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**ON THE DECOMPOSITION OF REDUCIBLE PRINCIPAL  
SERIES REPRESENTATIONS OF  $p$ -ADIC CHEVALLEY  
GROUPS**

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**In this paper we study the decomposition of principal series representations of  $p$ -adic Chevalley groups which are induced from a minimal parabolic subgroup, and determine the structure of the commuting algebras of these representations.**

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**Introduction.** Let  $G$  be a split reductive  $p$ -adic group,  $T$  a maximal split torus of  $G$  and  $B = TU$  a minimal parabolic subgroup of  $G$ . A (unitary) character  $\lambda$  of  $T$  may be extended trivially across  $U$  to define a character of  $B$ . The induced representation  $\text{Ind}_B^G \lambda$  is called a (unitary) principal series representation of  $G$ .

Let  $W$  be the Weyl group of  $G$  and choose  $w \in W$ . Then the representations  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w\lambda$  are equivalent. The problem of constructing explicit intertwining operators  $a(w, \lambda)$  between  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w\lambda$  has been studied for real semi-simple Lie groups by Kunze and Stein [24, 25, 26] Schiffmann [30], Knapp [14, 15, 16] Knapp and Stein [17, 18, 19, 20, 21, 22] Harish-Chandra [10] and others. For groups defined over a  $p$ -adic field  $\mathfrak{k}$ , these operators were first studied for  $\text{SL}(2)$  by Sally [28], and then for  $p$ -adic Chevalley groups by Winarsky [36, 37], who used them to determine necessary and

sufficient conditions for  $\text{Ind}_B^G \lambda$  to be reducible. A more general study of intertwining operators for  $p$ -adic groups has been carried out by Harish-Chandra, Silberger and others.

Let  $W_\lambda = \{w \in W | w\lambda = \lambda\}$ . By Bruhat theory [32], the length of the composition series of  $\text{Ind}_B^G \lambda$  is bounded by  $|W_\lambda|$ . Thus  $\text{Ind}_B^G \lambda$  is irreducible if  $\lambda$  is a nonsingular character of  $T$ , i.e.,  $W_\lambda = \{1\}$ .

Suppose that  $\lambda$  is a singular character of  $T$  and that  $w\lambda = \lambda$ ,  $1 \neq w \in W$ . Then  $\alpha(w, \lambda)$  is an intertwining operator for  $\text{Ind}_B^G \lambda$  which may or may not be scalar. By an unpublished theorem of Harish-Chandra, the operators  $\{\alpha(w, \lambda) | w \in W_\lambda\}$  span the commuting algebra  $C(\lambda)$  of  $\text{Ind}_B^G \lambda$ . However, these operators may not be distinct.

We determine a basis for  $C(\lambda)$  consisting of a subgroup of these operators. Following Knapp and Stein [14, 19], we write  $W_\lambda = R \ltimes W'$  as a semi-direct product, with  $W' = \{w \in W_\lambda | \alpha(w, \lambda) \text{ is scalar}\}$ . We show that, with appropriate normalizations, a cocycle condition holds and that  $w \mapsto \alpha(w, \lambda)$  is a homomorphism from  $W_\lambda$  to the group of invertible intertwining operators for  $\text{Ind}_B^G \lambda$ . We then give an elementary proof that the operators  $\{\alpha(w, \lambda) | w \in R\}$  are linearly independent. This is essentially Silberger's theorem [33] for the case of minimal parabolics. These facts combined with Harish-Chandra's theorem imply that  $\{\alpha(w, \lambda) | w \in R\}$  is a basis of the commuting algebra  $C(\lambda)$ , and further, that  $C(\lambda)$  is isomorphic to the group algebra  $C[R]$ .

For complex groups,  $\text{Ind}_B^G \lambda$  is always irreducible.

Knapp, in collaboration with Stein, [15, 16] has shown that for real groups,  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors of  $\mathbf{Z}_2$  bounded by the dimension of  $T$ . Thus  $\text{Ind}_B^G \lambda$  decomposes into  $|R|$  components, each occurring with multiplicity one.

For  $p$ -adic groups,  $\text{Ind}_B^G \lambda$  does not always decompose simply. We classify the nontrivial  $R$ -groups which occur.

*Type  $A_n$ .*  $R$  is abelian and  $|R|$  divides  $n + 1$ . If the largest cyclic subgroup of  $R$  has order  $m$ , then  $|R|$  divides  $[f^*: (f^*)^m]$ . Any finite abelian group with these properties occurs as an  $R$ -group.

*Type  $B_n$ .*  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  and  $|R|$  divides both  $2n$  and  $[f^*: (f^*)^2]$ .

*Type  $C_n$ .*  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors of  $\mathbf{Z}_2$  bounded by  $n$  and  $[f^*: (f^*)^2] - 1$ .

*Type  $D_n$ .*  $R$  may be nonabelian. (This general fact was first discovered by Knapp and Zuckerman.)

(a) Suppose  $n$  even. Then if  $R$  is abelian,  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors bounded by  $n - 1$  and by  $[f^*: (f^*)^2] - 1$ .

If  $R$  is nonabelian,  $R \cong (\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2) \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2)$  with the order of the first factor dividing both  $2n$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2]$  and the number of factors of  $\mathbf{Z}_2$  in the normal subgroup an odd number bounded by  $n - 1$  and by  $[\mathfrak{f}^*: (\mathfrak{f}^*)^3] - 1$ .

(b) Suppose  $n$  is odd. Then if  $R$  is abelian,  $R \cong \mathbf{Z}_3 \times \cdots \times \mathbf{Z}_2$  with the number of factors bounded by  $n - 1$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ , or  $R \cong \mathbf{Z}_4$ . If  $R$  is nonabelian, then  $R \cong \mathbf{Z}_4 \times (\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2)$  with the number of factors of  $\mathbf{Z}_2$  in the normal subgroup an even number bounded by  $n - 3$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 2$ .

*Type E<sub>6</sub>.*  $R \cong 1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_3 \times \mathbf{Z}_3$  or  $\mathbf{Z}_6$ . Further,  $\mathbf{Z}_3 \times \mathbf{Z}_3$  can occur if and only if  $p = 3$  or 3 divides  $q - 1$ .

*Type E<sub>7</sub>.*  $R$  may be nonabelian. If so,  $R \cong$  dihedral group  $D$  of order 8, or  $R \cong D \times \mathbf{Z}_2$ .  $D \times \mathbf{Z}_2$  can occur if and only if  $p = 2$  or 4 divides  $q - 1$ .

If  $R$  is abelian, then  $R \cong \mathbf{Z}_2^n$  with  $0 \leq n \leq 4$ ,  $\mathbf{Z}_3, \mathbf{Z}_4$ , or  $\mathbf{Z}_6$ .  $\mathbf{Z}_2^n$  will occur if and only if  $[k^*: (k^*)^2] \geq 2^n$ ,  $0 \leq n \leq 4$ .  $\mathbf{Z}_4$  occurs if and only if  $p = 2$ .  $\mathbf{Z}_3$  and  $\mathbf{Z}_6$  occur if and only if  $p = 3$  or 3 divides  $q - 1$ .

*Type E<sub>8</sub>.*  $R$  may be nonabelian. All nonabelian  $R$  are conjugate. The nonabelian  $R$ -group will occur if and only if  $[k^*: (k^*)^2] \geq 16$ . It has order 128, has 65 conjugacy classes, and  $R \bmod \langle w_0 \rangle$  is abelian.

If  $R$  is abelian, then  $R \cong \mathbf{Z}_2^n$  with  $0 \leq n \leq 4$ ,  $\mathbf{Z}_4, \mathbf{Z}_4 \times \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_3 \times \mathbf{Z}_3$ , or  $\mathbf{Z}_5$ .  $\mathbf{Z}_2^n$  occurs if and only if  $[k^*: (k^*)^2] \geq 2^{n+1}$ ,  $0 \leq n \leq 4$ .  $\mathbf{Z}_4$  occurs if and only if  $p = 2$  or 4 divides  $q - 1$ .  $\mathbf{Z}_4 \times \mathbf{Z}_2$  occurs if and only if  $p = 2$ .  $\mathbf{Z}_3 \times \mathbf{Z}_3$  occurs if and only if  $[k^*: (k^*)^3] \geq 3^{n+1}$ ,  $n = 1$  or 2.  $\mathbf{Z}_5$  occurs if and only if  $[k^*: (k^*)^5] \geq 25$ .

*Type F<sub>4</sub>.*  $R \cong \mathbf{Z}_2$  or  $\mathbf{Z}_3$ .  $\mathbf{Z}_3$  can occur as  $R$ -group if and only if  $p = 3$  or 3 divides  $q - 1$ .

*Type G<sub>2</sub>.*  $R \cong \mathbf{Z}_2$ .

The order of  $R$  depends on  $n$  and on the arithmetic of the field  $\mathfrak{f}$ , i.e., on the existence of enough multiplicative characters of order 2, or of order dividing  $n + 1$  in the case of type  $A_n$  and of order 3 in the case of type  $F_4$ .

We note that the methods in this paper also apply to Chevalley groups defined over the reals  $\mathbf{R}$  and the complex numbers  $\mathbf{C}$ . Since  $\mathbf{C}^*$  has no nontrivial characters of finite order,  $R = \{1\}$  and thus  $\text{Ind}_2^G \lambda$  is irreducible for Chevalley groups over  $\mathbf{C}$ . Since  $\mathbf{R}^*$  has only

one nontrivial character of finite order, we can recover the Knapp-Stein result for Chevalley groups over  $R$ . Further  $R \cong \mathbb{Z}_2$  or  $\{1\}$  except in the case of  $D_n$ ,  $n$  even, for which  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  can occur [19].

The organization of this paper is as follows. We establish notation and definitions in a preliminary section. In §1 of Chapter 1 we study the normalization and analytic continuation of the intertwining operators  $A(w, \lambda)$  and  $\alpha(w, \lambda)$  for Macdonald's "groups of  $p$ -adic type." In §2 we show that with appropriate normalizations the operators  $\alpha(w, \lambda)$  are well-defined and establish a cocycle relation for these operators with no condition on the lengths of the Weyl group elements. In §3 we follow Knapp [14, 15] to develop the theory of the  $R$ -group for  $p$ -adic Chevalley groups, and show that  $C(\lambda) = C[R]$ .

Chapter 2 is devoted to the classification of  $R$ -groups. In each section, we explicitly determine all  $R$  which occur for one type of root system, by constructing a list of  $\lambda$  and  $R$  and showing that every nontrivial  $R$ -group is conjugate to one on the list.

In Chapter 3 we use the intertwining operators to study the problem of decomposing  $\text{Ind}_B^G \lambda$  into irreducible components in a "Fourier transform realization" on  $L^2(V)$ , where  $V$  is the unipotent radical of the Borel subgroup opposed to  $B$ . A class of functions is found on which  $\alpha(w, \lambda)$  acts as multiplication by a function  $M(w, \lambda)$  and we show that the operators  $\{\alpha(w, \lambda) \mid w \in R\}$  are linearly independent.

Most of these results appeared in the author's thesis. I would like to express my gratitude and thanks to my advisor, Professor Paul J. Sally, Jr., for his help and guidance.

With some restrictions on the residual characteristic of  $k$ , independent work of Müller gives partial results describing the  $R$ -groups which occur for the classical Chevalley groups. See "Integrales d'entrelacement pour un groupe de Chevalley sur un corps  $p$ -adique" in the Springer Lecture Notes 739.

**Preliminaries and definitions.** Let  $\mathfrak{k}$  be a nonArchimedean local field. We will be concerned mainly with Chevalley groups  $G$  defined over  $\mathfrak{k}$ , although some of our results will apply to the  $\mathfrak{k}$ -rational points of any reductive algebraic group defined over  $\mathfrak{k}$ .

Let  $dx$  be Haar measure on  $\mathfrak{k}$  and  $|\cdot|$  the absolute value on  $\mathfrak{k}$  defined by  $d(ax) = |a|dx$ .

Let  $\mathcal{O} = \{x \in \mathfrak{k} \mid |x| \leq 1\}$  be the ring of integers of  $\mathfrak{k}$ ,  $\mathfrak{p}$  a prime element of  $\mathcal{O}$ , and  $\mathfrak{p} = \{x \in \mathfrak{k} \mid |x| < 1\}$  the unique nonzero prime ideal of  $\mathcal{O}$ . Then  $\mathcal{O}/\mathfrak{p}$  is a finite field with  $q$  elements, where  $q$  is a prime power.

