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INVARIANT HARMONIC ANALYSIS ON SPLIT RANK ONE GROUPS WITH APPLICATIONS

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Let G be a real connected noncompact semisimple Lie group with finite center; we shall denote the algebras of Lie groups L by the corresponding lower case German letter, \mathfrak{l} . We assume that if G_c is the simply connected complex analytic Lie group with Lie algebra \mathfrak{g}_c (here for any vector space V defined over R we denote its complexification by V_c ; in particular \mathfrak{g}_c is the complexification of \mathfrak{g}) then $G \subset G_c$. Fix a maximal compact subgroup K of G . Assume further that $rk(G/K)=1$. This paper has two principal sections. In §I we characterize the invariant transforms of functions in $\mathcal{C}^p(G: F)(F \subset K, |F| < \infty)$; §II deals with the characterization of the orbital integrals of such functions.

Let H be a θ -stable Cartan subgroup of G which is maximally split; in the case $rk(G) = rk(K)$ let B in a Cartan subgroup of G contained in K . It is known that $\hat{B} \subset \mathcal{L}_B \subseteq b^*$, and to each $\lambda \in \mathcal{L}_B'$, the regular elements of \mathcal{L}_B , there corresponds $\omega(\lambda) \in \tilde{G}^2$ (we denote by \tilde{G}^p the equivalence classes of those representations whose K -finite matrix coefficients are \mathcal{L}^p functions on G). Fix $\pi_\lambda \in \omega(\lambda)$ for each $\lambda \in \mathcal{L}_B'$. If $H = H_K A$ ($H_K = H \cap K$, A a vector group) $\chi \in \hat{H}_K'$ (the prime denoting the regular elements of \hat{H}_K) $\nu \in a^*$ then we can define $\pi_{\chi, \nu}$, a principal series representation of G .

We denote by \tilde{G} (resp. \hat{G}) the set of all infinitesimal equivalence classes of irreducible admissible (resp. unitary) representations of G . If π is an admissible representation we denote its global character by θ_π . It is known that θ_π is a distribution which is given by a function which is (real) analytic on the regular set of G ; we again denote this function by θ_π . If π, ν are equivalent irreducible admissible representations then $\theta_\pi = \theta_\nu$. Hence to each $\omega \in \tilde{G}$ there corresponds a character θ_ω ; characters of the class $\omega = \omega(\lambda) (\lambda \in \mathcal{L}_B')$ will be denoted θ_λ or θ_ω and characters of the class $[\pi_{\chi, \nu}]$ will be denoted $\theta_{\chi, \nu}$. In §I we introduce the transform

$$\hat{f}(\omega) = \int_G \theta_\omega(x) f(x^{-1}) dx = (\theta, f_\omega) \quad (\omega \in \tilde{G}).$$

We refer to this as the *invariant Fourier transform of f* . Let $\hat{f}(\omega(\lambda)) = \hat{f}(\lambda)$ and $\hat{f}(\chi: \nu) = \hat{f}([\pi_{\chi, \nu}])$. Then we show that there exists a list of properties involving; (a) holomorphy and growth properties of $\hat{f}(\chi)$ as a function on a strip $\mathcal{S}_c(2/p - 1)$; (b) relations between $\hat{f}(\lambda)$ and $\hat{f}(\chi: \nu)$ which reflect the fact that for $\lambda \in \mathcal{L}_B'$ such that

$\omega(\lambda) \notin \widehat{G}^p$ then π_λ is embedded in $\pi_{\chi, \nu}$ (for suitable χ and $\nu \in \mathcal{F}_c(2/p-1)$); which characterize the image of $\mathcal{E}^p(G: F)$ under the map $f \rightarrow \hat{f}$. We denote this space of functions by $\mathcal{E}^p(C(G): F)$. The most difficult part of the proof that $\mathcal{E}^p(C(G): F)$ is the image of $\mathcal{E}^p(G: F)$ is the question of surjectivity. We shall now detail the major steps in the proof of surjectivity.

Let $L \in \mathcal{E}^p(C(G): F)$ and let us form the *wave packets* (here let $L(\lambda) = L(\omega(\lambda))$, $L(\chi: \nu) = L([\pi_{\chi, \nu}])$)

$$\begin{aligned} \phi_L(x) = & \frac{1}{|W(G/B)|} \sum_{\lambda \in \mathcal{L}'_B} d(\lambda) \dim(\mathcal{H}_{\lambda, F}) L(\lambda) \theta^F(x) \\ & + D(G/A) \sum_{\chi \in \hat{H}'_K} d(\chi) \dim(\mathcal{H}_{\chi, F}) \int_{ia^*} L(\chi: \nu) \theta_{\chi, \nu}^F(x) \mu(\chi: \nu) d\nu \end{aligned}$$

where if $\omega \in \tilde{G}$, θ_ω^F denotes the sum of the K -Fourier coefficients of θ_ω corresponding to $\delta \in F$, $d(\lambda)$ equals the formal degree of π_λ , $D(G/A)$ and $\mu(\chi: \nu)$ are defined in [8]. Then it is known (cf. [8]) that $\phi_L \in \mathcal{E}^p(G: F)$ and $\hat{\phi}_L(\chi: \nu) = L(\chi: \nu)(\chi \in \hat{H}'_K, \nu \in ia^*)$, $\hat{\phi}_L(\lambda) = L(\lambda)$ ($\lambda \in \mathcal{L}'_B$). In fact more can be said. In the definition of ϕ_L let the sum over \mathcal{L}'_B (resp. the integral over ia^*) be denoted by ϕ_L^B (resp. ϕ_L^H). It can be shown that $\phi_L^B, \phi_L^H \in \mathcal{E}^2(G: F)$. Of course even if L satisfies the requisite properties ϕ_L^B, ϕ_L^H will not in general belong to $\mathcal{E}^p(G: F)$. This follows since if $S \subset \mathcal{L}'_B$, $\phi_{L, S} = \sum_{\lambda \in S} d(\lambda) \dim(\mathcal{H}_{\lambda, F}) L(\lambda) \theta_\lambda^F$, $\mathcal{L}_{B, p} = \{\lambda \in \mathcal{L}'_B: \omega(\lambda) \in \hat{G}^p\}$, $\mathcal{L}_{B, p}^\perp = \{\lambda \in \mathcal{L}'_B: \lambda \notin \mathcal{L}_{B, p}\}$, $\phi_{L, p} = \phi_{L, \mathcal{L}_{B, p}}$, $\phi_{L, p}^\perp = \phi_{L, \mathcal{L}_{B, p}^\perp}$, then $\phi_L^B = \phi_{L, p} + \phi_{L, p}^\perp$ and $\phi_{L, p}$ is perpendicular to $\phi_{L, p}^\perp$ in the L^2 -inner product. In fact we can write $\mathcal{E}^2(G: F) = \mathcal{E}^p(G: F) + \mathcal{E}^p(G: F)^\perp$ (orthogonal direct sum). Hence if $\phi_L^B, \phi_L^H \in \mathcal{E}^p(G: F)$ then $\phi_{L, p}^\perp \in \mathcal{E}^p(G: F) \cap \mathcal{E}^p(G: F)^\perp = \{0\}$. One would hope then to be able to show that $\phi_L^H + \phi_{L, p}^\perp \in \mathcal{E}^p(G: F)$. This is not in general possible. Instead we proceed as follows.

We produce given L an auxiliary function $\beta_L \in C_c^\infty(G: F)$ such that $(\beta_L)^\wedge(\lambda) = L(\lambda)(\lambda \in \mathcal{L}_{B, p}^\perp)$ and $\phi_{L-(\beta_L)}^\wedge \in \mathcal{E}^p(G: F)$. Hence if we denote by $\phi_{L_0}^B$ the wave packet formed with the modified function L_0 where $L_0(\lambda) = 0$ if $\lambda \in \mathcal{L}_{B, p}^\perp$, and $L_0(\lambda) = L(\lambda)(\lambda \in \mathcal{L}_{B, p})$ and set (here we use the notation of [8])

$$f_L = \phi_{L-(\beta_L)}^\wedge + (\beta_L)_A + \phi_{(\beta_L)^\wedge, p}^\perp + \phi_{L_0}^B$$

then we have by [8] that

$$\begin{aligned} \hat{f}_L(\chi: \nu) &= (\phi_{L-(\beta_L)}^\wedge)^\wedge(\chi: \nu) + ((\beta_L)_A)^\wedge(\chi: \nu) \\ &= L(\chi: \nu) - (\beta_L)^\wedge(\chi: \nu) + (\beta_L)^\wedge(\chi: \nu) \\ &= L(\chi: \nu). \end{aligned}$$

By the orthogonality relations for the discrete series characters we

have for $\lambda \in \mathcal{L}_{B,p}^\perp$

$$(f_L)^\wedge(\lambda) = (\phi^\perp_{(\beta_L)^\wedge})^\wedge(\lambda) = (\beta_L)^\wedge(\lambda) = L(\lambda)$$

and for $\lambda \in \mathcal{L}_{B,p}$ we have

$$(f_L)^\wedge(\lambda) = (\phi_{L_0}^B)^\wedge(\lambda) = L_0(\lambda) = L(\lambda).$$

In § II we introduce the invariant orbital integrals, F_f ; we denote the restriction of this function to the regular points of B (resp. H) by F_f^B (resp. F_f^H). The functions F_f^B , F_f^H are L^1 on B and H respectively. Hence we can take their Fourier transforms. It is known (cf. [19]) that \hat{F}_f^H has a simple relation with \hat{f}_H and $\hat{F}_f^B(\lambda)$ a complicated relation with \hat{f}_B . Nevertheless we are able to transcribe our conditions defining $\mathcal{C}^p(C(G): F)$ over onto F_f^B and F_f^H which then allows us to characterize these functions.

One remark is in order which is explicated in more detail in § II. The Fourier transforms of F_f^B and F_f^H are defined for all $\lambda \in \mathcal{L}_B$ and $\chi \in \hat{H}_k$ (not just the regular elements). We were then forced to extend our definition of the invariant transform to include the singular elements of $\lambda \in \mathcal{L}_B$ and $\chi \in \hat{H}_k$.

The importance of these characterizations is, apart from their natural place in the harmonic analysis of G , that they occur in the study of the Selberg trace formula. In fact if Γ is a discrete compact subgroup of G , L denotes the left regular representation of G on $L^2(G/\Gamma)$ then it is known that

$$L = \sum_{\omega \in \hat{G}} m_\omega \omega$$

(i.e., L is discretely decomposable into a direct sum of irreducible unitary representations with finite multiplicities). A natural problem is to determine the integers m_ω .

