 + \frac{1}{m}\right) \| c_1 - h(x) \|$$
$$< \infty .$$

Here h is an arbitrary member of G. Suppose then that $\bar{c}_1$ is a cluster point of $\{f_n g(c_1)\}$. Then, by Lemma 2, $G(\bar{c}_1) \subset C_1$.

Since C_1 is closed convex and bounded the center C_2 of $G(\bar{c}_1)$ with respect to C_1 exists and replacing y and X by $\bar{c}_1$ and C_1 respectively the preceding argument applies again to the effect that a $\bar{c}_2$ exists such that $G(\bar{c}_2) \subset C_2$. Proceeding inductively a sequence of centers $\{C_n\}$ and orbits $G(\bar{c}_n)$ lying within them arises, and it suffices to show that a positive integer $n \leq \dim X$ exists such that $C_n = C_{n+1}$. To this end we note that C_1 clearly cannot contain an open subset and that, similarly C_{k+1} cannot obtain any subset which is relatively open with respect to C_k (unless $C_{k+1} = C_k$). Restated, either $C_{k+1} = C_k$ or $\dim \operatorname{sp} C_{k+1} < \dim \operatorname{sp} C_k$ where $\operatorname{sp} S$ denotes the affine span of the set $S \subset X$ and $\dim$ denotes the dimension.

Clearly then after $n \leq \dim X$ steps we must arrive at a point ξ such that $G(\xi) = \xi$.

THEOREM 3. *Let G be an ultimately nonexpansive commutative semigroup of selfmappings of a reflexive locally uniformly convex Banach space X and suppose that $G(x)$ is precompact for some $x \in X^G$. Then the Chebyshev center of $G(x)$ is a singleton c and $G(c) = \{c\}$; i.e., c is fixed under G.*

Proof. Since X is strictly convex the Chebyshev center is a singleton c, by Lemma 4. To show that $G(c) = \{c\}$ let $f_n \in G$ be such that, for all $u \in \operatorname{cl} G(x)$,

$$\| f_n g(u) - f_n g(c) \| \leq \left(1 + \frac{1}{n}\right) \| u - c \|$$

for a fixed $g \in G$ and all $n = 1, 2, \cdots$ (cf. Proposition 2(b)). It suffices to show that $f_n g(c) \to c$.
Now

$$\| f_n g(c) - u \| \leq \| f_n g(c) - f_n gh(x) \| + \| f_n gh(x) - h(x) \| + \| h(x) - u \| .$$

Choosing $g, h \in G$ so that $\| h(x) - u \| < (1/2n) R_x$ and $\| f_n gh(x) - h(x) \| < (1/2n) R_x$, we obtain

$$\| f_n g(c) - u \| \leqq \left(1 + \frac{2}{n} \right) R_x .$$

Hence

$$\left\| \frac{f_n g(c) + c}{2} - u \right\| \leqq \frac{1}{2} \| f_n g(c) - u \| + \frac{1}{2} \| c - u \|$$

$$\leqq \frac{1}{2} \left(1 + \frac{2}{n} \right) R_x + \frac{1}{2} \| c - u \| .$$

Replacing u by $u_n \in \mathrm{cl}\, G(x)$ such that

$$\| z_n - u_n \| = R(z_n)$$

where $z_n = (f_n g(c) + c)/2$ then, whenever $z_n \neq c$,

$$R_x < R(z_n) \leqq \frac{1}{2} \left(1 + \frac{2}{n} \right) R_x + \frac{1}{2} \| c - u_n \|$$

$$\leqq \left(1 + \frac{1}{n} \right) R_x .$$

It clearly follows that

$$\| c - u_n \| \longrightarrow R_x .$$

Let $u_{n_i} \to \bar{u} \in \mathrm{cl}\, G(x)$ for some increasing sequence $\{ n_i \}$. Then,

$$\| c - \bar{u} \| = R_x$$

and $\| f_n g(c) - \bar{u} \| \to R_x$. Set $w_n = (f_n g(c) - \bar{u})(1 - R_x / \| f_n g(c) - \bar{u} \|)$. Then $w_n \to 0$ and

$$\| f_n g(c) - \bar{u} - w_n \| = R_x$$

for all $n = 1, 2, \cdots$. Finally,

$$\frac{1}{R_x} \left\| \frac{(c - \bar{u}) + (f_n g(c) - \bar{u} - w_n)}{2} \right\|$$

$$= \frac{1}{R_x} \left\| \left(\frac{c + f_n g(c)}{2} - \bar{u} \right) - \frac{w_n}{2} \right\| \longrightarrow 1 .$$

Since a Banach space is locally uniformly convex if, and only if, whenever members y_n, x in it are such that $\| y_n \| = \| x \| = 1$ and $\| y_n + x \| \to 2$ then $y_n \to x$, we conclude that

$$f_n g(c) - \bar{u} - w_n \longrightarrow c - \bar{u}$$

implying

$$f_n g(c) \longrightarrow c .$$

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