Normalize Haar measure on  $\mathfrak{k}$  so that volume  $(\mathcal{O}) = 1$ . Then  $\mathfrak{p}^n = \{x \in \mathfrak{k} \mid |x| \leq q^{-n}\}$  has volume  $q^{-n}$ . The collection  $\mathfrak{p}^n$ ,  $n \in \mathbf{Z}$ , forms a fundamental system of neighborhoods at 0 for the topology on  $\mathfrak{k}$ , which are both open and compact. Thus  $\mathfrak{k}$  is totally disconnected.

Haar measure on  $\mathfrak{k}^*$  is  $d^*x = |x|^{-1}dx$ .

Let  $U_0 = U = \mathcal{O}^* = \{x \in \mathcal{O} \mid |x| = 1\}$  be the units in  $\mathcal{O}$ . For each positive integer  $n$ , set  $U_n = 1 + \mathfrak{p}^n$ . Then the collection  $U_n$  forms a fundamental system of neighborhoods at 1 for  $\mathfrak{k}^*$  consisting of compact and open subgroups.

The additive group of  $\mathfrak{k}$  is self-dual. Fix a nontrivial additive character  $\chi$  of  $\mathfrak{k}$ . Then any character of  $\mathfrak{k}$  is of the form  $\chi_a(x) = \chi(ax)$ . Define the conductor  $\text{cond}(\chi)$  of  $\chi$  to be  $n$  if  $\chi$  is trivial on  $\mathfrak{p}^n$  and nontrivial on  $\mathfrak{p}^{n-1}$ .

Since any  $x \in \mathfrak{k}^*$  may be written as  $x = \Pi^n u$ ,  $n \in \mathbf{Z}$ ,  $u \in U$ , we see that  $\mathfrak{k}^* \cong \mathbf{Z} \times U$ . Thus  $(\mathfrak{k}^*)^\wedge \cong \mathbf{Z}^\wedge \times U^\wedge$  and any character of  $\mathfrak{k}^*$  is given by  $\lambda(\Pi^n u) = |\Pi^n|^s \lambda^*(u)$  where  $s \in \mathbf{C}$ ,  $\text{Re } s = 0$ , and  $\lambda^*$  is the restriction of  $\lambda$  to the compact group  $U$ . We obtain quasi-characters of  $\mathfrak{k}^*$  by  $\lambda(\Pi^n u) = |\Pi^n|^s \lambda^*(u)$  where  $s \in \mathbf{C}$ . Define  $\text{Re } \lambda = \text{Re}(s)$ .  $\lambda$  is *unramified* if  $\lambda^* = 1$ . Otherwise  $\lambda$  is *ramified*. Define  $\text{deg}(\lambda) = n$  if  $\lambda$  is trivial on  $U_n$  but nontrivial on  $U_{n-1}$ .

A gamma function  $\Gamma(\lambda)$  is associated to each nontrivial multiplicative quasi-character  $\lambda$  [29, 35]. If  $\lambda = |\cdot|^s \lambda^*$  is ramified of degree  $h$ , then  $\Gamma(\lambda) = P \cdot V \cdot \int \bar{\chi}(x) \lambda(x) |x|^{-1} dx = c_{\lambda^*} q^{h(s-1/2)}$ , where  $|c_{\lambda^*}| = 1$  and  $c_{\lambda^*} c_{\lambda^*} = \lambda^*(-1)$ . If  $\lambda = |\cdot|^s$  is unramified, then  $\Gamma(\lambda) = P \cdot V \cdot \int \bar{\chi}(x) |x|^{s-1} dx = (1 - q^{s-1}) / (1 - q^{-s})$  if  $\text{Re } \lambda > 0$ , and is the analytic continuation of this function into the left half-plane for  $\text{Re } \lambda \leq 0$ ,  $s \neq 0$ .

Let  $G$  be a Chevalley group over  $\mathfrak{k}$  [34]. Let  $L$  be the semi-simple Lie algebra over  $\mathbf{C}$  which determines  $G$  and  $\underline{h}$  a Cartan subalgebra. Then  $L = \underline{h} \oplus \sum_{\alpha \neq 0} L_\alpha$  where  $\alpha$  is a root. Denote the set of roots by  $\Phi$ .

Let  $w_\alpha$  denote the reflection in the hyperplane orthogonal to  $\alpha$  in the Euclidean space  $\mathbf{Z}[\Phi] \otimes \mathbf{R}$  and let the Weyl group  $W$  be the group generated by the  $w_\alpha$ ,  $\alpha \in \Phi$ .

$G$  is generated by subgroups  $U_\alpha = \{x_\alpha(t) \mid t \in \mathfrak{k}\}$ ,  $\alpha \in \Phi$ .  $U_\alpha$  carries a natural valuation  $U_{\alpha+n} = \{x_\alpha(t) \mid t \in \mathfrak{p}^n\}$ .

Let  $w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t)$  and  $h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1}$  for  $t \in \mathfrak{k}^*$ . Let  $T$  be the subgroup generated by all  $h_\alpha(t)$ ,  $\alpha \in \Phi$ . Then  $W = N(T)/T$  and  $w_\alpha(t)$  is a coset representative in  $N(T)$  for the reflection  $w_\alpha$ .

Fix an ordering on the root system  $\Phi$ . This determines a set of positive roots and a set of simple roots which forms a base for

$\Phi$ . Let  $U$  be the subgroup generated by all  $U_\alpha$ , where  $\alpha$  is a positive root.

Then  $T$  is a maximal torus of  $G$  and  $B = TU$  is a Borel subgroup of  $G$  with unipotent radical  $U$ .

For each root  $\alpha$ , there is a canonical homomorphism  $\varphi_\alpha$  from  $SL(2, \mathfrak{k})$  into the subgroup of  $G$  generated by  $U_\alpha$  and  $U_{-\alpha}$  such that

$$\begin{aligned} \varphi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} &= x_\alpha(t), & \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= w_\alpha(1), \\ \varphi_\alpha \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} &= x_{-\alpha}(t), & \text{and } \varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= h_\alpha(t). \end{aligned}$$

The kernel of  $\varphi_\alpha$  is either trivial or  $\{\pm I\}$ .

If  $\lambda$  is a character of  $T$ , we define for each root  $\alpha$  a character  $\lambda_\alpha$  of  $\mathfrak{k}^*$  by  $\lambda_\alpha(t) = \lambda(h_\alpha(t))$ . The Weyl group  $W$  acts on  $T$  and thus on characters of  $T$ . We note that  $w\lambda_\alpha(t) = w\lambda(h_\alpha(t)) = \lambda(w^{-1}h_\alpha(t)w) = \lambda(h_{w^{-1}\alpha}(t)) = \lambda_{w^{-1}\alpha}(t)$ . The one-parameter subgroups  $h_\alpha(t)$  form a root system  $\Phi^\vee$  dual to  $\Phi$  in  $\text{Hom}(\mathfrak{k}^*, T) \otimes \mathbf{R}$ .  $w$  acts on  $\lambda_\alpha$  as  $w$  acts on  $\alpha^\vee$ , as  $w^{-1}$  acts on  $\alpha$ . We use this observation to simplify notation and calculations in Chapter 2.

Let  $K$  be the subgroup of  $G$  generated by  $\{x_\alpha(t) \mid \alpha \in \Phi, t \in \mathcal{O}\}$ . Then  $K$  is a good maximal compact subgroup of  $G$  [4, 27], and there is an Iwasawa decomposition  $G = KB = KTU$ , nonuniquely.

More generally, suppose  $G$  is the group of  $\mathfrak{k}$ -rational points of a reductive algebraic group defined over  $\mathfrak{k}$ . A Borel subgroup  $B$  is a maximal connected solvable subgroup of  $G$ . A parabolic subgroup  $P$  is a subgroup of  $G$  containing a Borel subgroup. Let  $N$  be the unipotent radical of  $P$ ,  $A$  a maximal  $\mathfrak{k}$ -split torus in the radical of  $P$  and  $M = Z_G(A)$ . Then  $P$  has a Levi decomposition  $P = MN$ .

$B$  has Levi decomposition  $TU$  where  $T$  is the centralizer in  $G$  of a maximal  $\mathfrak{k}$ -split torus  $A$  of  $G$ .  $W = N(A)/Z(A)$  acts on  $A$  and thus on  $\text{Hom}(A, \mathfrak{k}^*)$ , which is dually paired over  $\mathbf{Z}$  with  $\text{Hom}(\mathfrak{k}^*, A)$ . If  $G$  is semi-simple, the root system  $\Phi = \Phi(G, A)$  spans  $\text{Hom}(A, \mathfrak{k}^*) \otimes \mathbf{R}$ , and we have the dual root system  $\Phi^\vee$  in  $\text{Hom}(\mathfrak{k}^*, A) \otimes \mathbf{R}$  [1].

Bruhat-Tits theory gives a generating set of valuated root data and the existence of good maximal compact subgroups of  $G$ , for which Iwasawa and Cartan decompositions hold [4, 27].

A topological group  $G$  is said to be *totally disconnected* (t.d.) if there exists a neighborhood basis at 1 for the topology on  $G$  consisting of open compact subgroups. A function on a t.d. group is *smooth*, or  $C^\infty$ , if it is locally constant.

Let  $G$  be a t.d. group and  $V$  a vector space over  $C$ . A *representation*  $(\Pi, V)$  of  $G$  is a mapping  $\Pi: G \rightarrow \text{End}(V)$  such that  $\Pi(1) = 1$  and  $\Pi(xy) = \Pi(x)\Pi(y)$  for all  $x, y \in G$ . A vector  $v \in V$  is *smooth* if

$x \mapsto \Pi(x)v$  is a smooth function on  $G$ . We say that  $\Pi$  is smooth if every  $v \in V$  is smooth.

If  $H$  is a subgroup of  $G$ , define  $V^H = \{v \in V \mid \Pi(h)v = v \text{ for all } h \in H\}$ . A representation  $(\Pi, V)$  of  $G$  is *admissible* if  $\Pi$  is smooth and  $\dim V^H < \infty$  for any open subgroup  $H$  of  $G$ .

A subspace  $W$  of  $V$  is *invariant* if  $\Pi(x)W = W$  for all  $x \in G$ . The representation  $(\Pi, V)$  is (algebraically) *irreducible* if  $V$  has no nontrivial invariant subspaces.

$(\Pi, V)$  is a *pre-unitary* representation if there is a positive-definite hermitian form on  $V$  which is preserved by all  $\Pi(x)$ ,  $x \in G$ . We may take the completion of  $V$  with respect to the inner product defined by this form to obtain a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$ , of which  $V$  is the subspace of smooth vectors.

We also require that  $x \mapsto \Pi(x)$  be continuous for unitary representations.  $(\Pi, \mathcal{H})$  is irreducible if there are no nontrivial closed invariant subspaces.

Let  $(\Pi, V)$  and  $(\Pi', V')$  be representations of  $G$ . An *intertwining operator* between  $\Pi$  and  $\Pi'$  is a linear map  $A: V \rightarrow V'$  with the property that  $A\Pi(x) = \Pi'(x)A$  for all  $x \in G$ .  $\Pi$  is *equivalent* to  $\Pi'$  if  $A$  can be chosen to be a bijection.

Define the *commuting algebra* of  $(\Pi, V)$  to be  $\{A: V \rightarrow V \mid A\Pi(x) = \Pi(x)A \text{ for all } x \in G\}$ .

If  $\pi, \pi'$  are unitary, we require an intertwining operator  $A$  to be a bounded linear operator.  $\pi$  and  $\pi'$  are (unitarily) equivalent if  $A$  can be chosen to be a unitary operator.

We will use the following criterion for reducibility.

**THEOREM.** *Suppose  $(\pi, V)$  is a unitary representation of  $G$ . Then  $\pi$  is irreducible if and only if its commuting algebra is one-dimensional [32].*

More detailed introductions to the representation theory of t.d. groups may be found in [6, 11, 13, 32].

## CHAPTER I

### INTERTWINING OPERATORS AND THE COMMUTING ALGEBRA

1. The intertwining operators  $A(w, \lambda)$  and  $a(w, \lambda)$ . Let  $P = MN$  be a parabolic subgroup of  $G$  and  $(\sigma, V)$  an admissible representation of  $M$ , extended trivially across  $N$ . Define the representation  $\text{Ind}_P^G \sigma$  to be left translation in the space of functions  $H_\sigma = \{f: G \rightarrow V \mid f \text{ is locally constant and } f(gmn) = \delta_P^{-1/2} \sigma^{-1}(m)f(g) \text{ for all } g \in G, m \in M, \text{ and } n \in N\}$ . Since  $G = KP$  with  $K$  compact,  $\text{Ind}_P^G \sigma$  is an admissible representation of  $G$ . The factor  $\delta_P^{-1/2}$  is used so that unitary



representations induce to unitary representations. One could also take functions which are square integrable mod  $P$ .