For $f \in \mathcal{C}^1(G)$ the operator $L(f)$ is of trace class and we have

$$\text{tr } L(f) = \sum_{\omega \in \hat{G}} m_\omega \hat{f}(\omega).$$

On the other hand, we can write

$$\text{tr } L(f) = \sum_{\{y\}} \mu(G_y/\Gamma_y) \int_{G/G_y} f(y^x) dG/G_y(x)$$

where $\{y\}$ runs through the conjugacy classes in Γ , $G_y = \text{Cent}_G(y)$, $\Gamma_y = \Gamma \cap G_y$, and $\mu(G_y/\Gamma_y)$ is the volume of G_y/Γ_y .

In order to obtain information about the m_ω (for instance obtaining limit formulas (cf [4])) one can attempt to express the right hand side of the last equation above in terms of invariant transforms for functions f on G whose Fourier transform can be

explicitly computed; the easiest way to do this is to *start* on the Fourier transform side and inverse transform back to the group. This of course requires the above characterizations.

For other papers on this subject see also [19].

Notation. We retain the notation of the introduction and all other notation not explained below is as in [15].

Let M be a differentiable manifold, W (resp. (W, γ)) be a finite dimensional vector space (resp. a finite dimensional double unitary K -module). The space of infinitely differentiable functions on M taking values in W and those of compact support (resp. the γ -spherical infinitely differentiable functions and those of compact support) will be denoted by $C(M: W)$ and $C_c^\infty(M: W)$ (resp. $C^\infty(M: W: \gamma)$, $C_c^\infty(M: W: \gamma)$) when $W = C$ we suppress the W in the notations $C^\infty(M: W)$ and $C_c^\infty(M: W)$.

If A is an arbitrary set $B \subseteq A$ we denote by $[B]$ the complement of B in A . Further if B is a finite subset of A then we denote the number of elements in B by the notation $|B|$.

If V is a vector space over R we shall denote by V_c its complexification; i.e., $V_c = V \otimes_R C$. Let V^* (resp. V_c^*) denote the real (resp. complex) dual of V (resp. V_c).

For an arbitrary Lie group L let us denote by \hat{L} the set of equivalence classes of irreducible unitary representations of L .

Suppose now that M is as above and there exists a topological action of K on M both on the right and left. If $\xi \in \hat{K}$ let χ_ξ denote the character of ξ and ρ_ℓ, ρ_r denote the left and right actions of K on M . We shall write for $F \subset \hat{K}$, $|F| < \infty$, $C^\infty(M: F)$ and $C_c^\infty(M: F)$ for the subspaces of $C^\infty(M)$ and $C_c^\infty(M)$ respectively of those functions f which satisfy the following:

$$d(\xi) \int_K \text{conj } \chi_\xi(k) f(\rho_\ell(k)m) dk = d(\xi) \int_K f(m \rho_r(k)) \text{conj } \chi_\xi(k) dk = f(m)$$

where $d(\xi)$ denotes the degree of ξ . If $A \subset C^\infty(M)$ then we shall write $A(F)$ for the corresponding subspace of $C^\infty(M: F)$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition of \mathfrak{g} , θ the corresponding Cartan involution of \mathfrak{g} (we also use θ for the involution of G). Let \mathfrak{h} be a θ stable Cartan subalgebra of \mathfrak{g} with maximal vector part. Put $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s}$ and assume $\dim \mathfrak{a} = 1$. If $\text{rank}(G) = \text{rank}(K)$ let B be a Cartan subgroup G contained in K .

For any Cartan subalgebra $\mathfrak{l} \subset \mathfrak{g}$ let $\Delta(\mathfrak{g}_c, \mathfrak{l}_c)$ denote the nonzero roots of the pair $(\mathfrak{g}_c, \mathfrak{l}_c)$. We denote by $W(\mathfrak{g}_c, \mathfrak{l}_c)$ the group generated by the reflections $s_\alpha (\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{l}_c))$. When \mathfrak{l} is understood we shall simply write W ; we refer to W as the Weyl group of the pair $(\mathfrak{g}_c, \mathfrak{l}_c)$.

If we have an action of A on B we denote by B^A the set of A -invariants in B .

Let $M = \text{Cent}_K(\alpha)$, $M' = \text{Norm}_K(\alpha)$. The M and M' are compact groups, and $W(A) = M'/M$ is a finite group. If $\chi \in \hat{M}$, (V, σ) is an M -module in the class χ , $\nu \in \alpha_c^*$ and $w \in M'$ then we shall write $w\sigma$ and $w\nu$ for the following

$$w\sigma(m) = \sigma(w^{-1}mw) \quad (m \in M)$$

$$w\nu(H) = \nu(w^{-1}Hw) \quad (H \in \alpha).$$

Obviously if σ' is also of class χ then $w\sigma'$ and $w\sigma$ are again of the same class, i.e., M' acts on \hat{M} . Note also that the action of M' on σ and ν depends only on the residue class of w in $W(A)$. If $s \in W(A)$ we shall have occasion to write $s\sigma$; $s\chi$ and $s\nu$ which then have their obvious meaning.

Let $P(A)$ denote the set of all parabolic subgroups of G whose split component is A . Fix $Q \in P(A)$, and let $Q = MAN_Q$ be its Langlands decomposition (note that M is as above). We put $\mathfrak{m}_1 = \mathfrak{m} + \alpha$ and note that $\mathfrak{h} \subset \mathfrak{m}_1$, and is a Cartan subalgebra of the reductive algebra \mathfrak{m}_1 . Let $W = W(\mathfrak{g}, \mathfrak{h}_e)$, $W_1 = W(\mathfrak{m}_1, \mathfrak{h}_e)$. Then in a natural way we may consider W_1 as a subgroup of W .

To each θ -stable Cartan subgroup of G we shall associate a series of representations as follows. Let Q be as above, $Q = MAN$. Let $\chi \in \hat{M}$, $\sigma \in \chi$, $\nu \in \alpha_c^*$ and put $\pi_{\chi, \nu} = \pi_{Q, \chi, \nu} = \text{Ind}_Q^G(\sigma \otimes \xi_\nu)$ where $\xi_\nu(\alpha) = e^{\nu(\log \alpha)}$ (as the exponential map restricted to \mathfrak{s} is a diffeomorphism then if $x = \exp X$ we write $X = \log x$), and $\sigma \otimes \xi_\nu$ is extended to Q by making it trivial on N . We shall assume that $\pi_{\chi, \nu}$ acts by right translation and it represents G on $\mathcal{H}_\chi = \mathcal{H}_{Q, \sigma}$, in the compact picture (i.e., functions defined on K), or on $\mathcal{H}_{\chi, \nu} = \mathcal{H}_{Q, \chi, \nu}$, in the noncompact picture (i.e., functions defined on G)).

Let $\pi_Q(\nu) = \langle \nu, \alpha \rangle (\nu \in \alpha_c^*)$ where α denotes the unique simple root of $\Delta(\mathfrak{g}, \alpha)$, $\langle \cdot, \cdot \rangle$ denotes the killing form on α_c^* . Let $\mathcal{F} = (-1)^{1/2} \alpha^*$, $\mathcal{F}_e = \alpha_c^*$ and $\mathcal{F}_r = \alpha^*$. We shall say that $\nu \in \mathcal{F}_e$ is regular if $\pi_Q(\nu) \neq 0$. We denote the set of regular elements of \mathcal{F} , \mathcal{F}_e , and \mathcal{F}_r by \mathcal{F}' , \mathcal{F}'_e , and \mathcal{F}'_r respectively.

It is known (cf. [11], Lemma 13.3) that $\pi_{\chi, \nu}$ is irreducible for all $\nu \in \mathcal{F}'$. Moreover $\pi_{\chi, \nu}$ is unitarily equivalent to $\pi_{s\chi, s\nu}$ for all $\nu \in \mathcal{F}'$, $s \in W(A)$. Hence there exists an isometry $\mathcal{A}_{Q|Q}: \mathcal{H}_{\chi, \nu} \rightarrow \mathcal{H}_{s\chi, s\nu}$ such that

$$\mathcal{A}_{Q|Q}(s: \chi: \nu) \pi_{\chi, \nu}(x) = \pi_{s\chi, s\nu}(x) \mathcal{A}_{Q|Q}(s: \chi: \nu) \quad (x \in G).$$

Moreover, for Q , s , and χ fixed, the function $\nu \rightarrow \mathcal{A}_{Q|Q}(s: \chi: \nu)$ has a meromorphic extension to \mathcal{F}_e .

Let $\theta_{\chi, \nu}$ denote the global character of $\pi_{\chi, \nu}$. It follows from

the above that $\theta_{s\chi, s\nu} = \theta_{\chi, \nu}$ for all $\chi \in \hat{M}$, $\nu \in \mathcal{F}'$, $s \in W(A)$.

Suppose that $rk(G) = rk(K)$ and B is as before a Cartan subgroup of G contained in K . Then there exists a lattice $\mathcal{L}_B \subset \mathfrak{b}_c^*$ such that \mathcal{L}_B is isomorphic to \hat{B} . Let $W(G/B)$ denote the finite group $\text{Norm}_G(B)/B$. Then $W(G/B)$ acts on \mathcal{L}_B' (the regular elements of \mathcal{L}_B). Let \mathcal{L}_B^+ be a fundamental domain for this action. To each $\lambda \in \mathcal{L}_B$ Harish-Chandra has associated a representation $\omega(\lambda)$ whose matrix elements are L^2 functions on G (hereafter let us write \hat{G}^2 and more generally \hat{G}^p for the equivalence classes of irreducible unitary representations of G whose K -finite matrix coefficients are L^p summable on G). It is known that if $\lambda_1, \lambda_2 \in \mathcal{L}_B$ then $\omega(\lambda_1)$ is equivalent to $\omega(\lambda_2)$ if and only if $\lambda_1 = s\lambda_2$ for some $s \in W(G/B)$. In particular, \mathcal{L}_B^+ uniquely parameterizes the class of representations corresponding to B . We shall denote by \mathcal{H}_λ the representation space of $\omega(\lambda)$.