From Bruhat theory, one knows that  $\text{Ind}_P^G \sigma$  and  $\text{Ind}_{P_1}^G \sigma_1$  have no composition factors in common if  $P$  and  $P_1$  are not conjugate in  $G$ . Further,  $\text{Ind}_P^G \sigma$  and  $\text{Ind}_{P_1}^G \sigma_1$  have a composition factor in common only if there exists a  $w \in W$  normalizing  $M$  such that  $w\sigma$  is equivalent to  $\sigma_1$ . In this case,  $\text{Ind}_P^G \sigma$  is equivalent to  $\text{Ind}_P^G w\sigma$ .

Jacquet's theorem states that any irreducible representation of a reductive  $p$ -adic group  $G$  is a subrepresentation of  $\text{Ind}_P^G \sigma$  for some parabolic subgroup  $P$ , where  $\sigma$  is a supercuspidal representation of  $M$  [13, 32].

Thus to give a complete list of the irreducible representations of  $G$ , one needs to decompose all  $\text{Ind}_P^G \sigma$ , with equivalent factors arising only in the case of the equivalent representations  $\text{Ind}_P^G \sigma$  and  $\text{Ind}_P^G w\sigma$ .

We study the problem of decomposing the representations  $\text{Ind}_B^G \lambda$ , where  $G$  is a Chevalley group over  $\mathfrak{k}$ ,  $B = TU$  is a Borel subgroup, and  $\lambda$  is a (unitary) character of  $T$ .

Let  $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$  for  $\lambda$  a quasi-character of  $T$ . By Bruhat theory, the length of the composition series of  $\text{Ind}_B^G \lambda$  is bounded by  $|W_\lambda|$  if  $\lambda$  is unitary.

Suppose  $w \in W$ . Intertwining operators  $A(w, \lambda)$  between  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w\lambda$  are defined initially for certain nonunitary  $\lambda$ . These operators are normalized to define operators  $a(w, \lambda)$  which can be extended by analytic continuation to meromorphic functions in  $\lambda$ .

Fix a coset representative  $\bar{w}$  in  $N(T)$  for  $w$ . Define [30, 37]

$$[A(\bar{w}, \lambda)f](g) = \int_{U \cap w \vee w^{-1}} f(gu\bar{w})du \quad \text{for } f \in H_\lambda.$$

We remark that if we choose a different coset representative  $\bar{w}'$  for  $w$ , then  $\bar{w}^{-1}\bar{w}' \in T$  and the operators differ by a scalar  $\lambda^{-1}\delta_B^{-1/2}(\bar{w}^{-1}\bar{w}')$ .

N. Winarsky has shown that  $A(\bar{w}, \lambda)f(g)$  converges absolutely for quasi-characters  $\lambda$  in the domain  $D(w) = \{\lambda \mid \text{Re } \lambda_\alpha > 0 \text{ for } \alpha \in R(w)\}$ , where  $R(w) = \{\alpha \in \Phi \mid \alpha > 0 \text{ and } w\alpha < 0\}$ , and that  $A(\bar{w}, \lambda): H_\lambda \rightarrow H_{w\lambda}$  intertwines  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w\lambda$ . Further, if the condition  $l(w'w'') = l(w') + l(w'')$  on lengths holds, then the cocycle condition  $A(\bar{w}'\bar{w}'', \lambda) = A(\bar{w}', \bar{w}''\lambda) \circ A(\bar{w}'', \lambda)$  holds [37].

These results are true for  $G$  a reductive  $p$ -adic group. The proofs are as in [30, 37] once we have the following.

LEMMA 1. *Let  $G$  be a reductive  $p$ -adic group. Let  $\text{Re } \lambda = |\lambda|$  and let  $\chi_{\text{Re } \lambda}$  be the  $K$ -fixed vector in  $H_{\text{Re } \lambda}$  defined by  $\chi_{\text{Re } \lambda}(ktu) = \text{Re}(\lambda)^{-1}\rho^{-1}(t)$ . Suppose  $\text{Re } \lambda_\alpha > 0$ . Then*

$$\int_{U_\alpha} \chi_{\text{Re } \lambda}(u\bar{w}_\alpha)du < \infty .$$

*Proof.* By Bruhat-Tits theory, the derived group of  $G$  possesses a system of valuated root data, with properties which Macdonald has taken as axioms for a “group of  $p$ -adic type” [4, 27].

$B = TU$  is a minimal parabolic, where  $T$  is now the centralizer of a maximal  $\mathfrak{k}$ -split torus  $A$  in  $G$ . There is a homomorphism  $\nu$  with kernel  $T \cap K$  from  $N(A)$  to the affine Weyl group of  $G$ , which is the group generated by reflections in the hyperplanes determined by the set of affine roots  $\{\alpha + r \mid \alpha \in \Phi, r \in \mathbf{Z}\}$ . Let  $Y_r = U_{-\alpha-r}/U_{-\alpha-r+1}$ . Then

$$\begin{aligned} \int_{U_\alpha} \chi_{\text{Re } \lambda}(u\bar{w}_\alpha)du &= \int_{U_{-\alpha}} \chi_{\text{Re } \lambda}(\bar{w}_\alpha v)dv \\ &= \int_{U_{-\alpha}} \chi_{\text{Re } \lambda}(v)dv \\ &= \int_{U_{-\alpha+0}} dv + \sum_{r=1}^\infty \int_{Y_r} \chi_{\text{Re } \lambda}(v)dv . \end{aligned}$$

We may write  $v \in U_{-\alpha-r}$  as  $v = u_1 n u_2$ , where  $u_1, u_2 \in U_{\alpha+r} \subset U \cap K$  ( $r$  is a positive interger) and  $\nu(n) = w_{\alpha-r}$ . If  $n_\alpha \in K$  with  $\nu(n_\alpha) = w_\alpha$ , then  $n_\alpha n \in T$  and  $\nu(n_\alpha n) = t_\alpha^r$  where  $t_\alpha$  is the translation  $x \mapsto x + \alpha^r$  in the affine Weyl group. Let  $q_\alpha = (U_{\alpha-1}; U_\alpha)$  and  $q_{\alpha/2} = q_{\alpha+1}q_\alpha^{-1}$ .

Thus

$$\begin{aligned} \int_{Y_r} \chi_{\text{Re } \lambda}(v) &= \int_{Y_r} \chi_{\text{Re } \lambda}(u_1 n u_2) = \int_{Y_r} \chi_{\text{Re } \lambda}(n) = \int_{Y_r} \chi_{\text{Re } \lambda}(n_\alpha n) \\ &= \text{Re } \lambda^{-1} \rho^{-1}(t_\alpha^r) \cdot \text{vol}(Y_r) \\ &= \text{Re } \lambda^{-1}(t_\alpha)^r q_{\alpha/2}^{-r/2} q_\alpha^{-r} [q_{\alpha/2}^{[r/2]} q_\alpha^r - q_{\alpha/2}^{[r-1/2]} q_\alpha^{r-1}] . \end{aligned}$$

Thus the sum over  $r$  is a geometric series with common ratio  $\text{Re } \lambda(t_\alpha)^{-2}$ , which converges if and only if  $s = \text{Re } \lambda_\alpha > 0$ .

The value of the sum is then given by Harish-Chandra’s  $c$ -function  $c_0(\alpha, s) = c(\alpha/2, s)c(\alpha, s)$ . The reader is referred to Macdonald [27].

Let  $V$  be the unipotent radical of the Borel opposed to  $B$ . Since  $G = VB$  up to a set of Haar measure zero, functions in  $H_\lambda$  are determined by their values on  $V$  and we may realize  $\text{Ind}_B^G \lambda$  on  $L^2(V)$ . Assume that  $\text{Re } \lambda_\alpha > 0$ . If  $G$  is a Chevalley group, then  $U_{-\alpha}$  is one-dimensional, and a calculation realizing the representation on  $L^2(V)$  via the Fourier transform in  $U_{-\alpha}$ , as in the  $\chi$ -realization of Gelfand, Graev and Pyatetskii-Shapiro [7] or Sally [28] for  $\text{SL}(2)$ , shows that  $A(\bar{w}_\alpha, \lambda)$  acts as multiplication by  $\lambda_\alpha^{-1} \Gamma(\lambda_\alpha)$ , where  $\bar{w}_\alpha = w_\alpha(1)$ . We may then use the analytic continuation of the gamma function to define the intertwining operator  $A(w_\alpha, \lambda)$  for any quasi-character  $\lambda$  such that  $\Gamma(\lambda_\alpha)$  is defined, i.e., for  $\lambda_\alpha \neq 1$ .

If we normalize  $A(\bar{w}_\alpha, \lambda)$  by  $\Gamma(\lambda_\alpha)$  by setting  $\alpha(\bar{w}_\alpha, \lambda) = (1/\Gamma(\lambda_\alpha))A(\bar{w}_\alpha, \lambda)$ , then by analytic continuation  $\alpha(\bar{w}_\alpha, \lambda)$  defines an intertwining operator between  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w_\alpha \lambda$  for all  $\lambda$ .

Suppose  $w \in W$  has length  $l$  and  $w = w_{\alpha_1} \cdots w_{\alpha_l}$  is a reduced product of basic reflections,  $\alpha_i$  simple. The appropriate normalizing factor for  $A(\bar{w}, \lambda) = A(\bar{w}_{\alpha_1}, w_{\alpha_2} \cdots w_{\alpha_l} \lambda) \circ \cdots \circ A(\bar{w}_{\alpha_l}, \lambda)$  is

$$\prod_{i=1}^l \Gamma(w_{\alpha_{i+1}} \cdots w_{\alpha_i} \lambda_{\alpha_i}) = \prod_{\alpha \in R(w)} \Gamma(\lambda_\alpha).$$

Denote this product by  $\Gamma_w(\lambda)$  and define

$$\alpha(\bar{w}, \lambda) = \frac{1}{\Gamma_w(\lambda)} A(\bar{w}, \lambda).$$

An argument similar to that in Winarsky [37] gives the analytic continuation of  $A(\bar{w}, \lambda)$  and  $\alpha(\bar{w}, \lambda)$  in the case of a semi-simple  $p$ -adic algebraic group.

**THEOREM 1.** *Let  $G$  be a connected semi-simple  $p$ -adic group and suppose  $f \in H_\lambda$  is locally constant. The map  $\lambda \mapsto (A(\bar{w}, \lambda)f)(k)$  of  $D(w)$  into  $\mathbb{C}$  is analytic for  $k \in K$ . It extends to  $\mathbb{C}^n$  as a meromorphic function. When  $\lambda$  is not a pole of the extension, the operators  $A(\bar{w}, \lambda)$  intertwine the representations  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w \lambda$ .*

*Proof.* The unramified part of  $\lambda$  is determined by  $n$  unramified characters  $|\cdot|^{s_\alpha}$ ,  $\alpha$  simple, each of which is identified with the complex number  $s_\alpha$ . Multiply this by a representation  $\lambda^*$  of  $\ker \nu$ . Considering  $\lambda^*$  fixed and letting the unramified part of  $\lambda$  vary, we identify  $\lambda$  with a point in  $\mathbb{C}^n$ .

It is enough to prove the theorem in the case  $w = w_\alpha$  is a simple reflection. Again, we follow Macdonald [27]. Choose a coset representative  $n_\alpha \in K$  for  $w_\alpha$  with  $\nu(n_\alpha) = w_\alpha$ . Write  $v \in Y_r = U_{-\alpha-r} \backslash U_{-\alpha-r+1}$  as  $v = u_1 n_\alpha^{-1} t_\alpha^r u_2$ , with  $u_1, u_2 \in U_{\alpha+r}$  and  $\nu(t_\alpha)$  translation by  $\alpha^r$ . Suppose that  $f$  is constant on cosets of  $U_{-\alpha+m}$  in  $K$ .

Then

$$\begin{aligned} A(n_\alpha, \lambda)f(k) &= \int_{U_{-\alpha}} f(kn_\alpha v)dv \\ &= \int_{U_{-\alpha+m-1}} f(kn_\alpha v)dv + \sum_{r=m}^\infty \int_{Y_r} f(kn_\alpha u_1 n_\alpha^{-1} t_\alpha^r u_2) \\ &= \int_{U_{-\alpha+m-1}} f(kn_\alpha v)dv + \sum_{r=m}^\infty \int_{Y_r} f(kn_\alpha u_1 n_\alpha^{-1}) \lambda^{-1} \rho^{-1}(t_\alpha^r). \end{aligned}$$

But  $n_\alpha u_1 n_\alpha^{-1} \in U_{-\alpha+m}$  and  $f$  is assumed constant on this. The sum over  $r$  is thus

$$\begin{aligned}
 f(k) & \sum_{r=m}^{\infty} \int_{T_r} \lambda^{-1} \rho^{-1}(t_\alpha^r) \\
 & = \begin{cases} 0 & \text{if } \lambda_\alpha \text{ is ramified} \\
 f(k) \sum_{r=m}^{\infty} \lambda(t_\alpha)^{-r} q_{\alpha/2}^{-r/2} q_\alpha^{-r} (q_{\alpha/2}^{[r/2]} q_\alpha^r - q_{\alpha/2}^{[(r-1)/2]} q_\alpha^{r-1}) & \text{if } \lambda_\alpha \text{ is unramified.} \end{cases}
 \end{aligned}$$

For  $\lambda_\alpha$  unramified, this is a geometric series with common ratio  $\lambda(t_\alpha)^{-2}$ , which converges if and only in  $\text{Re } \lambda_\alpha > 0$ . In this case the sum is given by

$$f(k) \cdot \frac{(1 - q_\alpha^{-1})(1 + \lambda(t_\alpha)^{-1} q_{\alpha/2}^{-1/2}) \lambda(t_\alpha)^{-m}}{1 - \lambda(t_\alpha)^{-2}}.$$

We note that if  $G$  is split, then  $q_\alpha = q$  and  $q_{\alpha/2} = 1$  and the above sum agrees with Winarsky's.

Thus  $\lambda \mapsto A(n_\alpha, \lambda) f(k)$  extends to a meromorphic function of  $s_\alpha$  with simple poles at  $\lambda(t_\alpha) = \pm 1$  for  $q_{\alpha/2} \neq 1$  and at  $\lambda(t_\alpha) = 1$  for  $q_{\alpha/2} = 1$ , if  $|\cdot|^{s_\alpha}$  is unramified, and extends to an analytic function if  $\lambda_\alpha$  is ramified. By analytic continuation, the intertwining relation holds if  $\lambda$  is not a pole of the extension.

If we normalize  $A(\bar{w}_\alpha, \lambda)$  by Harish-Chandra's  $c$ -function  $c_0(\alpha, \lambda_\alpha)$  and  $A(\bar{w}, \lambda)$  by  $c_w(\lambda) = \prod_{\alpha \in R(w)} c_0(\alpha, \lambda_\alpha)$  then  $\lambda \mapsto a(\bar{w}, \lambda) = (1/c_w(\lambda)) A(\bar{w}, \lambda)$  extends to a meromorphic function on  $C^n$  which is holomorphic in a neighborhood of  $\{(c_1, \dots, c_n) \in C^n \mid \text{Re } c_i = 0, i = 1, \dots, n\}$  and defines an intertwining operator between  $\text{Ind}_B^G \lambda$  and  $\text{Ind}_B^G w\lambda$  if  $\lambda$  is not a pole.

An argument similar to that of [37] shows that  $\text{Ind}_B^G \lambda$  is reducible if there exists a  $w \in W, w \neq 1$  with  $w\lambda = \lambda$  such that  $\lambda$  is not a pole of  $c_w(\lambda)$ .

**2. The cocycle condition for  $a(w, \lambda)$ .** We now choose certain coset representatives for each  $w \in W$ . Fix any coset representatives  $n_\alpha$  for the basic reflections  $w_\alpha, \alpha$  simple. Suppose  $w \in W$  has length  $l$  and  $w = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_l}$  is a reduced product of basic reflections. We take  $n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_l}$  as the coset representative of  $w$  and define

$$\begin{aligned}
 A(w, \lambda) & = A(n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_l}, \lambda) \quad \text{and} \\
 a(w, \lambda) & = a(n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_l}, \lambda).
 \end{aligned}$$

This is well-defined by the following.

**THEOREM 1.** *Fix coset representatives  $n_\alpha \in N(T)$  for the basic reflections  $w_\alpha, \alpha$  simple. Suppose  $w$  is expressed as a reduced product  $w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_l}$  of basic reflections,  $l(w) = l$ . Then the coset*

representative  $n_{\alpha_1} n_{\alpha_2} \cdots n_{\alpha_l}$  of  $w$  is independent of the expression  $w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_l}$ .

*Proof.* For Chevalley groups, see page 242 of [34]. For connected reductive  $p$ -adic groups, see page 112 of [4].

We now fix a set of coset representatives as above and write  $A(w, \lambda)$  instead of  $A(\bar{w}, \lambda)$ . For the calculations in Chapter 3, we have taken  $n_\alpha = w_\alpha(1)$  as the coset representative for the basic reflection  $w_\alpha$ ,  $\alpha$  simple.

Recall the cocycle condition  $A(w'w'', \lambda) = A(w', w''\lambda) \circ A(w'', \lambda)$  if  $l(w'w'') = l(w') + l(w'')$ . In this case we also have  $\Gamma_{w'w''}(\lambda) = \Gamma_{w'}(w''\lambda)\Gamma_{w''}(\lambda)$  since  $R(w'w'') = R(w'') \cup w''^{-1}R(w')$ . Thus  $\alpha(w'w'', \lambda) = \alpha(w', w''\lambda) \circ \alpha(w'', \lambda)$  if  $l(w'w'') = l(w') + l(w'')$ .

We will show that with the above choice of coset representatives, the cocycle condition holds for the normalized operators  $\alpha(w, \lambda)$  with no condition on the lengths of  $w'$  and  $w''$ .

We have seen that under the  $\chi$ -realization in  $U_{-\alpha}$ ,  $A(w_\alpha, \lambda)$  acts as multiplication by  $\lambda_\alpha^{-1}\Gamma(\lambda_\alpha)$ . Thus  $A(w_\alpha, w_\alpha\lambda) \circ A(w_\alpha, \lambda) = \Gamma(\lambda_\alpha)\Gamma(\lambda_\alpha^{-1})$  is scalar and  $\alpha(w_\alpha, w_\alpha\lambda) \circ \alpha(w_\alpha, \lambda) = I$  is the identity.

Thus  $\alpha(w_\alpha, w_\alpha\lambda)$  is the inverse of  $\alpha(w_\alpha, \lambda)$ , i.e., the cocycle condition holds in this case.

**THEOREM 2.** *The cocycle condition  $\alpha(w'w'', \lambda) = \alpha(w', w''\lambda) \circ \alpha(w'', \lambda)$  holds with no condition on the lengths of  $w'$  and  $w''$ .*

*Proof.* We first recall that with our choice of coset representatives the operators are well-defined. This is in fact equivalent to the cocycle condition.

The proof is by induction on the length of  $w'$ . Suppose  $l(w') = 1$ , say  $w' = w_\alpha$ ,  $\alpha$  simple. If  $l(w_\alpha w'') = l(w'') + 1$ , then we are done. Otherwise  $l(w_\alpha w'') = l(w'') - 1$ . Suppose  $w'' = w_{\beta_1} w_{\beta_2} \cdots w_{\beta_l}$  is a reduced expression for  $w''$  as a product of simple reflections. Then by Coxeter's exchange condition [2],  $w'w'' = w_\alpha w_{\beta_1} w_{\beta_2} \cdots w_{\beta_l} = w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}$ , where  $\beta_j$  is omitted.

Since  $w_\alpha$  has order 2,  $w'' = w_{\beta_1} \cdots w_{\beta_l} = w_\alpha w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}$ , and these are both reduced expressions for  $w''$ . Then since  $\alpha(w'', \lambda)$  does not depend on the reduced expression chosen for  $w''$ , we get

$$\begin{aligned} &\alpha(w_\alpha, w''\lambda) \circ \alpha(w'', \lambda) \\ &= \alpha(w_\alpha, w''\lambda) \circ \alpha(w_\alpha w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}, \lambda) \\ &= \alpha(w_\alpha, w''\lambda) \circ \alpha(w_\alpha, w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}\lambda) \circ \alpha(w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}, \lambda) \\ &= I \circ \alpha(w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}, \lambda) \\ &= \alpha(w_\alpha w'', \lambda), \end{aligned}$$

since  $l(w_\alpha w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}) = 1 + l(w_{\beta_1} \cdots \hat{w}_{\beta_i} \cdots w_{\beta_l})$  and  $w_{\beta_1} \cdots \hat{w}_{\beta_j} \cdots w_{\beta_l}$  is a reduced expression for  $w_\alpha w''$ .

Thus the theorem is true if  $w'$  has length 1. Suppose  $w'$  has length  $> 1$  and write  $w' = w_\alpha w_1$  with  $\alpha$  simple and  $l(w_1) = l(w') - 1$ . Then

$$\begin{aligned} \alpha(w'w'', \lambda) &= \alpha(w_\alpha w_1 w'', \lambda) = \alpha(w_\alpha, w_1 w'' \lambda) \alpha(w_1 w'', \lambda) \\ &= \alpha(w_\alpha, w_1 w'' \lambda) \alpha(w_1, w'' \lambda) \alpha(w'', \lambda) \\ &\quad \text{by the induction hypothesis,} \\ &= \alpha(w_\alpha w_1, w'' \lambda) \alpha(w'', \lambda) \text{ since } l(w_\alpha) = 1, \\ &= \alpha(w', w'' \lambda) \alpha(w'', \lambda). \end{aligned}$$

Thus the cocycle condition is true with no condition on the lengths of  $w'$  and  $w''$ . We remark that one could also use the relations  $(w_\alpha w_\beta)^{n(\alpha, \beta)} = 1$  defining  $W$  as a Coxeter group to prove the cocycle condition.

We note that to prove the theorem, we need only normalize the operators so that  $\alpha(w_\alpha, w_\alpha \lambda)$  is the inverse of  $\alpha(w_\alpha, \lambda)$ . For Chevalley groups we may do this with either gamma functions or  $c$ -functions.

For Macdonald's "groups of  $p$ -adic type" we may use the  $c$ -functions to do this, at least for unramified  $\lambda$ . In any case,  $\alpha(w^{-1}, w\lambda) \alpha(w, \lambda)$  is scalar. If  $\lambda$  is unramified and  $f_\lambda$  is the  $K$ -fixed vector in  $H_\lambda$  with  $f_\lambda(e) = 1$ , then  $A(w, \lambda) f_\lambda = c_w(\lambda) f_{w\lambda}$  and  $A(w^{-1}, w\lambda) A(w, \lambda) f_\lambda = c_{w^{-1}}(w\lambda) c_w(\lambda) f_\lambda$ . So if  $\alpha(w, \lambda) = (1/c_w(\lambda)) A(w, \lambda)$ , we see that  $\alpha(w^{-1}, w\lambda) \alpha(w, \lambda) = I$ .

Thus the cocycle relation holds with no condition on lengths for "groups of  $p$ -adic type" and unramified characters  $\lambda$ .

Finally, we note that the cocycle condition implies that  $w \mapsto \alpha(w, \lambda)$  is a representation of  $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$ .

**3. The Knapp-Stein  $R$ -group.\*** We define a subgroup  $R$  of  $W_\lambda$  such that the commuting algebra of  $\text{Ind}_B^G \lambda$  is given as the group algebra  $C[R]$ . The theory of the  $R$ -group was developed by Knapp and Stein for real semi-simple Lie groups. The following  $p$ -adic analogue is another illustration of Harish-Chandra's "Lefschetz principle," which says that whatever is true for real reductive groups is also true for  $p$ -adic groups.

Let  $\Delta' = \{\alpha > 0 \mid \lambda_\alpha \equiv 1\}$ . Then  $\pm \Delta'$  is a sub-root system of the root system  $\Phi$ .

Let

$$\begin{aligned} R &= \{w \in W_\lambda \mid \alpha > 0 \text{ and } \lambda_\alpha \equiv 1 \text{ imply that } w\alpha > 0\} \\ &= \{w \in W_\lambda \mid w(\Delta') = \Delta'\}. \end{aligned}$$

---

\* Suppose that  $G$  is a Chevalley group.

Let  $W'$  be the reflection group associated to  $\pm\mathcal{A}'$ , i.e., the group generated by the reflections  $\{w_\alpha \mid \alpha \in \mathcal{A}'\}$ .

**THEOREM 1.**  $W_\lambda$  can be written as a semi-direct product  $W_\lambda = R \times W'$ , where  $R$  and  $W'$  are defined above. Further,  $W'$  is the group  $\{w \in W_\lambda \mid \alpha(w, \lambda) \text{ is scalar}\}$ .