Let us now fix $F \subset \hat{K}$, $|F| < \infty$. If L is a Lie group, N a compact subgroup of L , π a representation of L which when restricted to N is unitary, then we shall write for $\delta \in \hat{N}$, $[\pi: \delta]_N$ for the multiplicity of δ in the direct sum decomposition of π restricted to N . With this notation let us put

$$\begin{aligned}\hat{M}(F) &= \{\chi \in \hat{M}: [\delta: \chi]_M \geq 1 \text{ for some } \delta \in F\}, \\ \hat{G}^p(F) &= \{\omega \in \hat{G}^p: [\omega: \delta]_K \geq 1 \text{ for some } \delta \in F\}\end{aligned}$$

then $|\hat{M}(F)| < \infty$, and $|\hat{G}^p(F)| < \infty$, and we have by the Frobenius reciprocity theorem that $[\pi_{\chi, \nu}: \delta] \neq 0$ for some $\delta \in F$ if and only if $\chi \in \hat{M}(F)$.

Let π be a representation of G on a Hilbert space \mathcal{H} . If $\delta \in \hat{K}$ let us write \mathcal{H}_δ for the isotypic component of \mathcal{H} corresponding to δ . Further, if $F \subset \hat{K}$ and $|F| < \infty$ let us put

$$\mathcal{H}_F = \sum_{\delta \in F} \mathcal{H}_\delta.$$

Let

$$d(m) = d_Q(m) = (\det \text{Ad}_{m|_{\mathfrak{n}_Q}})^{1/2} \quad (m \in MA)$$

and

$$\rho(H) = \rho_Q(H) = \frac{1}{2} \text{tr}(\text{ad}_{H|_{\mathfrak{n}_Q}}) \quad (H \in \underline{a}).$$

Let

$$A^+ = A^+(Q) = \{a \in A: e^{\alpha(\log a)} > 1\}$$

where $\alpha = \alpha_Q$ is the unique simple root in $\mathcal{A}(\mathfrak{g}, \mathfrak{a})$.

We shall denote the enveloping algebra of \mathfrak{g}_c by \mathcal{G} ; if \mathfrak{l} is a

subalgebra of \mathfrak{g} we shall denote the subalgebra of \mathfrak{G} generated by \mathfrak{l}_c and \mathfrak{l} by \mathfrak{l} . The symmetric algebra of \mathfrak{g}_c will be denoted by $\mathcal{S}(\mathfrak{g}_c)$; elements of $\mathcal{S}(\mathfrak{g}_c)$ will be treated as directional derivatives of function on \mathfrak{g}_c .

1.1. Some properties of representations. Let $0 < p \leq 2$, $a \in \mathfrak{G}$ and $r \in R$. For $f \in C^\infty(G)$ let

$$\nu_{a,r}^p(f) = \sup_G E^{-2/p}(1 + \sigma)^r |af|.$$

Put

$$\mathcal{E}^p(G) = \{f \in C^\infty(G) : \nu_{a,r}^p(f) < \infty \text{ for all } a \in \mathfrak{G}, r \in R\}.$$

Note that we use only one sided derivatives but we shall now restrict to a K -finite subspace of $\mathcal{E}^p(G)$ on which the two-sided-derivative seminorms and the one sided induce the same topology.

Fix $F \subseteq \hat{K}$, $|F| < \infty$. For $\delta \in F$ let χ_δ denote the character of δ and set $\xi_\delta = d(\delta) \text{conj} \chi_\delta$. Set

$$\mathcal{E}^p(G; F) = \left\{ f \in \mathcal{E}^p(G) : \int_K \xi_F(k) f(k^{-1}x) dk = \int_k \xi_F(k^{-1}) f(xk) dk = f(x) \right\}.$$

The following is material which we will need in order to form the auxillary function mentioned in the introduction. The following results can be found in [10]; all unexplained notations are as in that paper.

Let π be an admissible representation of finite length. Set $\mathcal{F}_c^- = \{\mu = \nu + i\gamma \in \mathcal{F}_c : \nu(H) \leq 0 \text{ for all } H \in \alpha^-(Q)\}$ where if α is the unique simple root of the pair $\Delta(\mathfrak{q}, \alpha)$ then $\alpha^-(A) = \{H \in \mathfrak{a} : \alpha(H) < 0\}$. Also set

$$I_Q(\pi) = \{\nu \in \mathcal{F}_c : \text{Hom}_{(\mathfrak{g}, K)}(\pi : \pi_{Q, \nu}) \neq \{0\} \text{ for some } \chi \in \hat{M}\}.$$

We denote by $\mathcal{E}_Q^0(\pi)$ the set of leading exponents of π along Q .

LEMMA 1. (*Milicic*). *Let π be an admissible representation of finite length. Then the set $\mathcal{E}_Q^0(\pi)$ equals the set of minimal exponents in $I_Q(\pi)$.*

REMARK. The ordering in Lemma 2 is as follows. Let L be the lattice generated by $\Delta(\mathfrak{q}, \alpha)$, L^+ the cone of sums of positive roots. We write $\lambda \gg \mu$ ($\lambda, \mu \in \mathcal{F}_c$) if $\lambda - \mu \in L^+$.

Suppose that $rk(G) = rk(K)$, $\mathfrak{b} = LA(B)$. Put $\Phi = \Delta(\mathfrak{g}, \mathfrak{b})$ and for $\alpha \in \Phi$ set

$$k(\alpha) = 1/4 \sum_{\beta \in \Phi} |(\alpha, \beta)|.$$

LEMMA 2. (*Milicic* [10]). *Let $\kappa > 0$. Then for every $\lambda \in L_B^+$*

the following are equivalent:

- (1) $|(A, \alpha)| \geq \kappa k(\alpha)$ for all noncompact $\alpha \in \Phi$
- (2) every leading exponent of π_A along Q lies in $\kappa\rho_Q + \mathcal{F}^-$.

LEMMA 3. ([10], [17]) (i) A necessary and sufficient condition that $\pi_A \in \hat{G}^p$ is that $|(A, \alpha)| > (2/p - 1)k(\alpha)$ for all noncompact α .

(ii) If $\pi_A \in \hat{G}^p$ and $c(x)$ is a K -finite matrix coefficient of π_A then there exists a constant $D > 0$ such that

$$|c(x)| \leq D\Xi^{2/p+\varepsilon_Q}(x)$$

for some $\varepsilon_0 > 0$.

LEMMA 4. Let $\omega \in \hat{G}^2$ and $0 < p < 2$. Then

$$I_Q(\pi_\omega) \cap \mathcal{F}_c\left(\frac{2}{p} - 1\right) \neq \phi$$

if and only if $w \in \hat{G}^2/\hat{G}^p$.

Proof. It is a surprising fact (although in the rank one case it is more or less obvious) that one can deduce global estimates from leading exponents. That is, if for all $\lambda \in \mathcal{E}_Q^0(\pi)$, $\operatorname{Re} \lambda - \gamma\rho_Q \leq 0$ (on $\alpha^-(Q))\gamma > 0$, then for any K -finite matrix coefficient c , there exists a constant $D > 0$ such that $|c(x)| \leq D\Xi^r(x)$. Hence from Lemmas 2 and 3 above it is easy to deduce that $\mathcal{E}_Q^0(\pi_\omega) \cap \mathcal{F}_c(2/p - 1) \neq \phi$ if and only if $\omega \in \hat{G}^2/\hat{G}^p$. The lemma then follows from Lemma 1 above.

From the estimates of [15] it is easy to deduce that if $\chi \in \hat{M}$, $\nu \in \mathcal{F}_c(2/p - 1)$ then $\alpha \rightarrow \theta_{\chi, \nu}(\alpha)$ is a continuous linear functional on $\mathcal{E}^p(G)$. The following lemma is also a simple consequence of these estimates.

LEMMA 5. Let $\chi \in \hat{M}$, $\nu \in \mathcal{F}_c(2/p - 1)$. If $\theta_{\chi, \nu} = \theta_1 + \cdots + \theta_\ell$ where each θ_i is an irreducible character then $\alpha \rightarrow \theta_i(\alpha)$ is a continuous linear functional on $\mathcal{E}^p(G)$.

If $\chi \in \hat{M}$, $\nu \in \mathcal{F}_c$, and $G(\chi; \nu; x) = \theta_{\chi, \nu}^F(x)$ then for each $\chi \in \hat{M}$, $G(x)$ is analytic on $\mathcal{F}_c \times G$. Let us denote by $\theta_{\chi, \nu, k}^F$ the distribution given by the function $G(\chi; \nu; \partial^k(\nu); x)$.

Fix $0 < p < 2$. Let U_p denote the union of the sets (1), (2), and (3) specified in § 7 of [15] intersected with $\mathcal{F}_c(2/p - 1)$. Set $V_p = U_p \cup (\cup I_Q(\omega) \cap \mathcal{F}_c(2/p - 1))$. For every $\chi \in \hat{M}(F)$, $t \in W(A)$, $\zeta \in V_p$, $\omega \in \hat{G}^2(F) \setminus \hat{G}^p(F)$ we can write $\theta_{\chi, t\zeta}^F$ as a sum of irreducible characters. Let \mathcal{E}_p denote the set of F -Fourier components of all

these characters union with the distributions $\theta_{\lambda, t, \zeta, k}^F(\chi, t, \zeta)$ as above and $0 \leq k \leq 0_i(\zeta) - 1$ (cf. [15])). Recall that $\theta_\omega \in \mathcal{E}_p$ ($\omega \in \hat{G}^2(F) \setminus \hat{G}^p(F)$).

Let \mathcal{B}_p be a basis for the linear space spanned by \mathcal{E}_p chosen as follows. To the characters $\theta_\omega (\omega \in \hat{G}^2(F) \setminus \hat{G}^p(F))$ adjoint linearly independent elements from the *characters* in the set \mathcal{E}_p . Next adjoint to this set linearly independent elements from the set $\{\theta_{\lambda, t, \zeta, k}: \lambda \in \hat{M}(F), t \in W(A), \zeta \in V_p, 1 \leq k \leq 0_i(\zeta) - 1\}$. For λ, t, ζ, k as usual let us define constants $C_p(\theta_{\lambda, t, \zeta, k}^F: \theta) (\theta \in \mathcal{B}_p)$ by the equation

$$(1) \quad \theta_{\lambda, t, \zeta, k}^F = \sum_{\theta \in \mathcal{B}_p} C_p(\theta_{\lambda, t, \zeta, k}^F: \theta) \theta.$$

Further for any $\theta' \in \mathcal{E}_p$ let $C_p(\theta': \theta)$ be defined by

$$(2) \quad \theta' = \sum_{\theta \in \mathcal{B}_p} C_p(\theta': \theta) \theta.$$

For $0 < p_j < 2, p_j$ sufficiently small ($j = 1, 2$) $U_{p_1} = U_{p_2}$ and $V_{p_j} \supset \bigcup_{\omega \in \hat{G}^2(F) \setminus \hat{G}^p(F)} I_Q(\omega)$. Let us fix such a p and replace the notations $V_p, \mathcal{E}_p, \mathcal{B}_p$, and C_p by $V, \mathcal{E}, \mathcal{B}$, and C .

Fix an open neighborhood V_0 of $1 \in G$ with compact closure. By their linear independence and analyticity on the regular elements of G , we can choose for each $\theta \in \mathcal{B}_p, \alpha_\theta \in C_c^\infty(G: F)$ such that $\text{supp } \alpha_\theta \subseteq V_0$ and if

$$(3) \quad (\alpha_\theta, \theta') = \int_G \alpha_\theta(x^{-1}) \theta'(x) dx,$$

then $(\alpha_\theta, \theta') = 0$ if $\theta' \in \mathcal{B}_p, \theta' \neq \theta$, and $(\alpha_\theta, \theta) = 1$. Further let us require that $(\alpha_\theta, \theta_\omega) = 0 (\omega \in G^p)$. This last condition is permissible by Lemma 4. The functions α_θ are by no means uniquely determined. However, we do have the following result. Define $(\alpha, \theta_{\lambda, t, \zeta, k}^F)$ as in (3).

PROPOSITION 1. *With the above notation,*

$$(\alpha_\theta, \theta_{\lambda, t, \zeta, k}^F) = C(\theta_{\lambda, t, \zeta, k}^F: \theta).$$

More generally for any $\alpha \in C_c(G: F)$,

$$(\alpha, \theta_{\lambda, t, \zeta, k}^F) = \sum_{\theta \in \mathcal{B}_p} C(\theta_{\lambda, t, \zeta, k}^F: \theta) (\alpha, \theta).$$

Proof. (1) is obvious from (1) above. (2) follows from the bilinearity of the symbol (\cdot, \cdot) .

1.2. Definition and properties of the invariant transform. Let $C(G)$ denote the space of characters of quasi-simple admissible representations of G which are of finite length. For $\theta \in C(G)$ and $\alpha \in C_c^\infty(G: F)$ let us write

$$\hat{\alpha}(\theta) = (\alpha, \theta) = \int_{G'} \alpha(x^{-1})\theta(x)dx.$$

Here we use the fact that such characters are given by (analytic) functions on the regular set G' of G . For $\theta = \theta_{\lambda, \nu}$, $\lambda \in \hat{M}$, $\nu \in \mathcal{F}_c$ we shall sometimes write $\hat{\alpha}(\lambda: \nu)$ in place of $\hat{\alpha}(\theta_{\lambda, \nu})$ and for $\omega \in \hat{G}^2$ we shall frequently write $\hat{\alpha}(\omega)$ in place of $\hat{\alpha}(\theta_\omega)$. We shall refer to $\hat{\alpha}$ as the *invariant Fourier transform of α* .

If G is a complex valued function with domain $D \subseteq \hat{M} \times \mathcal{F}_c$ such that $\hat{M}(F) \times \mathcal{F} \subset D$ then set

$$(1) \quad \phi_G(\lambda: x) = \int_{\mathcal{F}} \theta_{\lambda, \nu}^F(x) G(\lambda: \nu) \mu(\lambda: \nu) d\nu \quad (x \in G)$$

and

$$(2) \quad \phi_G(x) = D(G/A) \sum d(\lambda) d_F(\lambda)^{-1} \phi_G(\lambda: x) \quad (\text{sum over } \lambda \in \hat{M}),$$

where $d_F(\lambda) = \dim \mathcal{H}_{\lambda, F}$.

Let $\lambda \in \hat{M}(F)$ and denote by $E_F(\lambda)$ the orthogonal projection of \mathcal{H}_λ onto $\mathcal{H}_{\lambda, F}$.

LEMMA 1. Let $\alpha \in C_c^\infty(G: F)$. Then $\hat{\alpha}(\lambda: \cdot)$ is an entire function on \mathcal{F}_c of exponential type. Further,

$$(1) \quad \hat{\alpha}(s\lambda: s\nu) = \hat{\alpha}(\lambda: \nu) \quad (s \in W(A), \lambda \in \hat{M}, \nu \in \mathcal{F}_c)$$

$$(2) \quad \hat{\alpha}(\lambda: \nu) \equiv 0 \text{ if } \lambda \notin \hat{M}(F).$$

$$(3) \quad \hat{\alpha}(\omega) = 0 \text{ if } \omega \notin \hat{G}^2(F)$$

$$(4) \quad \text{defining } \phi_\alpha \text{ as in (2) above we have}$$

$$(3) \quad \hat{\phi}_\alpha(\lambda: \nu) = \hat{\alpha}(\lambda: \nu) \quad (\lambda \in \hat{M}(F), \nu \in \mathcal{F}_c).$$

Proof. The holomorphy and growth properties of $\hat{\alpha}(\lambda: \nu)$ (as a function on \mathcal{F}_c) follow easily from the corresponding properties of $\mathcal{F}_H(\alpha)$ (cf. [15]).

(1) follows from the identity $\theta_{\lambda, \nu} = \theta_{s\lambda, s\nu}$. Statements (2) and (3) are obvious. To prove (4) let us call the left side of (3) $g(\lambda: \nu)$. Then

$$g(\lambda_0: \nu_0) = D(G/A) \sum_{\lambda \in \hat{M}(F)} d(\lambda) d_F(\lambda)^{-1} (\phi_\alpha^\wedge, \theta_{\lambda_0, \nu_0}).$$

By Theorem 2.1 of [8] we have (with $W(\lambda_0) = \{s \in W(A): s\lambda_0 = \lambda_0\}$)

$$g(\lambda_0: \nu_0) = D(G/A) \sum_{s \in W/W(\lambda_0)} d(s\lambda_0) d_F(s\lambda_0)^{-1} (\phi_\alpha^\wedge(s\lambda_0), \theta_{\lambda_0, \nu_0})$$

where the sum over $W/W(\lambda_0)$ means over a complete set of representatives. Again applying Theorem 22.1 of [8] we have

$$g(\chi_0: \nu_0) = D(G/A) \sum_{s \in W/W(\chi_0)} d(s\chi_0) d_F(s\chi_0)^{-1} r c^2 d(s\chi_0)^{-1} F_0 \psi_{E_F(s\chi_0)}(1) \\ \times \sum_{t \in W(s\chi_0)} \hat{\alpha}(s\chi_0: ts\nu_0).$$

By (3) of Lemma 9.1 of [8] we have the above

$$= |W|^{-1} \sum_{s \in W/W(\chi_0)} \sum_{t \in W/W(s\chi_0)} \hat{\alpha}(s\chi_0: ts\nu_0) \\ = |W|^{-1} \sum_{s \in W/W(\chi_0)} \hat{\alpha}(s\chi_0: s\nu_0) |W(s\chi_0)| \\ = |W|^{-1} |W(\chi_0)| [W: W(\chi_0)] \hat{\alpha}(\chi_0: \nu_0).$$

The last lines of equalities follow from (1) above and the obvious fact that $W(s\chi_0) = sW(\chi_0)s^{-1}$.

In the preceding section we defined the sets \mathcal{E} and \mathcal{B} and the constants $C(\theta_{\lambda, t\zeta, k}: \theta)$. For any scalar valued function L defined on the space $C(G)$ such that for each $\chi \in \hat{M}$, $\nu \rightarrow L(\chi: \nu)$ is C^∞ let us put

$$(4) \quad \beta_L(x) = \sum_{\theta \in \mathcal{B}} L(\theta) \alpha_\theta(x) \quad (x \in G)$$

where if $\theta = \theta_{\lambda, t\zeta, k}$ with $k > 0$ then by $L(\theta)$ we intend $L(\chi: t\zeta; \partial^k(\nu))$. As $|\mathcal{B}| < \infty$ then $\beta_L \in C_c^\infty(G: F)$.

PROPOSITION 1. *Let $\alpha \in C_c^\infty(G: F)$. Then,*

$$(1) \quad \hat{\beta}_\alpha(\chi: t\zeta; \partial^k(\nu)) = \hat{\alpha}(\chi: t\zeta; \partial^k(\nu))$$

for all $\chi \in \hat{M}(F)$, $t \in W(A)$, $\zeta \in V$, $0 \leq k \leq 0_i(\zeta) - 1$.

$$(2) \quad \hat{\beta}_\alpha(\omega) = \hat{\alpha}(\omega) \quad (\omega \in \hat{G}^2).$$

Proof. By Proposition I.1.1 we have

$$\hat{\beta}_\alpha(\chi: t\zeta; \partial^k(\nu)) = \sum_{\theta \in \mathcal{B}} \hat{\alpha}(\theta) (\alpha_\theta, \theta_{\lambda, t\zeta, k}^F) \\ = \sum_{\theta \in \mathcal{B}} \hat{\alpha}(\theta) \sum_{\theta' \in \mathcal{B}} C(\theta_{\lambda, t\zeta, k}^F: \theta') (\alpha_\theta, \theta') \\ = \sum_{\theta \in \mathcal{B}} C(\theta_{\lambda, t\zeta, k}^F: \theta) \hat{\alpha}(\theta) = \hat{\alpha}(\chi: t\zeta; \partial^k(\nu)).$$

Recalling that $\theta_\omega \in \mathcal{B}$ for all $\omega \in \hat{G}^2$ we obviously have (2), i.e.,

$$\hat{\beta}_\alpha(\theta_\omega) = \sum_{\theta \in \mathcal{B}} \hat{\alpha}(\theta) (\alpha_\theta, \theta_\omega) = \hat{\alpha}(\theta_\omega).$$

PROPOSITION 2. *Let $\alpha \in C_c^\infty(G: F)$, and $\alpha_0 = \alpha - \beta_\alpha$. Then $\phi_{\alpha_0} \in C_c^\infty(G: F)$ and hence $\phi = \phi_{\alpha_0} + \beta_\alpha \in C_c^\infty(G: F)$. Furthermore, $\hat{\phi}(\chi: \nu) = \hat{\alpha}(\chi: \nu)$ for all $\chi \in \hat{M}$, $\nu \in \mathcal{F}_c$.*

Proof. The proof that $\phi_{\alpha_0} \in C_c^\infty(G: F)$ follows from the observation that for all $\chi \in \hat{M}$, $t \in W(A)$, $\zeta \in V$, $0 \leq k \leq 0_i(\zeta) - 1$ we have

by (1) of Proposition 1 that,

$$\hat{\alpha}_0(\chi: t\zeta; \partial^k(\nu)) = 0.$$

By Johnson's theorem (cf. [15]) $\phi_{\alpha_0}^*(a)$ (for $\alpha(\log a) \gg 0$) can be expressed in terms of the residues of

$$\Phi(\nu: a) C_{\bar{q}|\bar{q}t}(1: t\nu) \cdot tG(\chi: t\nu) \psi_{E_F(\chi)}(1:1:1) \quad (t \in W(A)).$$

Arguing as in the corollary to Proposition 7.2 of [15] leads to the compactness of the support of $\phi_{\alpha_0}^*$.