*Proof.* First we show that  $W' \leq W_\lambda$ . Let  $\alpha \in \mathcal{A}'$  and show that  $w_\alpha \lambda_\beta = \lambda_\beta$  for all roots  $\beta$ . But since  $\lambda_\alpha \equiv 1$ ,  $w_\alpha \lambda_\beta = \lambda_{w_\alpha^{-1}\beta} = \lambda_\beta \lambda_\alpha^{-\langle \alpha^\vee, \beta^\vee \rangle} = \lambda_\beta$ .

Now suppose  $w \in W_\lambda$  has length  $l$  and write  $w = w_{\alpha_1} \cdots w_{\alpha_l}$  as a reduced product of basic reflections. If  $w \in R$  then we are done. Otherwise there exists  $\alpha \in \mathcal{A}'$  with  $w\alpha < 0$ . Then  $\alpha = w_{\alpha_l} \cdots w_{\alpha_{i+1}}(\alpha_i)$  for some  $i$ ,  $1 \leq i \leq l$ . Let  $r = w_{\alpha_1} \cdots \hat{w}_{\alpha_i} \cdots w_{\alpha_l}$  where  $w_{\alpha_i}$  is omitted. Then

$$\begin{aligned} w &= w_{\alpha_1} \cdots \hat{w}_{\alpha_i} \cdots w_{\alpha_l} w_{\alpha_l} \cdots \hat{w}_{\alpha_i} \cdots w_{\alpha_1} w_{\alpha_1} \cdots w_{\alpha_l} \\ &= r w_{\alpha_l} \cdots w_{\alpha_{i+1}} w_{\alpha_i} w_{\alpha_{i+1}} \cdots w_{\alpha_l} \\ &= r w_{w_{\alpha_l} \cdots w_{\alpha_{i+1}}(\alpha_i)} \\ &= r w_\alpha. \end{aligned}$$

Then  $w_\alpha \in W'$  since  $\alpha \in \mathcal{A}'$ . Since  $l(r) < l(w)$  we may use induction on  $l(w)$  to complete the proof that  $W_\lambda = R \times W'$ .

Finally, we show that  $W' = \{w \in W_\lambda \mid \alpha(w, \lambda) \text{ is scalar}\}$ . In the  $\mathcal{X}$ -realization,  $\alpha(w_\alpha, \lambda)$  acts as multiplication by  $\lambda_\alpha^{-1}$ , if we use  $w_\alpha(1)$  as coset representative for  $w_\alpha$  and normalize the operator by the gamma function. Thus  $\alpha(w_\alpha, \lambda) = I$  if and only if  $\alpha \in \mathcal{A}'$ . Then  $\alpha(w, \lambda) = I$  for all  $w \in W'$ . The cocycle condition shows that  $w \rightarrow \alpha(w, \lambda)$  is a homomorphism from  $W_\lambda$  into the group of invertible intertwining operators for  $\text{Ind}_B^G \lambda$ , and Winarsky [37] shows that  $\alpha(w, \lambda)$  is nonscalar if  $w \in R$ ,  $w \neq 1$ . These observations complete the proof of the theorem.

We note that Winarsky's condition for reducibility is essentially that  $R$  is nontrivial.

By an unpublished theorem of Harish-Chandra, the commuting algebra  $C(\lambda)$  of  $\text{Ind}_B^G \lambda$  is spanned by  $\{\alpha(w, \lambda) \mid w \in W_\lambda\}$ . By the above, it is spanned by  $\{\alpha(w, \lambda) \mid w \in R\}$ . But these operators are linearly independent, by our calculations in Chapter 3, or by an appeal to Silberger's theorem [33], which states that

$$\dim C(\lambda) = |W_\lambda|/|W'|.$$

Thus the operators  $\{\alpha(w, \lambda) \mid w \in R\}$  form a basis for  $C(\lambda)$ . Finally, since  $\alpha(w'w'', \lambda) = \alpha(w', \lambda)\alpha(w'', \lambda)$  for  $w'$  and  $w''$  in  $R \leq W_\lambda$ , we have the following

**THEOREM 2.** *The commuting algebra  $C(\lambda)$  of the (unitary) principal series representation  $\text{Ind}_B^G \lambda$  is isomorphic to the group algebra  $C[R]$ .*

**COROLLARY 1.**

(a)  $\dim C(\lambda) = |R|$ .

(b) *The number of inequivalent irreducible components of  $\text{Ind}_B^G \lambda$  is equal to the dimension of the center of  $C[R]$ , which equals the number of conjugacy classes in  $R$ .*

(c)  *$\text{Ind}_B^G \lambda$  decomposes with multiplicities equal to 1 if and only if  $R$  is abelian.*

(d) *If  $C[R] = M_{n_1}(\mathbf{C}) \oplus \cdots \oplus M_{n_k}(\mathbf{C})$ , then  $n_1, \dots, n_k$  are the multiplicities of the irreducible components of  $\text{Ind}_B^G \lambda$ .*

## CHAPTER II

### CLASSIFICATION OF THE $R$ -GROUPS

The  $R$ -groups which occur for Chevalley groups of each type  $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$  and  $G_2$ , are determined. They are abelian except in the cases of  $D_n$ , for which non-abelian  $R$  occur for every  $n \geq 4$ , and in the cases  $E_7$  and  $E_8$ .

The orders of the  $R$ -groups which can occur depend on  $n$  and on the arithmetic of the field  $\mathfrak{k}$ . Further, the existence of the non-abelian  $E_8$   $R$ -group depends on the arithmetic of  $\mathfrak{k}$ .

Let  $\lambda$  be a character of  $T$  and let

$$\begin{aligned} \mathcal{A}' &= \{\alpha > 0 \mid \lambda_\alpha \equiv 1\} \\ &= \{\alpha > 0 \mid \alpha(w_\alpha, \lambda) \text{ is scalar}\}. \end{aligned}$$

Then

$$\begin{aligned} R &= \{w \in W_\lambda \mid \alpha > 0 \text{ and } \lambda_\alpha \equiv 1 \text{ imply that } w\alpha > 0\} \\ &= \{w \in W_\lambda \mid w(\mathcal{A}') = \mathcal{A}'\}. \end{aligned}$$

We note that the second definition of  $R$  shows that it is a group.

Identify  $\lambda_\alpha$  with  $\alpha^\nu$  in the root system  $\Phi^\nu$  dual to  $\Phi$  and let  $\mathcal{L} = \mathcal{L}_\lambda = \mathbf{Z}[\sum_\alpha m_\alpha \cdot \alpha^\nu \mid \prod_\alpha \lambda_\alpha^{m_\alpha} = 1, m_\alpha \in \mathbf{Z}]$ . Then  $w \in W_\lambda$  if and only if  $\alpha^\nu - w\alpha^\nu \in \mathcal{L}$  for all simple roots  $\alpha^\nu$ .  $\mathcal{L}$  contains the set  $\{\alpha^\nu \mid \alpha \in \mathcal{A}'\} =$  positive elements in  $\mathcal{L} \cap \Phi^\nu$ , which we sometimes denote by  $\mathcal{A}'$ .

$w$  acts on  $\lambda_\alpha$  as  $w$  acts on  $\alpha^\nu$ , as  $w^{-1}$  acts on  $\alpha$ . Since  $w \in R$  if and only if  $w^{-1} \in R$ ,

$$R = \{w \in W_\lambda \mid \alpha^\nu \in \Phi^\nu, \alpha^\nu \in \mathcal{L} \text{ and } \alpha^\nu > 0 \text{ imply that } w\alpha^\nu > 0\}.$$

We do the calculations to classify  $R$  in the root system  $\Phi^\nu$  dual



to  $\Phi$ . Note that not all of  $w^i(\alpha^v - w\alpha^v)$ ,  $0 \leq i < \text{ord } w$ , can be positive, since their sum is zero. Thus, if  $\alpha^v - w\alpha^v \in \mathcal{L}_\lambda \cap \Phi_+^v$  for some root  $\alpha$ , then  $w \notin R_\lambda$ . Note that this condition is invariant under conjugation, replacing  $\lambda$  by  $w\lambda$ , although  $R_{w\lambda}$  may not be equal to  ${}^wR_\lambda = wR_\lambda w^{-1}$ .

We use this observation to determine which elements of  $W$  can form an  $R$ -group for some  $\lambda \in T^\wedge$ . Once we have a possible  $R$ , we look for a character  $\lambda$  with  $R \leq W_\lambda$  as  $R$ -group. The existence of such a  $\lambda$  depends on the arithmetic of  $\mathfrak{k}$ . Our proof explicitly constructs a list of  $\lambda$  and  $R$  and shows that any nontrivial  $R$ -group is conjugate under  $W$  to one on the list.

We proceed according to the classification of types of root systems [2].

1. *Type  $A_n$ .*  $\Phi = \Phi^v = \{e_i - e_j \mid 1 \leq i \neq j \leq n + 1\}$  is self-dual and the Weyl group  $W \cong S_{n+1}$  acts as permutations of the  $e_i$ .

**THEOREM  $A_n$ .**  *$R$  is abelian and  $|R|$  divides  $n + 1$ . If the largest cyclic subgroup of  $R$  has order  $m$ , then  $|R|$  divides  $[\mathfrak{k}^*: (\mathfrak{k}^*)^m] = \text{order of the subgroup of } (\mathfrak{k}^*)^\wedge \text{ consisting of characters of order dividing } m$ .*

*Conventions.* We identify  $e_i - e_j \in \Phi^v$  with the character  $\lambda_{e_i - e_j}$  and consider  $Z[\Phi^v]/\mathcal{L}$  as a subgroup of  $(\mathfrak{k}^*)^\wedge$  by the map  $\sum m_\alpha \alpha^v \mapsto \prod \lambda_\alpha^{m_\alpha}$ .

**LEMMA 1.**  *$w \mapsto e_i - e_{wi}$  is an injective homomorphism from  $R$  into  $(\mathfrak{k}^*)^\wedge$ , independent of  $i$ .*

*Proof.* Let  $w \in W_\lambda$ . Then  $e_i - e_j = w(e_i - e_j) = e_{wi} - e_{wj}$  implies that  $e_i - e_{wi} = e_j - e_{wj}$ , so that the map is independent of  $i$ . Note that means congruence mod  $\mathcal{L}$  and that we have used the fact that  $w \in W_\lambda$  if and only if  $\alpha^v - w\alpha^v \in \mathcal{L}$  for all  $\alpha^v \in \Phi^v$ .

Let  $w, w' \in W_\lambda$ . Then  $e_i - e_{ww'i} = e_i - e_{w'i} + e_{w'i} - e_{w(w'i)} = e_i - e_{w'i} + e_i - e_{wi}$  shows that the map is a homomorphism.

If  $w \neq 1$  then we may replace everything by a conjugate to assume that  $w1 \neq 1$ . Then if  $e_1 - e_{w1} \in \mathcal{L}$ , we have  $e_1 - e_{w1} \in \mathcal{A}'$  and  $w^{-1}(e_1 - e_{w1}) < 0$ , so that  $w \notin R$ . Thus the map is injective on  $R$ .

Thus  $R$  is isomorphic to a subgroup of  $(\mathfrak{k}^*)^\wedge$  and is abelian. Further, if the largest cyclic subgroup of  $R$  has order  $m$ , then any element of  $R$  has order dividing  $m$  and the image of  $R$  is contained in the subgroup of characters of  $\mathfrak{k}^*$  of order dividing  $m$ . Thus  $|R|$  divides  $[\mathfrak{k}^*: (\mathfrak{k}^*)^m]$ .

Since  $R$  is abelian,  $\text{Ind}_R^G \lambda$  decomposes simply. This is shown for  $G = \text{SL}(n, \mathfrak{k})$  by Howe and Silberger [12].

We note that if  $\mathfrak{k} = R$ , then the image of  $R$  is a finite subgroup of  $(R^*)^\wedge$ , so has order 1 or 2 [17].

LEMMA 2. *The stabilizer of any  $e_i$  in  $R$  is trivial. Thus  $|R|$  divides  $n + 1$ .*

*Proof.* Suppose  $w \in R$  fixes some  $i$ . Then  $e_i - e_{wi} = 0$  and the image of  $w$  under the above map is trivial. Thus  $w = 1$ . So the action of  $R$  partitions  $\{1, 2, \dots, n + 1\}$  into orbits of cardinality  $|R|$  and  $|R|$  divides  $n + 1$ .

Note that any finite subgroup of  $(\mathfrak{k}^*)^\wedge$  with order dividing  $n + 1$  is the image of some  $R$ -group.

REMARK. The homomorphism  $w \mapsto e_i - e_{wi}$  is suggested by the following. In Chapter 3 we realize  $\text{Ind}_B^G \lambda$  and  $\alpha(w, \lambda)$  on  $L^2(V)$ . We exhibit a class of functions in  $L^2(V)$  on which  $\alpha(w, \lambda)$  acts as multiplication by  $M(w, \lambda) = \lambda_{\alpha}^{-1}$  in the  $U_{-\alpha}$  coordinate,  $\alpha$  simple. Then  $\alpha(w, \lambda) = \alpha(w_{\alpha_1} \cdots w_{\alpha_i}, \lambda)$  acts as multiplication by the function  $M(w, \lambda) = M(w_{\alpha_1}, w_{\alpha_2} \cdots w_{\alpha_i}, \lambda) \cdots M(w_{\alpha_i}, \lambda)$ .

Then  $w \mapsto M(w, \lambda)$  is a homomorphism, as is  $w \mapsto M(w, \lambda)$  evaluated at some  $U_{-\alpha}$ ,  $\alpha$  simple. The above map  $M(w, \lambda)$  is evaluated at  $U_{-\alpha}$ ,  $\alpha = e_1 - e_2$ .

We note that the linear independence of distinct characters of  $\mathfrak{k}^*$  implies that the  $M(w, \lambda)$  evaluated at  $U_{-\alpha}$  are linearly independent for  $w \in R$ , and therefore the operators  $\{\alpha(w, \lambda) | w \in R\}$  are linearly independent.

2. *Type B<sub>n</sub>.*  $\Phi = \{\pm e_i \pm e_j, \pm e_k | 1 \leq i < j \leq n, 1 \leq k \leq n\}$ . The dual root system  $\Phi^v = \{\pm e_i \pm e_j, \pm 2e_k | 1 \leq i < j \leq n, 1 \leq k \leq n\}$  is type  $C_n$ . The Weyl group  $W \cong S_n \times \mathbf{Z}_2^n$  acts on  $\Phi$  and  $\Phi^v$  by permutations and sign changes on the  $e_i$ .

THEOREM B<sub>n</sub>.  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with  $|R|$  dividing both  $2n$  and  $[\mathfrak{k}^*: (\mathfrak{k}^*)^2]$ .

Suppose  $w = sc \in W_\lambda$  with  $s \in S_n$  and  $c \in \mathbf{Z}_2^n$ . We may replace  $w$  by a conjugate under  $S_n$  to assume the cycles in  $s$  consist of consecutive integers, and then by a conjugate by a sign change to assume that  $c$  changes the sign of at most one  $e_i$  in each orbit of  $s$ .

LEMMA 1. *If  $w = sc \in R$ , then a nontrivial cycle of  $s$  can not have only one sign change associated to it.*

*Proof.* We may assume the cycle is  $(k k + 1 \cdots n - 1 n)$ ,  $k < n$ ,

and that the sign change is on  $2e_n$ . Repeated application of  $w^{-1}$  sends  $e_{n-1} - e_n$  to  $e_k - e_{k+1}$ , which  $w^{-1}$  then sends to  $-e_n - e_k$ . Thus  $e_k + e_{n-1} \in \mathcal{L}$ .

If  $k = n - 1$ , then  $2e_{n-1} \in \mathcal{L} \cap \Phi^v$ . But then  $2e_{n-1} > 0$  and  $w^{-1}(2e_{n-1}) = -2e_n < 0$  contradicts  $w \in R$ .

Otherwise  $k < n - 1$  and  $e_k + e_{n-1} \in \mathcal{L} \cap \Phi^v$ . But then  $e_k + e_{n-1} > 0$  and  $w^2(e_k + e_{n-1}) = w(e_{k+1} + e_n) = e_{k+2} - e_k < 0$  contradicts  $w \in R$ .

**LEMMA 2.** *Any nontrivial cycle of  $s \in S_n$  must be a transposition if  $w = sc \in R$ .*

*Proof.* We may assume that the cycle is  $(k \cdots n - 1 n)$ , and by the above lemma, that there are no sign changes associated to this cycle, i.e.,  $c(2e_i) = 2e_i$  for  $k \leq i \leq n$ .

Then  $w(e_{n-1} + e_n) = e_n + e_k$  implies that  $e_k - e_{n-1} \in \mathcal{L}$ . If  $k < n - 1$ , then  $e_k - e_{n-1} \in \mathcal{L} \cap \Phi^v$ , with  $e_k - e_{n-1} > 0$  and  $w^{-1}(e_k - e_{n-1}) = e_n - e_{n-2} < 0$ , contradicting  $w \in R$ . Thus  $k = n - 1$  and the cycle is a transposition.

By the two lemmas, any  $w = sc \in R$  is conjugate to a product of disjoint transpositions and sign changes, so  $w^2 = 1$  and  $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ . Further, no such  $w \neq 1$  can fix an  $e_i$ . This follows by the argument for type  $A_n$  if  $s \neq 1$ . If  $s = 1$ ,  $w = c \neq 1$  changes the sign of some  $e_j$ . Then if  $w(e_i) = e_i$  we have  $e_j - e_i - w(e_j - e_i) = 2e_j \in \mathcal{L} \cap \Phi^v$ . But then  $2e_j > 0$  and  $w(2e_j) < 0$  contradicts  $w \in R$ .

Thus  $R$  permutes  $\{\pm e_i \mid 1 \leq i \leq n\}$  with  $\text{stab}_R(\pm e_i) = \{1\}$ , so  $|R|$  divides  $2n$ .

We now have that any  $w \in R$  is conjugate to one of 1, (12)(34)  $\cdots$   $(n - 1 n)$ , (12)  $\cdots$   $(k - 1 k)c_{k+1} \cdots c_n$ , or  $c_1 c_2 \cdots c_n$ , where  $c_i$  is the sign change on  $e_i$ .

If we evaluate  $M(w, \lambda)$  at  $U_{-\alpha}$ ,  $\alpha = e_1 - e_2$ , we get the homomorphism  $c_1 \cdots c_n \mapsto 2e_1$  (i.e.,  $\lambda e_1$ ) and  $w \mapsto e_i - we_i$  if  $w = sc$  with  $s(i) \neq i$ . We note that none of these characters can be trivial if  $w \in R$ , so  $w \mapsto e_i - we_i$  is an injective homomorphism from  $R$  into the group of characters of  $\mathfrak{k}^*$  generated by those of order 2. Thus  $|R|$  divides  $[\mathfrak{k}^* : (\mathfrak{k}^*)^2]$ .

Of course, one may directly check that  $w \mapsto e_i - we_i$  is independent of  $i$  and is an injective homomorphism from  $R$  into the subgroup  $Z[\Phi^v]/\mathcal{L}$  of  $(\mathfrak{k}^*)^\wedge$  without reference to  $M(w, \lambda)$ .

We note that if  $\mathfrak{k} = \mathbf{R}$ , then  $|R| = 1$  or 2, and that if  $\mathfrak{k}$  is non-Archimedean with odd residual characteristic, then  $|R| = 1, 2$ , or 4.

3. *Type  $C_n$ .*  $\Phi = \{\pm e_i \pm e_j, \pm 2e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$ . The dual root system  $\Phi^v = \{\pm e_i \pm e_j, \pm e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$  is type  $B_n$ . The Weyl group  $W \cong S_n \times \mathbb{Z}_2^n$  acts on  $\Phi$  and  $\Phi^v$  by permutations and sign changes on the  $e_i$ .

**THEOREM C<sub>n</sub>.**  $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors of  $\mathbf{Z}_2$  bounded by  $n$  and by  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ .

Suppose  $w = sc \in W_\lambda$ ,  $s \in S_n$  and  $c \in \mathbf{Z}_2^n$ . We may replace  $w$  by a conjugate under a sign change to assume that  $c$  changes the sign of at most one  $e_i$  in each orbit of  $s$ .

**LEMMA 1.** Suppose  $w = sc \in R$ ,  $s \in S_n$ ,  $c \in \mathbf{Z}_2^n$ . Then  $s = 1$ .

*Proof.* If  $s$  has a nontrivial cycle, by conjugation we may assume it is  $(k \cdots n - 1 \ n)$  and that  $c$  changes the sign of at most one  $e_i$  in the corresponding orbit.

Suppose  $c(e_i) = e_i$  for  $k \leq i < n$  and  $c(e_n) = -e_n$ . Then  $w^{-1}(e_n) = e_{n-1}$  implies that  $e_{n-1} - e_n \in \mathcal{L} \cap \Phi^v$ . But repeated application of  $w^{-1}$  sends  $e_{n-1} - e_n$  to  $e_k - e_{k+1}$ , which  $w^{-1}$  sends to  $-e_n - e_k < 0$ , contradicting  $w \in R$ .

Now suppose  $c(e_i) = e_i$  for  $k \leq i \leq n$ . Then  $w^{-1}(e_n) = e_{n-1}$  and  $e_{n-1} - e_n \in \mathcal{L} \cap \Phi^v$ . But then  $w(e_{n-1} - e_n) = e_n - e_k < 0$  contradicts  $w \in R$ .

Thus  $s = 1$  if  $w = sc \in R$ , and  $R$  is contained in the group of sign changes in  $W$ . Hence  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors bounded by  $n$ .

Let  $w \in R$ . By conjugation we may assume that  $w = c_k c_{k+1} \cdots c_n$ .

**LEMMA 2.** If  $c_k c_{k+1} \cdots c_n \in R$ , then  $c_i \in R$ ,  $k \leq i \leq n$ .

*Proof.*  $e_i$  and  $e_i - e_j$ ,  $k \leq i \neq j \leq n$ , correspond to characters of order 2, and  $\mathcal{L}$  contains  $\mathbf{Z}[2e_i | k \leq i \leq n]$ . Then  $\alpha - c_i \alpha \in \mathcal{L}$  for all simple  $\alpha$ , so  $c_i \in W_\lambda$ ,  $k \leq i \leq n$ . Since  $R(c_i) \subseteq R(c_k \cdots c_n)$  does not intersect  $\Delta'$ , we have that  $c_i \in R$ . (Recall that  $R(w) = \{\alpha > 0 | w\alpha < 0\}$ .)

Thus any  $R$  is conjugate to  $\langle c_k, c_{k+1}, \dots, c_n \rangle$  for some  $k$ ,  $1 \leq k \leq n$ , taking  $c_k \cdots c_n$  above with as many sign changes as possible.

Note that each  $e_i$  corresponds to a character of order 2,  $k \leq i \leq n$ , and that these characters must be distinct, since  $e_i - e_j$  does not correspond to the trivial character,  $k \leq i \neq j \leq n$ . Conversely, we may define a character  $\lambda$  with  $R$ -group  $\langle c_k, c_{k+1}, \dots, c_n \rangle$  by assigning a distinct character of order 2 to each  $e_i$ ,  $k \leq i \leq n$ .

Thus the number of factors of  $\mathbf{Z}_2$  in  $R$  is bounded by  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ .

Note that there can be more reducibility in the case of type  $C_n$  than in the case of type  $B_n$ .

$B_n$ :  $|R|$  divides  $2n$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2]$ .

$C_n$ :  $|R|$  divides  $2^n$  and  $2^{[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1}$ .

If  $\mathfrak{f} = R$ , we again get  $|R| = 1$  or  $2$ .

4. *Type  $D_n$ .*  $\Phi = \Phi^v = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$  is self-dual and the Weyl group  $W \cong S_n \times \mathbf{Z}_2^{n-1}$  acts as permutations and even sign changes on the  $e_i$ .

**THEOREM  $D_n$ .**

(a) *Suppose  $n$  is even. Then if  $R$  is abelian,  $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors bounded by  $n - 1$  and by  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ . If  $R$  is nonabelian, then  $R \cong (\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2) \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2)$  with the order of the first factor dividing both  $2n$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2]$ , and the number of factors of  $\mathbf{Z}_2$  in the normal subgroup an odd number bounded by  $n - 1$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ .*

(b) *Suppose  $n$  is odd. Then if  $R$  is abelian,  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  with the number of factors of  $\mathbf{Z}_2$  bounded by  $n - 1$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ , or  $R = \mathbf{Z}_4$ . If  $R$  is nonabelian,  $R \cong \mathbf{Z}_4 \times (\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2)$  with the number of factors of  $\mathbf{Z}_2$  in the normal subgroup an even number bounded by  $n - 3$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 2$ .*

The actions on the normal factors of the semi-direct products are described explicitly in the course of the proof.