For the last part we note that by (3) we have

$$\hat{\phi}(\chi: \nu) = \hat{\phi}_{\alpha_0}(\chi: \nu) + \hat{\beta}_{\hat{\alpha}}(\chi: \nu) = \hat{\alpha}_0(\chi: \nu) + \hat{\beta}_{\hat{\alpha}}(\chi: \nu) = \hat{\alpha}(\chi: \nu).$$

1.3. The characterization of the invariant transform. Let notation be as in the preceding sections. For $L: C(G) \rightarrow \mathbb{C}$ such that $L(\chi: \nu) = L(\theta_{\chi, \nu})$ is a C^∞ -function of ν for all $\chi \in \hat{M}$ let us define for $u \in \mathcal{S}(\mathcal{F}_c)$, $\alpha \in \mathbb{R}$,

$$\nu_{u, \alpha}^p(L) = \sup (1 + |\nu|)^\alpha |\mathcal{L}(\chi: \nu; u)|$$

where the sup is taken over $\hat{M} \times \text{Int } \mathcal{F}_c(2/p - 1)$.

Let

$$C^p(G) = \left\{ \theta_{\chi, \nu}: \chi \in \hat{M}, \nu \in \text{Int } \mathcal{F}_c \left(\frac{2}{p} - 1 \right) \right\} \cup \{ \theta_\omega: \omega \in \hat{G}^2 \} \cup \mathcal{B}_p.$$

DEFINITION 1. Let $\mathcal{E}^p(C(G): F)_0$ denote the linear space of all complex valued functions defined on $C^p(G)$ having the following properties:

(1) for each $\chi \in \hat{M}$ the function $L(\chi)$ is holomorphic on $\text{Int } \mathcal{F}_c(2/p - 1)$

(2) $L(s\chi: s\nu) = L(\chi: \nu)$ ($s \in W(A)$)

(3) $L(\chi: \nu) \equiv 0$ if $\chi \notin \hat{M}(F)$

(4) $L(\omega) = 0$ if $\omega \notin \hat{G}^2(F)$

(5) $\nu_{u, \alpha}^p(L) < \infty$ for all $u \in \mathcal{S}(\mathcal{F}_c)$, $\alpha \in \mathbb{R}$.

Note that by the uniform continuity of $\nu \rightarrow L(\chi: \nu; u)$, $L(\chi: \nu; u)$ can be extended to a continuous function on $\mathcal{F}_c(2/p - 1)$ which we again denote by $L(\chi: \nu; u)$.

DEFINITION 2. Let $\mathcal{E}^p(C(G): F)$ denote the subspace of functions $L \in \mathcal{E}^p(C(G): F)_0$ which satisfy the additional condition,

$$L(\chi: t\zeta; \partial^k(\nu)) = \sum_{\theta \in \mathcal{B}_p} C_p(\theta_{\chi, t\zeta, k}: \theta) L(\theta)$$

for all $\chi \in \hat{M}$, $t \in W(A)$, $\zeta \in V_p$ (cf. the remark preceding the defini-

tion), and $0 \leq k \leq 0_i(\zeta) - 1$. Here if $\theta = \theta_{\chi', t', \zeta', k'} \in \mathcal{B}_p$ with $k' > 0$ then by $L(\theta)$ we intend the k' th derivative of $\nu \rightarrow L(\theta_{\chi', \nu})$ evaluated at $\nu = t'\zeta'$.

We give $\mathcal{E}^p(C(G): F)$ the topology generated by the following seminorms; let $L \in \mathcal{E}^p(C(G): F)$ and let $u \in \mathcal{S}(\mathcal{F}_c)$, $\alpha \in \mathbf{R}$ and set

$$\mu_{\alpha, r}^p(L) = \nu_{u, \alpha}^p(L) + \left(\sum_{\omega \in \hat{G}^2} |L(\omega)|^2 \right)^{1/2}.$$

PROPOSITION 1. *The map $\alpha \rightarrow \hat{\alpha}$ is a continuous map of $\mathcal{E}^p(G: F)$ into $\mathcal{E}^p(C(G): F)$.*

Proof. The fact that $\hat{\alpha}$ is defined on $\hat{M} \times \text{Int } \mathcal{F}_c(2/p - 1)$ and on $\theta_\omega(\omega \in \hat{G}^2)$ together with the holomorphy of $\hat{\alpha}(\chi)$ on $\text{Int } \mathcal{F}_c(2/p - 1)$ all follow from the corresponding properties of the transform $\alpha \rightarrow \mathcal{F}_H(\alpha)$ (cf. [15]) and the fact that $\hat{\alpha}(\chi: \nu) = \text{tr } \mathcal{F}_H(\alpha)(\chi: \nu)$. Property (5) follows from the preceding remark together with the observation that the trace map is obviously continuous and $\alpha \rightarrow \mathcal{F}_H(\alpha)$ is continuous (cf. [15]). This latter comment shows moreover that given $u \in \mathcal{S}(\mathcal{F}_c)$, $\alpha \in \mathbf{R}$, there exists η , a continuous seminorm on $\mathcal{E}^p(G: F)$, such that $\nu_{u, \alpha}^p(\hat{\beta}) \leq \eta(\beta)$ ($\beta \in \mathcal{E}^p(G: F)$). The condition of Definition 2 is obvious from (i) of § I.1.

Finally, the continuity follows on noting that properties (2) through (4) are obvious for $\hat{\alpha}$ ($\alpha \in \mathcal{E}^p(G: F)$) and hence as $|\hat{G}^2(F)| < \infty$ there exists η' a continuous seminorm on $\mathcal{E}^p(G: F)$ such that

$$\left| \sum_{\omega \in \hat{G}^2} |\hat{\alpha}(\omega)|^2 \right|^{1/2} \leq \eta'(\alpha) \quad (\alpha \in \mathcal{E}^p(G: F)).$$

The continuity is now obvious from the first part of the proof.

THEOREM 1. *The map $\alpha \rightarrow \hat{\alpha}$ is surjective.*

Proof. Let $L \in \mathcal{E}^p(C(G): F)$ and define (cf. § I.2)

$$\beta_L(x) = \sum_{\theta \in \mathcal{B}_p} L(\theta) \alpha_\theta(x).$$

An easy computation using Proposition 1 of § I.1 shows that $\hat{\beta}_L(\chi: t\zeta; \partial^k(\nu)) = L(\chi: t\zeta; \partial^k(\nu))$ and $\hat{\beta}_L(\omega) = L(\omega)$ ($\omega \in \hat{G}^2 \setminus \hat{G}^p$). In particular if we put

$$L_0 = L - \hat{\beta}_L$$

and set (cf. (2) of § I.2 for the definitions of ϕ_L)

$$\psi_{L_0}(x) = \phi_{L_0}(x) + \sum_{\omega \in \hat{G}^2} d(\omega) d_F(\omega)^{-1} L_0(\omega) \theta_\omega^F(x)$$

then reasoning as in [15] we see that $\psi_{L_0} \in \mathcal{C}^p(G; F)$. Hence

$$\psi_L(x) = \psi_{L_0}(x) + \beta_L(x) \in \mathcal{C}^p(G; F).$$

But then

$$\begin{aligned}\hat{\psi}_L(\chi; \nu) &= \hat{\phi}_{L_0}(\chi; \nu) + \hat{\beta}_L(\chi; \nu) \\ &= L(\chi; \nu)\end{aligned}$$

and if $\omega \in \hat{G}^2/\hat{G}^p$

$$\hat{\psi}_L(\omega) = \hat{\beta}_L(\omega) = L(\omega).$$

Whereas if $\omega \in \hat{G}^p$ then as $\hat{\beta}_L(\omega) = 0$ (i.e., $(\alpha_\theta, \theta_\omega) = 0$ if $\omega \in \hat{G}^p$)

$$\hat{\psi}_L(\omega) = \hat{\psi}_{L_0}(\omega) = \sum_{\omega' \in \hat{G}^2} d(\omega') d_F(\omega')^{-1} L_0(\omega') (\theta_{\omega'}^F, \theta_{\omega'}^F).$$

Again using the fact that $\hat{\beta}_L(\omega) = 0$ ($\omega \in \hat{G}^p$) and the orthogonality properties of discrete series we obtain (cf. Lemma 1, pg. 93 of [5])

$$\hat{\psi}_L(\omega) = d(\omega) d_F(\omega)^{-1} L(\omega) (\theta_{\omega}^F, \theta_{\omega}^F) = L(\omega).$$

II.1. Characterization of F_f . We shall make in this section some further assumptions on G ; further in the case of equal rank we shall pick a compact Cartan subgroup $B \subset K$, and from it construct H . This construction will give us an explicit Cayley transform of \mathfrak{b}_e onto \mathfrak{h}_e . We shall then show how to change the domain of \hat{f} from $C^p(G)$ to $L_B \cup \hat{H}_k \times \mathfrak{a}_k^* \cup \mathcal{B}_p$. This change actually extends the domains of these functions as well as we now explain. It is known that $\hat{M} \cong \hat{H}'_K/W_1$ where $W_1 = W(\mathfrak{m}_e, \mathfrak{h}_{ke})(\mathfrak{m} = \text{cent}_k(\alpha))$ whereas in § I, $\mathfrak{h} = \mathfrak{h}_k + \alpha$, $A = \exp \alpha$, $(LA(H_K) = \mathfrak{h}_k)$, \hat{H}'_K denotes the set of regular elements of \hat{H}_K (we shall make exact definitions below) and \hat{H}'_K/W_1 denotes the equivalence classes of \hat{H}'_K under W_1 . Also it was pointed out in § I that $\hat{G}^2 \cong \mathcal{L}'_B/W$; hence with these new definitions \hat{f} will be defined for both singular and regular elements of \mathcal{L}_B and \hat{H}_K respectively. This is necessitated because one must use all the characters of \hat{B} and \hat{H}_K in order to expand functions on B and H_k .