**LEMMA 1.** *Suppose  $w = sc \in R$ ,  $s \in S_n$  and  $c \in \mathbf{Z}_2^{n-1}$ . Then  $s^2 = 1$ .*

*Proof.* Suppose  $s$  has a cycle of length  $\geq 3$ . Replacing  $w$  by a conjugate under  $S_n$ , we may assume the cycle is  $(kk + 1 \cdots n)$ ,  $k < n - 1$ . Then by conjugating  $w$  by a sign change, we may assume that  $c$  changes the sign of at most 2 of the  $e_i$  in each orbit of  $s$ .

If  $c$  involves no sign changes on  $e_k, \dots, e_n$ , then  $w^{-1}(e_{n-1} + e_n) = e_{n-2} + e_{n-1}$  implies that  $e_{n-2} - e_n \in \mathcal{A}'$ . But then  $w(e_{n-2} - e_n) < 0$  contradicts  $w \in R$ .

If  $c$  involves only one sign change on  $e_k, \dots, e_n$ , we may suppose it is on  $e_n$ . Then  $w(e_{n-1} + e_n) = e_n - e_k$  implies  $e_k + e_{n-1} \in \mathcal{A}'$ . But then  $w^2(e_k + e_{n-1}) = e_{k+2} - e_k < 0$  contradicts  $w \in R$ .

Finally, if there are two sign changes involved, we may suppose they are on  $e_{n-1}$  and  $e_n$ . Then  $w(e_{n-1} - e_n) = -e_n + e_k$  implies that  $e_k - e_{n-1} \in \mathcal{A}'$ . But then  $w^{-1}(e_k - e_{n-1}) = -e_n - e_{n-2} < 0$  contradicts  $w \in R$ .

Note that  $w = sc \in R$ ,  $s^2 = 1$  implies that  $w^2 = (scs^{-1})c$  is a sign change in  $R$  and thus  $w^4 = 1$ . If we let  $R'$  be the group of sign changes in  $R$ , then  $R' \trianglelefteq R$  and  $R/R' \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ .

**LEMMA 2.** *Suppose  $c_k c_{k+1} \cdots c_n \in R$  with  $k > 1$ . Then  $R$  contains all even sign changes on  $\{e_k, e_{k+1}, \dots, e_n\}$ .*

*Proof.* If  $k > 1$ , then  $c_k c_{k+1} \cdots c_n \in R$  if and only if  $e_i - e_j$  corresponds to a character of order 2 for  $k \leq i < j \leq n$ , and  $e_{n-1} \pm e_n$  correspond to the same character. Then  $c_i c_{i+1} \in W_\lambda$  and  $R(c_i c_{i+1}) \subseteq R(c_k c_{k+1} \cdots c_n)$  imply that  $c_i c_{i+1} \in R$ , for  $k \leq i < n$ .

Thus  $R'$  consists of all even sign changes on  $\{e_k, \dots, e_n\}$ , and  $|R'| \leq 2^{n-1}$ .

Further, since the characters corresponding to  $e_i - e_j$  are non-trivial for  $k \leq i < j < n$ , the characters corresponding to  $e_i - e_n$  are distinct,  $k \leq i < n$ . Thus  $|R'| \leq 2^{\lceil (n-k)/2 \rceil - 1}$ .

Now, suppose  $w = sc \in R$  with  $s \neq 1$ . By conjugation we may assume  $s = (12)(34) \cdots (k-1 k)$  with  $k \leq n$ . Then  $c(e_i) = -e_i$  for  $k < i \leq n$ ; first, we may assume  $c(e_k) = +e_k$  by conjugation by  $c_k c_n$  if necessary. Then if  $c(e_i) = e_i$  for some  $k < i \leq n$ ,  $w(e_k - e_i) = e_{k-1} - e_i$  would imply  $e_{k-1} - e_k \in \Delta'$ . But then  $w(e_{k-1} - e_k) < 0$  contradicts  $w \in R$ . Thus we have shown

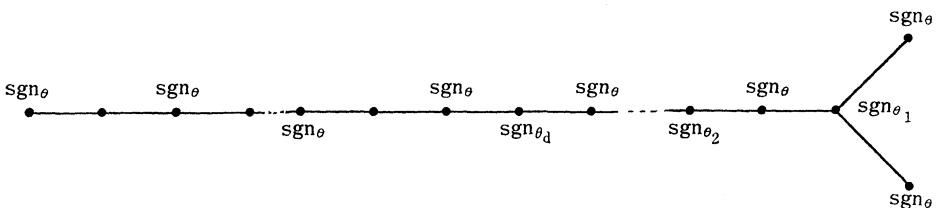
LEMMA 3.  $\text{stab}_R(\pm e_i) \leq R'$ .

Further, any element of order 2 in  $R$  is conjugate to one of  $c \in \mathbf{Z}_2^{n-1}$ ,  $(12)(34) \cdots (n-1 n)$ ,  $(12)(34) \cdots (n-1 n)c_{n-1} c_n$ , or  $(12) \cdots (k-1 k)c_{k+1} \cdots c_n$  for some  $k$ . Any element of order 4 in  $R$  is conjugate to  $(12)(34) \cdots (m-1 m)c_k c_{k+2} \cdots c_m c_{m+1} c_{m+2} \cdots c_n$  for some  $m, k$  with  $2 \leq k \leq m \leq n$ , where the sign change changes the sign of the  $e_i$ ,  $m < i \leq n$  and of the  $e_j$ ,  $j$  even,  $k \leq j \leq m$ .

If  $R$  has no elements of order 4, then  $R \cong \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$  is abelian. Suppose there is an element of order 4. We distinguish the cases  $n$  even and  $n$  odd.

*Case 1.* Suppose  $n$  is even.

Then any element of  $R$  of order 4 is conjugate to  $w = (12)(34) \cdots (m-1 m)c_k c_{k+2} \cdots c_m c_{m+1} \cdots c_n$  with  $k \leq m - 2$ . Suppose for a moment that  $m = n$ . Then since  $w \in W_\lambda$  if and only if  $\alpha - w\alpha \in \mathcal{L}$ , the  $\lambda_\alpha$  must satisfy relations corresponding to  $e_1 - e_2 \equiv e_3 - e_4 \equiv \cdots \equiv e_{n-1} - e_n \equiv e_{n-1} + e_n$ ,  $2(e_1 - e_2) \equiv 0$ , and  $2(e_i - e_j) \equiv 0 \pmod{\mathcal{L}}$  for  $k \leq i < j \leq n$ . Further, if  $w \in R$  then  $\mathcal{L} \cap \{e_i - e_j | k - 1 \leq i < j \leq n\} = \emptyset$ , and thus  $\lambda$  is given by



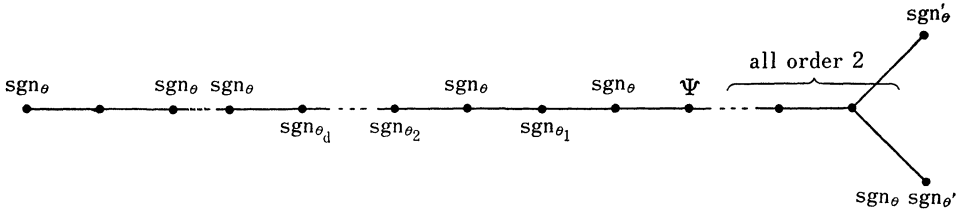
where  $k = n - 2d$ .

Since  $w^2 = c_{k-1} c_k \cdots c_n \in R$ , by Lemma 2,  $R$  contains the group  $Z_2^{2d+1}$  of all even sign changes on  $\{e_{k-1}, e_k, \dots, e_n\}$ . Then  $(12)(34) \cdots (n-1n) \in R$  and  $R \cong \langle (12)(34) \cdots (n-1n) \rangle \times \langle c_n c_{n-1}, c_{n-1} c_{n-2}, \dots, c_k c_{k-1} \rangle$  is nonabelian.

If there are other sign changes in  $R$ , we may assume they involve  $e_l, e_{l+1}, \dots, e_{k-2}$ , where  $l$  is odd. Then the group  $R'$  of all sign changes in  $R$  consists all even sign changes on  $\{e_l, e_{l+1}, \dots, e_n\}$  and each  $e_i - e_j, l \leq i < j \leq n$  corresponds to a character of order 2.

Now, if  $m < n$ ,  $w = (12) \cdots (m-1m) c_k c_{k+2} \cdots c_m. c_{m+1} \cdots c_n$  acts on  $D_m = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq m\} \subset \Phi$  as the above. Then  $e_1 - e_2 \equiv e_3 - e_4 \equiv \dots \equiv e_{m-1} - e_m \equiv e_{m-1} + e_m \pmod{\mathcal{L}}$ , and  $e_i - e_j$  corresponds to a character of order 2 for  $k-1 \leq i < j \leq m$  and for  $m+1 \leq i < j \leq n$ .

Thus  $e_{m-1} - e_m \equiv (e_{n-1} - e_n) + (e_{n-1} + e_n) \not\equiv 0$  and  $2(e_m - e_{m+1}) \equiv e_{m-1} - e_m \pmod{\mathcal{L}}$ , and  $\lambda$  is given by



with  $\Psi^2 = sgn_\theta$ .

Again,  $R$  contains  $sc = (12) \cdots (m-1m) c_{m+1} \cdots c_n$  and we may assume by conjugation the  $R'$  consists of all even sign changes on  $\{e_i, \dots, e_m\}$ .

Then  $R \cong \langle sc \rangle \times R'$  is nonabelian, as before. Suppose there are other  $s'c'$  in  $R$ . If  $s' = s$  then  $sc'$  is in the subgroup  $\langle sc \rangle \times R'$ . If  $s' \neq s$  then  $R \cong \langle sc, s'c' \rangle \times R'$ . We keep adding new elements of  $R$  until

LEMMA 4.  $R = \{1, sc, s'c', \dots\} \times R'$  with the permutations  $s^{(i)}$  distinct.

Further, the order of the first subgroup divides  $2n$  by Lemma 3, and also  $R' \leq Z_2^{n-1}$ . Thus  $|R|$  divides  $n \cdot 2^n$ .

Formally define a character corresponding to  $2e_n$  to be  $-(e_{n-1} - e_n) + (e_{n-1} + e_n)$  and then use  $2e_i = 2(e_i - e_n) + 2e_n$  to define a character corresponding to  $2e_i$ . If  $w = sc \in R, c \in Z_2^{n-1}$  with  $s \neq 1$ , then  $e_i - we_i$  is a character of order 2 and  $w \mapsto e_i - we_i$  is an injective homomorphism on the first (nonnormal) factor of  $R$ . Thus the order of this factor divides  $[f^*: (f^*)^2]$ .

We have already seen that the number of factors of  $Z_2$  in  $R'$  is bounded by  $n - 1$  and  $[f^*: (f^*)^2] - 1$ .

Finally, if  $n$  is even and  $R$  is abelian, we may write  $R = \{1, sc, s'c', \dots\} \times R' \cong \mathbf{Z}_2 \times \dots \times \mathbf{Z}_2$  as in Lemma 4, with the  $s^{(i)}$  distinct. We show that the number of factors of  $\mathbf{Z}_2$  is bounded by  $n - 1$  and  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2] - 1$ . If  $R = R'$ , this is true. If  $R' = \{1\}$ , then  $|R|$  divides  $2n$  by Lemma 3 and divides  $[\mathfrak{f}^*: (\mathfrak{f}^*)^2]$  by the above. Suppose that both factors are nontrivial and that  $R$  is abelian. We may assume that  $(12)(34) \dots (n - 1 n)$  or  $(12) \dots (k - 1 k)c_{k+1} \dots c_n$  is in  $R$ .

Suppose that  $(12) \dots (k - 1 k)c_{k+1} \dots c_n$  and  $c_l \dots c_{k-1} c_k \in R, k < n$ . Then also  $c_{k-1} c_k \in R$ . Then if  $s'c' \in R, s' \neq 1, (12) \dots (k - 1 k)$ , we may assume that  $c'(e_i) = e_i, i = k - 1, k$ . Then  $s'(k) \neq k - 1$  by Lemma 3 and  $s'c'$  does not commute with  $c_{k-1} c_k$ , contradicting the assumption that  $R$  is abelian. Thus no other  $s'c'$  are in  $R$ . Further, if  $l < k - 1$ , then  $c_{k-2} c_{k-1} \in R$ , contradicting  $R$  abelian, and  $c_{k-1} c_k$  is the only sign change in  $R$ . Thus  $|R| = 4$ .