Let us assume that we have a group of equal rank; let B be a compact Cartan subgroup, \mathfrak{b} its Lie algebra. Fix a singular imaginary root β of the pair $(\mathfrak{g}, \mathfrak{b})$ and a point $\Gamma \in \mathfrak{b}$ such that $\pm\beta$ are the only roots of the pair $(\mathfrak{g}, \mathfrak{b})$ which vanish at Γ . Let $\mathfrak{g} = \text{Cent}_{\mathfrak{g}}(\Gamma)$, and $\mathfrak{c}_\Gamma, \mathfrak{l}_\Gamma$ the center and derived algebra respectively of \mathfrak{g}_Γ .

\mathfrak{l}_Γ is isomorphic over \mathbf{R} to $\mathfrak{sl}(2, \mathbf{R})$, and we may select a basis H^*, X^*, Y^* for \mathfrak{l} (over \mathbf{R}) such that $[H^*, X^*] = 2X^*$, $[H^*, Y^*] = -2Y^*$, $[X^*, Y^*] = H^*$. Then $\mathfrak{b} = \mathbf{R}(X^* - Y^*) + \mathfrak{l}_\Gamma$ and $\mathfrak{h} = \mathbf{R}H^* + \mathfrak{c}_\Gamma$ form a complete set of nonconjugate Cartan subalgebras of \mathfrak{g} . Put

$\mu = \exp[\sqrt{-1}(\pi/4)(X^* + Y^*)] \in G_c$. Then $b'_c = \mathfrak{h}_c$, if $\alpha = \beta''$, we have $\alpha(H^*) = 2$ and α vanishes identically on c_r ; we shall thus again denote by α the restriction of α to $a = RH^*$. Order the space of real linear functionals λ on $RH^* + \sqrt{-1}c_r$ by requiring that $\lambda > 0$ whenever $\lambda(H^*) > 0$. We then obtain a set of positive roots for the pair $(\mathfrak{g}, \mathfrak{b})$ by requiring that the μ -transform of such a root be positive when considered as a root of $(\mathfrak{g}, \mathfrak{h})$.

Let H be the Cartan subgroup of G associated with \mathfrak{h} , and let H^0 be the identity component of H . Then, setting $H_K = H \cap K$, $H_K^0 = H^0 \cap K$ and $A = \{\exp tH^*: t \in \mathbf{R}\}$, we have $H = H_K A$, and $H^0 = H_K^0 A$. Put $Z(A) = K \cap \exp\{iRH^*\}$. Then $Z(A) = \{1, \gamma\}$ is a group of order two with $\gamma = \exp[\pi(X^* - Y^*)] = \exp(i\pi H^*) \neq 1$. We have $H_K = Z(A)H_K^0$.

Set $b_1 = c_r$, $b_2 = R(X^* - Y^*)$ and let B_1, B_2 be the analytic subgroups of B corresponding to b_1 and b_2 respectively. B_1 and B_2 are compact and $B_1 \cap B_2 \subset Z(A)$. Since $H_K = B_1 \cup \gamma B_1$, ($B_1 = H_K^0$), it follows that H_K has one or two connected components according to whether γ lies in $B_1 \cap B_2$ or not. If $M = \text{Cent}_K(A)$, M^0 its identity component, then $M = M^0 \cup \gamma M^0$.

If no simple factor of G is isomorphic to $SL(2, \mathbf{R})$, it follows from the classification of real rank one groups that M is connected, or equivalently that $B_1 \cap B_2 = \{1, \gamma\}$. In this case $B_1 = H_K$ is a maximal torus in M . As in [11] we shall now assume that M is connected.

Choose a normalized invariant measure $d_{G/B}(\dot{x})$ as in [19] v. II. If we choose a Haar measure $d_B(b)$ on B normalized so that the volume of B is one, then a Haar measure dx on G is fixed by the formula

$$\int_G f(x) dx = \int_{G/B} \int_B f(xb) db d_{G/B}(\dot{x}) \quad (f \in C_c(G)).$$

Let $d_A(h)$ be the Haar measure on A which is the transport via the exponential map of the canonical Haar measure on the Lie algebra \mathfrak{a} associated with the Euclidean structure derived from the Killing form of \mathfrak{g} . Since $A = \{\exp tH^*: t \in \mathbf{R}\}$ we have $d_A(h) = c_A dt$ where c_A is a positive constant and dt is normalized Lebesgue measure on \mathbf{R} . Normalize Haar measure $d_{H_K}(h)$ on H_K so that the volume of H_K is one. Now a Haar measure $d_H(h)$ on H is fixed by the formula $d_H(h) = d_{H_K}(h_1) d_A(h_2)$ ($h = h_1 h_2$, $h_1 \in H_K$, $h_2 \in A$). A G -invariant measure $d_{G/H}(\dot{x})$ on G/H is then determined by the formula

$$\int_G f(x) dx = \int_{G/H} \int_H f(xh) d_H(h) d_{G/H}(\dot{x}) \quad (f \in C_c(G)).$$

If $x \in G$ write $b^{\hat{x}} = xbx^{-1}$ ($b \in B$), where \hat{x} denotes the image of x under the canonical map of G onto G/B ; similarly let $h^{\hat{x}} = xhx^{-1}$ ($h \in H$) where now \hat{x} is the image of x under the canonical map of G onto G/H . If b, h are as above, $f \in C_c^\infty(G)$ let

$$F_f^B(b) = \Delta_B(b) \int_{G/B} f(b^{\hat{x}}) d_{G/B}(\hat{x})$$

$$F_f^H(h) = \Delta_H(h) \varepsilon_R^H(h) \int_{G/H} f(h^{\hat{x}}) d_{G/H}(\hat{x}),$$

(for definitions of Δ_B , Δ_H , and ε_R^H one may refer to [19]). By Weyl's integration formula we have

$$\int_G f(x) dx$$

$$= |W(G/B)|^{-1} \int_B \overline{\Delta_B(b)} F_f^B(b) d_B(b) + |W(G/H)|^{-1} \int_H \overline{\Delta_H(h)} \varepsilon_R^H(h) F_f^H(h) d_H(h).$$

For $f \in C_c^\infty(G)$ the invariant integrals have the following properties;

- (1) $F_f^B(wb) = \det(w) F_f^B(b)$ ($w \in W(G/B)$, $b \in B'$)
- (2) $F_f^H(h_1 h_2) = F_f^H(h_1 h_2^{-1})$ ($h_1 \in H_K$, $h_2 \in A$)
- (3) $F_f^H(wh) = \det(w) F_f^H(h)$ ($h \in H'$, $w \in W(M/H_K)$).

Further, it is known that $F_f^B \in C^\infty(B')$ (here G' denotes the set of regular elements of G and for any subset L of G , $L' = L \cap G'$) and in general F_f^B does not extend to a C^∞ function on all of B . The function $F_f^H \in C^\infty(H')$ and extends to a compactly supported C^∞ function on all of H since (g, \mathfrak{h}) has no singular imaginary roots.

Recall that $\hat{B} \cong \mathcal{L}_B$. The Weyl group $W(g, \mathfrak{b})$ acts on \mathcal{L}_B and hence on \hat{B} by the prescription

$$w\lambda(H) = \lambda(w^{-1}H), \quad w\xi_\lambda(h) = \xi_{w\lambda}(h) \quad (H \in \mathfrak{b}, \lambda \in \mathcal{L}_B)$$

(here $\xi_\lambda(\exp H) = e^{\lambda(H)}$).

We say that $\lambda \in \mathcal{L}_B$ is *regular* if $w\lambda \neq \lambda$ for all $w \neq 1$ in $W(g, \mathfrak{b}_c)$; otherwise we say λ is *singular*. The set of regular λ will be denoted by \mathcal{L}_B' and the set of singular λ by \mathcal{L}_B^s . The character ξ_λ is called regular or singular accordingly.

To each $\lambda \in \mathcal{L}_B$, there is associated a central eigendistribution θ_λ on G characterized uniquely by certain properties (cf. [19]). θ_λ is locally summable on G and analytic on G' . We have

$$\theta_\lambda(b) = \Delta_B(b)^{-1} \sum_{w \in W(G/B)} \det(w) \xi_{w\lambda}(b), \quad b \in B'.$$

If $\lambda \in \mathcal{L}_B^s$ and if λ is fixed by a nontrivial element of $W(G/B)$, then $\theta_\lambda \equiv 0$ on B' .

For $A \in \mathcal{L}'_B$, put $s = 1/2 \dim(G/K)$ and $\varepsilon(A) = \text{sgn} \{ \prod_{\beta \in P_B} (A, \beta) \}$ where P_B denotes a set of positive roots of $(\mathfrak{g}, \mathfrak{h})$. Then

$$\theta_A = (-1)^s \varepsilon(A) \theta_A$$

is a character of a representation $\omega(A)$ in the discrete series for G and all discrete series characters are obtained in this way.

For $\lambda \in \hat{H}_K$, the unitary character group of H_K , denote by $\log \lambda$ the linear function on $\mathfrak{h}_1 = \mathfrak{h}_k$ defined by

$$\lambda(\exp H) = e^{\langle H, \log \lambda \rangle} \quad (H \in \mathfrak{h}_1).$$

Let P_I^+ be the set of positive imaginary roots of the pair $(\mathfrak{g}, \mathfrak{h})$, and W_1 be the subgroup of $W(\mathfrak{g}_o, \mathfrak{h}_o) = W$ which is generated by the Weyl reflections associated with elements of P_I^+ . W_1 may be identified with the Weyl group of $W(\mathfrak{m}_o, \mathfrak{h}_{ko})$. An element $\lambda \in \hat{H}_K$ is called *regular* if $w\lambda \neq \lambda$ for all $w \neq 1$ in W_1 and *singular* otherwise. If $\lambda \in \hat{H}_K$ is singular put $\varepsilon(\lambda) = 1$, and if λ is regular put

$$\varepsilon(\lambda) = \text{sgn} \left\{ \prod_{\alpha \in P_I^+} (\log \lambda, \alpha) \right\}.$$

The unitary character group \hat{A} of A is isomorphic to \mathbf{R} and, for $\nu \in \mathbf{R}$, we define the corresponding unitary character on A by

$$h^{i\nu} = e^{i\nu(\log h)} \quad (h \in A).$$

If $\phi \in C_c^\infty(A)$ define its Fourier transform by

$$\hat{\phi}(\lambda; \nu) = (2\pi)^{-1/2} \int_{H_K} \int_A \lambda(h_1) h_2^{i\nu} \phi(h_1 h_2) d_{H_K}(h_1) d_A(h_2) \quad (\lambda \in \hat{H}_K, \nu \in \mathbf{R}).$$

If λ is singular it follows from (3) that $\hat{\phi}(\lambda; \nu) \equiv 0$. We have the following inversion formula for all ϕ as above;

$$(*) \quad \phi(h_1 h_2) = C_A^{-1} (2\pi)^{-1/2} \sum_{\lambda \in \hat{H}_K} \overline{\lambda(h_1)} \int_{-\infty}^{\infty} e^{-i\nu(\log h_2)} \hat{\phi}(\lambda; \nu) d\nu,$$

where $d\nu$ is normalized Lebesgue measure on \mathbf{R} .

The following now gives the relation between the Fourier transform of the invariant integral F_f^H and the invariant transform of f with respect to principal series characters. Suppose $\lambda \in \hat{H}_K$ is a regular character, $\nu \in \hat{A}$, and $r_I = |P_I^+|$. Then the distribution,

$$(4) \quad \theta_{\lambda, \nu}(f) = (2\pi)^{1/2} (-1)^{r_I} \varepsilon(\lambda) \hat{F}_f^H(\lambda; \nu) \quad (f \in C_c^\infty(G))$$

is the character of a principal series representation which we denote by $\pi_{\chi(\lambda), \nu}$. We have then in the notation of § I that

$$(5) \quad \theta_{\lambda, \nu} = \theta_{\chi(\lambda), \nu} \quad (\forall \text{ regular } \lambda \text{ in } \hat{H}_K).$$

If λ is singular define $\theta_{\lambda,\nu}$ by the right hand side of (4). Of course $\theta_{\lambda,\nu} \equiv 0$ for singular λ . It follows from the general theory of finite dimensional representations of semisimple Lie groups that

$$(6) \quad \theta_{s\lambda,\nu} = \theta_{\lambda,\nu} \quad (s \in W_1)$$

where if $\lambda(H) = e^{\langle H, \log \lambda \rangle}$ then $s\lambda(H)$ is defined to be $e^{\langle H, s \log \lambda \rangle}$.

Note also that from (2) we have

$$(7) \quad \theta_{\lambda,\nu} = \theta_{\lambda,-\nu} \quad (\lambda \in \hat{H}_K, \nu \in R).$$

If $\phi \in L^1(B)$ define

$$\hat{\phi}(A) = \int_B \xi_A(b) \phi(b) db \quad (A \in L_B).$$

We shall now need the following result.

THEOREM 1. ([11] Theorem 3.19.). *Suppose that $b_0 \in B'$. For $w \in W(G/B)$, we write $w^{-1}b_0 = b_1(w)b_2(w)$ where $b_1(w) \in B_1$ and $b_2(w) = \exp(\theta_w(X^* - Y^*)) \in B_2$. Then, if $f \in C_c^\infty(G)$ we have*

$$F_f^B(b_0) = (-1)^r \sum_{A \in L_B} \theta_A(f) \overline{\xi_A(b_0)} + I_f(b_0)$$

where

$$\begin{aligned} & I_f(b_0) \\ &= \frac{1}{2} (-1)^{r_I} |W(g_c, \mathfrak{h}_c)|^{-1} \sum_{w \in W(G/B)} \det(w) \sum_{\lambda \in \hat{H}_K} \varepsilon(\lambda) \\ & \quad \times \left\{ \overline{\lambda(b_1(w))} \int_{-\infty}^{\infty} \theta_{\lambda,\nu}(f) \eta_1(\nu; \theta_w) d\nu + \overline{\lambda(\gamma b_1(w))} \int_{-\infty}^{\infty} \theta_{\lambda,\nu}(f) \eta_2(\nu; \theta_w) d\nu \right\} \end{aligned}$$

where

$$\begin{aligned} \eta_1(\nu; \theta_w) &= \sinh(\nu(\theta_w \mp \pi)) / \sinh(\nu\pi) \\ \eta_2(\nu; \theta_w) &= \sinh(\nu\theta_w) / \sinh(\nu\pi). \end{aligned}$$

Here in the definition of η_1 we choose the minus sign if $0 < \theta_w < \pi$ for all $w \in W(G/B)$ and the plus sign otherwise.

Now for $f \in \mathcal{C}^p(G; F)$ (cf. § I) let us put for $A \in \mathcal{L}_B$, $\lambda \in \hat{H}_K$, $\nu \in R$

$$\hat{f}(A) = \theta_A(\check{f}), \hat{f}(\lambda; \nu) = \theta_{\lambda,\nu}(\check{f}).$$

Henceforth we shall assume that the invariant transform is defined on $\mathcal{D}_p = \mathcal{L}_B \cup \hat{H}_K \times \mathcal{F}_c(2/p - 1) \cup \mathcal{B}_p$ ($0 < p < 2$). Using the isomorphisms $\hat{G}^2 \cong \mathcal{L}'_B/W(G/B)$ and $\hat{M} \cong \hat{H}'_K/W_1$ it is not hard to give the following characterization of the invariant transform of $\mathcal{C}^p(G; F)$. First we need one fact; by [1] there exists for each

$\Lambda \in \mathcal{L}_B^s$ a distribution S_Λ living on $\mathcal{L}_B' \cup \hat{H}_K' \times \mathcal{F}$ such that

$$\theta_\Lambda(f) = S_\Lambda((f)^\wedge) \quad (f \in \mathcal{C}(G)).$$

That is, the values of \hat{f} on \mathcal{L}_B^s are determined by its values on the *characters* of the tempered representations of G . It follows from this that if $f \in \mathcal{C}^p(G; F)$ then

$$(*) \quad \hat{f}(\Lambda) = S_\Lambda(\hat{f}).$$

Let now $\mathcal{C}^p(C(G); F)$ be defined as those functions $L: \mathcal{D}_p \rightarrow \mathbb{C}$ satisfying properties of Definitions 1 and 2 of § I.3 subject to the changes that $\lambda \in \hat{M}$ is to be replaced by $\lambda \in \hat{H}_K$, ω by $\lambda \in \mathcal{L}_B$, and $\hat{M}(F)$, $\hat{G}^2(F)$ are to be replaced by the sets $\hat{H}_K(F)$ and $\mathcal{L}_B(F)$ which have their obvious meaning. We must then add two further conditions reflecting (*) which we number conditions 6 and 7.

$$(6) \quad L(\omega\lambda; \nu) = L(\lambda; \nu)(\omega \in W_1), \quad L(\omega\Lambda) = L(\Lambda) \quad (\omega \in W(G/B))$$

$$(7) \quad L(\Lambda) = S_\Lambda(L) \quad (\Lambda \in \mathcal{L}_B^s).$$

We shall now designate the normalized orbital integrals of $f \in \mathcal{C}(G)$ by F_f ; the argument of F_f will make it clear whether we are considering F_f^H or F_f^B . Similarly when we take the Fourier transform of F_f the arguments again will make it clear whether we are transforming on H or B .

PROPOSITION 1. For all $f \in \mathcal{C}^p(G; F)$ ($0 < p < 2$)

$$(i) \quad \hat{F}_f(\lambda; \nu) = (2\pi)^{-1/2}(-1)^{r_1} \varepsilon(\lambda)(\check{f})^\wedge(\lambda; \nu) \quad (\lambda \in \hat{H}_K, \nu \in \mathcal{F}_c(2/p-1))$$

(ii) Let

$${}^\circ F_f(b) = F_f(b) - I_f(b) \quad (b \in B')$$

then

$$({}^\circ F_f)^\wedge(\Lambda) = (-1)^r(f)^\wedge(\Lambda) \quad (\Lambda \in \mathcal{L}_B).$$

Proof. (i) is just a reformulation of (4); (ii) follows from Theorem 1 and Fourier inversion on B .

Let $0 < p < 2$ and for $\phi: B' \cup H \rightarrow \mathbb{C}$, ϕ of class C^∞ , $u \in \mathfrak{X}$, $v \in \mathfrak{B}$, $n \in \mathbb{Z}$, and set

$$\eta_{u,v,n}^p(\phi) = \sup_{H_K \times A} e^{(2/p-1)\rho(\log h_2)} (1 + \sigma(h_2))^n |\phi(h_1 h_2; u)| + \sup_{B'} |\phi(b; v)|.$$

Let θ be the character of a quasi-simple representation of G . Set $\check{\theta}(\alpha) = \theta(\check{\alpha})$ ($\alpha \in C_c^\infty(G)$) and,

$$\begin{aligned} \theta_B &= [W(G/B)]^{-1} \bar{\mathcal{A}}_B \cdot \theta, & \theta_B^* &= [W(G/B)]^{-1} \bar{\mathcal{A}}_B \cdot \check{\theta} \\ \theta_H &= [W(G/H)]^{-1} \bar{\mathcal{A}}_H \cdot \varepsilon_H^\eta \cdot \theta, & \theta_H^* &= [W(G/H)]^{-1} \bar{\mathcal{A}}_H \cdot \varepsilon_H^\eta \cdot \check{\theta}. \end{aligned}$$

Then for all $f \in \mathcal{C}(G)$ we have

$$\begin{aligned}
\theta(f) &= \int_{B'} \theta_B(b) F_f(b) d_B(b) + \int_H \theta_H(h) F_f(h) d_H(h) \\
&= \int_{B'} \theta_B^*(b) F_f(b) d_B(b) + \int_H \theta_H^*(h) F_f(h) d_H(h) \\
&= (\theta_B^*, F_f) + (\theta_H^*, F_f).
\end{aligned}$$

The second line follows from; $|\Delta_H(h)|^2 = |\Delta_H(h^{-1})|^2$, $|\Delta_B(b)|^2 = |\Delta_B(b^{-1})|^2$.

PROPOSITION 2. *Let $f \in \mathcal{C}^p(G: F)$ ($0 < p < 2$).*

(a) $\hat{F}_f(\lambda)$ is a holomorphic function on $\text{Int}(\mathcal{F}_e(2/p - 1))$ for all $\lambda \in \hat{H}_K$.

(b) $\hat{F}_f(\lambda) \equiv 0$ if $\lambda \notin \hat{H}_K(F)$.

(c) $\hat{F}_f(s\lambda: s\nu) = \varepsilon(s\lambda)/\varepsilon(\lambda) \hat{F}_f(\lambda: \nu)$ ($s \in W(A)$, $\lambda \in \hat{H}_K$, $\nu \in \mathcal{F}_e(2/p - 1)$).

(d) $\hat{F}_f(\lambda: t\zeta; \partial^k(\nu)) = (2\pi)^{-1/2} (-1)^{r_I} \varepsilon(\lambda) \sum_{\theta \in \mathcal{D}_p} C_p(\theta_{\lambda, t\zeta, k}; \theta) ((\theta_B^*, F_f) + (\theta_H^*, F_f))$ for all $t \in W(A)$, $\zeta \in V_p$, $0 \leq k \leq 0_i(\zeta) - 1$.

(e) ${}^\circ \hat{F}_f(A) = 0$ if $A \notin \mathcal{L}_B(F)$.

(f) $\eta_{u, v, n}^p(F_f) < \infty$ for all $u \in \mathfrak{G}$, $v \in \mathfrak{B}$, $n \in \mathbf{Z}$.

Proof. Statements (a) through (e) all follow from (i) of Proposition 1 together with the corresponding properties of \hat{f} .

For (f) we note that

$$F_f(h_1 h_2) = C_A^{-1} (2\pi)^{-1} (-1)^{r_I} \sum_{\lambda \in \hat{H}_K} \varepsilon(\lambda) \overline{\lambda(h_1)} \int_{-\infty}^{\infty} e^{-i\nu(\log h_2)} (f)^\vee(\lambda: \nu) d\nu.$$

Hence, if $u \in \mathfrak{G}$ then we can write $u = \sum_i \xi_i \eta_i$ where $\xi_i \in \mathfrak{G}_k$ and $\eta_i \in \mathfrak{A}$. From this and the fact that $\hat{f}(\lambda: \nu) \equiv 0$ for $\lambda \notin \hat{H}_K(F)$ we have,

$$\begin{aligned}
F_f(h_1 h_2; u) \\
= \text{const} \sum_j \sum_{\lambda \in \hat{H}_K(F)} \varepsilon(\lambda) \overline{\xi_j(\log \lambda)} \lambda(h_1) \int_{-\infty}^{\infty} e^{-i\nu(\log h_2)} \eta_j(-i\nu) (f)^\vee(\lambda: \nu) d\nu.
\end{aligned}$$

Using the holomorphy on $\text{Int}(\mathcal{F}_e(2/p - 1))$ and continuity on $\mathcal{F}_e(2/p - 1)$ we have on letting $\rho_p = (2/p - 1)$ that

$$\begin{aligned}
F_f(h_1 h_2; u) \\
= \text{const} \sum_j \sum_{\lambda} \varepsilon(\lambda) \overline{\xi_j(\log \lambda)} \lambda(h_1) e^{-\rho_p(\log h_2)} \\
\times \int_{-\infty}^{\infty} e^{-i\nu(\log h_2)} \eta_j(-\rho_p - i\nu) (f)^\vee(\lambda: -\rho_p - i\nu) d\nu.
\end{aligned}$$

One can easily deduce from Lemma 8.1 of [15] that the function $\nu \rightarrow (f)^\vee(\lambda: -\rho_p - i\nu)$ belongs to the Schwartz space of R for all $\lambda \in \hat{H}_K(F)$ and that there exists for every $n \in N$ (here we are combining the fact that the map $f \rightarrow \hat{f}$ is continuous with Lemma 8.1 *ibid.*) a continuous seminorm ν on $\mathcal{C}^p(G: F)$ such that

$$(1 + \sigma(h))^n \sum_j \left| \int_{-\infty}^{\infty} e^{-i\nu(\log h_2)} \eta_j(-\rho_p - i\nu) (f)^\vee(\lambda: -\rho_p - i\nu) d\nu \right| \leq \nu(f).$$

As $|\hat{H}_K(F)| < \infty$ then it is clear that there exists a continuous seminorm ν_1 on $\mathcal{C}^p(G; F)$ such that

$$\sup_{H_K \times A} e^{(2/p-1)\rho(\log h_2)} (1 + \sigma(h_2))^n |F_f(h_1 h_2; u)| \leq \nu_1(f).$$

Let $v \in \mathfrak{B}$. Then from Theorem 11 of [18] it follows that vF_f has finite jump discontinuities on $B - B'$; moreover, the jumps can be bounded by derivatives of $F_f|_H$. Given the continuity of the map $f \rightarrow F_f$, as a map of $\mathcal{C}^p(G; F)$ into $C^\infty(B')$ (cf. [18] § 12.2), and the continuity of the map $f \rightarrow F_f$ as a map of $\mathcal{C}^p(G; F)$ into $C^\infty(H)$ we see that there exists a continuous seminorm ν_2 on $\mathcal{C}^p(G; F)$ such that

$$|F_f(b; v)| \leq \nu_2(f).$$

Hence (f) follows.

DEFINITION 1. Let $D = B' \cup H$ and for every $0 < p < 2$ let $I^p(D; F)$ denote the space of all functions $\phi: D \rightarrow C$ such that $\phi|_H \in C^\infty(H)$, $\phi|_{B'} \in C^\infty(B')$, $\phi(h_1 h_2) = \phi(h_1 h_2^{-1})$ ($h_1 \in H$, $h_2 \in A$), $\phi(wh) = \det(w)\phi(h)$ ($h \in H$, $w \in W_1$), $\phi(wb) = \det(w)\phi(b)$ ($w \in W(G/B)$, $b \in B'$), and if $b \in B'$ and,

$$(9) \quad I_\phi(b) = (i/2) |W(\mathfrak{g}_e, \mathfrak{h}_e)|^{-1} \sum_{w \in W(G/B)} \det(w) \sum_{\lambda \in \hat{H}_K} (2\pi)^{1/2} \cdot \\ \left\{ \overline{\lambda(b_1(w))} \cdot \int_{-\infty}^{\infty} \hat{\phi}(\lambda; \nu) \eta_1(\nu; \theta_w) d\nu + \overline{\lambda(\gamma b_1(w))} \int_{-\infty}^{\infty} \hat{\phi}(\lambda; \nu) \eta_2(\nu; \theta_w) d\nu \right\}$$

(η_1, η_2 defined in Theorem 1), and

$$(10) \quad {}^\circ\phi(b) = \phi(b) - I_\phi(b)$$

then

$${}^\circ\hat{\phi}(\lambda) = 0 \text{ if } \lambda \notin \mathcal{L}_B(F).$$

Further we require that ϕ satisfy properties (a) through (d) and property (f) of Proposition 2 (with ϕ replacing F_f).

We topologize $I^p(D; F)$ using the seminorms $\eta_{v,n}^p$. As a result of Proposition 2 and its proof we have the following.

COROLLARY. *The map $f \rightarrow F_f$ is a continuous map of $\mathcal{C}^p(G; F)$ into $I^p(D; F)$ ($0 < p < 2$).*

We now come to the principal result of § II.

THEOREM 2. *The map $f \rightarrow F_f$ is a continuous surjection of $\mathcal{C}^p(G; F)$ onto $I^p(D; F)$.*

Proof. All that remains to be shown is the surjectivity. Let $\phi \in I^p(D; F)$ and put

$$\begin{aligned} L(\lambda; \nu) &= (2\pi)^{1/2}(-1)^{r_I \varepsilon(\lambda)} \hat{\phi}(\lambda; \nu) & (\lambda \in \hat{H}_K, \nu \in \mathcal{F}_e(2/p - 1)) \\ L(\theta) &= (\theta_B^*, \phi) + (\theta_H^*, \phi) & (\theta \in B_p) \\ L(A) &= (-1) \hat{\phi}(A) & (A \in \mathcal{L}_B) \end{aligned}$$

here ${}^\circ \phi$ is defined as in (10).

As $\phi(h_1 h_2)$ is of exponential type in h_2 for each $h_1 \in H_K$ it follows from the classical Paley-Wiener theorem applied to the vector group A that for each $\lambda \in \hat{H}$, $\nu \rightarrow L(\lambda; \nu)$ extends to a holomorphic function on $\text{Int}(\mathcal{F}_e(2/p - 1))$ and that for each $u \in \mathcal{S}(\mathcal{F}_e)$, $\alpha \in \mathbf{R}$

$$\sup |L(\lambda; \nu; u)| (1 + |\nu|)^\alpha < \infty$$

where the sup is taken over $(\lambda, \nu) \in \hat{H}_K \times \text{Int}(\mathcal{F}_e(2/p - 1))$. Further, since $\varepsilon(w\lambda) = \varepsilon(\lambda)$ ($w \in W_1$, $\lambda \in \hat{H}_K$) it follows from property (c) of Proposition 2 that,

$$\begin{aligned} L(w\lambda; \nu) &= L(\lambda; \nu) \quad \text{and} \quad L(s\lambda; s\nu) = L(\lambda; \nu) \\ L(\lambda; \nu) &\equiv 0 \quad (\lambda \notin \hat{H}_K(F)) \quad \text{and} \quad L(A) = 0 \quad (A \notin \mathcal{L}_B(F)). \end{aligned}$$

We have for $t \in W(A)$, $\zeta \in V_p$, $\lambda \in \hat{H}_K$, $0 \leq k \leq 0_i(\zeta) - 1$ that

$$\begin{aligned} L(\lambda; t\zeta; \partial^k(\nu)) &= (2\pi)^{1/2}(-1)^{r_I \varepsilon(\lambda)} \hat{\phi}(\lambda; t\zeta; \partial^k(\nu)) \\ &= \sum_{\theta \in \mathcal{O}_p} C_p(\theta_{\lambda, t\zeta, k}; \theta) ((\theta_B^*, \phi) + (\theta_H^*, \phi)) \\ &= \sum_{\theta \in \mathcal{O}_p} C_p(\theta_{\lambda, t\zeta, k}; \theta) L(\theta). \end{aligned}$$

It follows from Theorem 1 of § I.3 that as $L \in \mathcal{E}^p(C(G); F)$ there exists $f \in \mathcal{E}^p(G; F)$ such that $\hat{f} = L$. Hence by Proposition 1,

$$\hat{F}_f(\lambda; \nu) = (2\pi)^{-1/2}(-1)^{r_I \varepsilon(\lambda)} \hat{f}(\lambda; \nu) = \hat{\phi}(\lambda; \nu).$$

Therefore

$$F_f(h) = \phi(h) \quad (h \in H).$$

We also have

$${}^\circ \hat{F}_f(A) = (-1) {}^\circ \hat{f}(A) = {}^\circ \hat{\phi}(A).$$

It follows that

$${}^\circ F_f(b) = {}^\circ \phi(b) \quad (b \in B').$$

Hence,

$$F_f(b) = {}^\circ F_f(b) + I_f(b) = {}^\circ \phi(b) + I_\phi(b) = \phi(b) \quad (b \in B').$$

The last line of equalities following directly from the first part of the proof and (9) and (10).

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