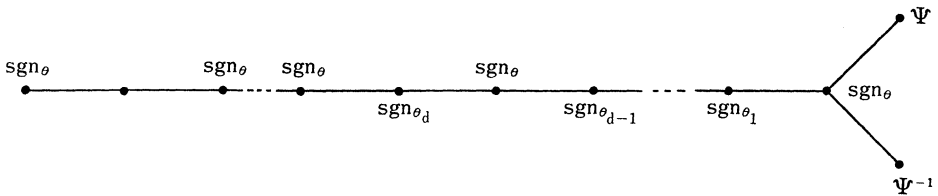
Suppose that  $(12)(34) \dots (n - 1 n)$  and  $c_i \dots c_{n-1} c_n \in R$ . Then if  $1 < i < n - 1, c_{n-2} c_{n-1} \in R$ , contradicting the assumption  $R$  is abelian. Thus  $i = 1$  or  $n - 1$ . If  $i = 1$ , the  $R' = \{1, c_1 c_2 \dots c_n\}$  and  $|R|$  divides  $2n$  and  $2[\mathfrak{f}^*: (\mathfrak{f}^*)^2]$ . If  $i = n - 1$ , then  $|R| = 4$ , as above.

We note that if  $\mathfrak{f} = R$ , then one can have  $|R| = 1, 2$ , or  $4$  in the case of  $D_n, n$  even.

*Case 2. Suppose  $n$  is odd.*

In this case any element of order 4 in  $R$  is conjugate to  $w = (12) \dots (m - 1 m). c_k c_{k+2} \dots c_m \cdot c_{m+1} \dots c_n, \text{ with } k \leq m < n$ .

If  $m = n - 1$  and  $w = (12) \dots (n - 2 n - 1)c_{n-1} c_n$ , then  $\lambda$  is given by

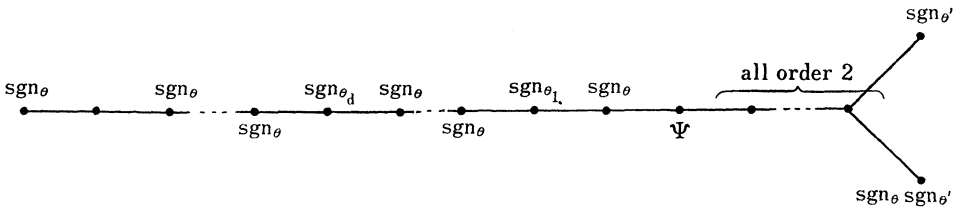


with  $\Psi^2 = \text{sgn}_\theta$ .

Then  $w^2 = c_{k-1} c_k \dots c_{n-1} \in R$  and thus  $R$  contains all even sign changes on  $\{e_{k-1}, e_k, \dots, e_{n-1}\}$ . Thus each  $e_i - e_j, k - 1 \leq i < j \leq n - 1$  corresponds to a character of order 2 and also  $sc = (12) \dots (n - 2 n - 1)c_{n-1} c_n \in R$ . We have  $R \supseteq \langle sc \rangle \times \langle c_{k-1} c_k, c_k c_{k+1}, \dots, c_{n-3} c_{n-2} \rangle \cong \mathbf{Z}_4 \times \mathbf{Z}_2^{n-k-1}$ .

If  $m < n - 1$  and  $w = (12) \dots (m - 1 m)c_k c_{k+2} \dots c_m c_{m+1} \dots c_n \in R$ , then  $\lambda$  is given by





with  $\Psi^2 = \text{sgn}_\theta$ .

In this case,  $w^2 = c_{k-1} c_k \cdots c_m \in R$  and  $R$  contains all even sign changes on  $\{e_{k-1}, \dots, e_m\}$ . Then  $sc = (12) \cdots (m-1 m) c_m c_{m+1} \cdots c_n \in R$  and each  $e_i - e_j$ ,  $k-1 \leq i < j \leq m$ , and  $m+1 \leq i < j \leq n$ , corresponds to a character of order 2. Also  $2(e_m - e_{m+1}) \equiv e_{m-1} - e_m \equiv (e_{n-1} - e_n) + (e_{n-1} + e_n) \pmod{\mathcal{L}}$ .

$R \cong \langle sc \rangle \times \langle c_{k-1} c_k, c_k c_{k+1}, \dots, c_{m-2} c_{m-1} \rangle = Z_4 \times Z_2^{m-k}$ . Let  $R''$  be the group of the even sign changes on  $\{e_1, e_2, \dots, e_{m-1}\}$  which occur in  $R$ . Then  $R = \langle sc \rangle \times R''$  by

LEMMA 5. *If  $s'c' \in R$ , then  $s' = 1$  or  $s' = s = (12) \cdots (m-1 m)$ .*

*Proof.* If  $sc$  and  $s'c'$  are in  $R$ , then  $s's = ss'$  by Lemma 1, so that  $s'$  permutes the odd number of fixed points  $m+1, \dots, n$  of  $s$ .  $s'$  must permute them faithfully by Lemma 3. But this contradicts Lemma 1, which implies that  $s'$  must be a product of transpositions.

Thus  $R = \langle sc \rangle \times R'' \cong Z_4 \times R''$ .

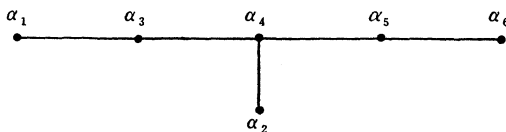
Since  $R'' \leq Z_2^{m-2} \leq Z_2^{n-3}$ , the number of factors of  $Z_2$  in  $R''$  is bounded by  $n-3$ . It is also bounded by  $[f^*: (f^*)^2] - 2$ , since the number of factors of  $Z_2$  in  $R' = \langle c_{m-1} c_m \rangle \times R''$  is bounded by  $[f^*: (f^*)^2] - 1$ .

Finally, suppose that  $R$  is abelian and  $n$  is odd. Then either there are no factors of  $Z_2$  above, i.e.,  $k = m$ ,  $R'' = \{1\}$  and  $R \cong Z_4$ , or  $R$  is contained in the group  $Z_2^{n-1}$  of even sign changes.

5. Type  $E_6$ .

THEOREM  $E_6$ .  $R \cong 1, Z_2, Z_3, Z_3 \times Z_3$  or  $Z_6$ . Further,  $Z_3 \times Z_3$  occurs as a reducibility group if and only if  $p = 3$  or 3 divides  $q-1$ .

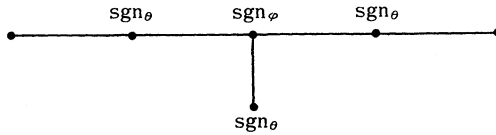
Arrange the simple roots in the traditional Dynkin diagram



The roots spanned by  $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$  form a subsystem of type  $D_5$ , giving an inclusion of Weyl groups  $W(D_5) < W(E_6)$ . Comparing orders,  $2^7 \cdot 3 \cdot 5$  and  $2^7 \cdot 3^4 \cdot 5$  respectively, we see that a 2-Sylow subgroup of  $W(D_5)$  is also a 2-Sylow subgroup of  $W(E_6)$ . By conjugation, we may assume that a 2-Sylow subgroup of  $W_\lambda$  is contained in  $W(D_5)$ .

Let  $\alpha_6 = e_1 - e_2, \dots, \alpha_3 = e_4 - e_5$  and  $\alpha_2 = e_4 + e_5$ . Then by our  $D_5$  results, potential candidates in  $R \cap W(D_5)$  are conjugate to  $c_2 c_3 c_4 c_5, c_4 c_5, (12) c_2 c_3 c_4 c_5,$  or  $(12) (34) c_4 c_5$ . Adding the condition  $w\alpha_1 \equiv \alpha_1,$  only  $c_2 c_3 c_4 c_5 = w_{\alpha_5} w_{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5} w_{\alpha_3} w_{\alpha_2}$  can be in an  $E_6$   $R$ -group.

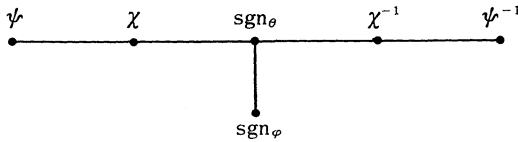
If  $c_2 c_3 c_4 c_5 \in R,$  then  $\lambda$  is given by



with  $\text{sgn}_\theta \neq \text{sgn}_\phi$ . There can be no other elements of order 2 in  $R$  with  $c_2 c_3 c_4 c_5;$  if there were another, by conjugation we could assume it is  $c_1 c_2 c_3 c_4$ . But the product  $c_1 c_5$  cannot be in an  $R$ -group.

Thus, if there is an element with order a power of 2 in  $R,$  it has order 2 and is unique, hence is in the center of  $R$ .

Note that the longest Weyl element  $w_0$  and the character



are conjugate to the above.

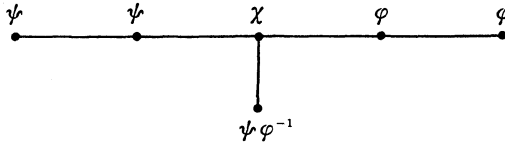
There is only one conjugacy class of elements of order 5 in  $W(E_6)$  and none of its elements can be in an  $R$ -group. Thus,  $R$  is the direct product of a 2-Sylow subgroup (1 or  $Z_2$ ) and a 3-Sylow subgroup. Examining conjugacy classes of elements of order 3 or 9, [5], any element in  $R$  with order a power of 3 is conjugate to one of  $w_{\alpha_1} w_{\alpha_3} w_{\alpha_6} w_{\alpha_5}$  or  $w_{\alpha_1} w_{\alpha_3} w_{\alpha_5} w_{\alpha_6} w_{\alpha_2} w^{12321}_2,$  where  $^{12321}_2$  represents the root  $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ .

For  $w_{\alpha_1} w_{\alpha_3} w_{\alpha_6} w_{\alpha_5} \in R,$   $\lambda$  is given by



with  $\psi$  of order 3, giving  $R \cong Z_3$ . Further, if  $\lambda_{\alpha_2} = \psi$  and  $\lambda_{\alpha_4} \neq \psi^{\pm 1}$  have order 3, then  $R = \langle w_{\alpha_1} w_{\alpha_3} w_{\alpha_6} w_{\alpha_5}, w_{\alpha_1} w_{\alpha_3} w_{\alpha_2} w^{12321}_2 \rangle \cong Z_3 \times Z_3$ . If instead  $\lambda_{\alpha_2} \neq \lambda_{\alpha_4}$  have order 2, then  $R = \langle w_{\alpha_1} w_{\alpha_3} w_{\alpha_6} w_{\alpha_5}, w_0 \rangle \cong Z_6$ .

For  $w_{\alpha_1}w_{\alpha_3}w_{\alpha_5}w_{\alpha_6}w_{\alpha_2}w^{12321} \in R$ ,  $\lambda$  is given by



with each character having order 3,  $\psi \neq \varphi$  and  $\lambda \notin \langle \psi, \varphi \rangle$ . Then  $R \cong \mathbf{Z}_3$ , or there is also an element of type  $2A_2$  [5] in  $R \cong \mathbf{Z}_3 \times \mathbf{Z}_3$ , and we are in one of the above cases.

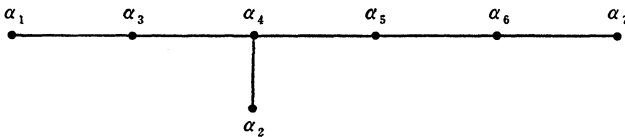
Note that if  $G$  is a Chevalley group over  $k = R$ , then  $R = 1$  and  $\text{Ind}_B^G \lambda$  is irreducible.

6. Type  $E_7$ .

**THEOREM  $E_7$ .**  *$R$  may be nonabelian. If so,  $R \cong$  dihedral group  $D$  of order 8, or  $R \cong D \times \mathbf{Z}_2$ .  $D \times \mathbf{Z}_2$  can occur if and only if  $p = 2$  or 4 divides  $q - 1$ .*

*If  $R$  is abelian, then  $R \cong \mathbf{Z}_2^n$  with  $0 \leq n \leq 4$ ,  $\mathbf{Z}_3$ ,  $\mathbf{Z}_4$  or  $\mathbf{Z}_6$ .  $\mathbf{Z}_2^3$  and  $\mathbf{Z}_4$  occur if and only if  $p = 2$ .  $\mathbf{Z}_2^4$  occurs if and only if  $[k^* : (k^*)^2] \geq 16$ .  $\mathbf{Z}_3$  and  $\mathbf{Z}_6$  occur if and only if  $p = 3$  or 3 divides  $q - 1$ .*

Arrange the simple roots in the diagram

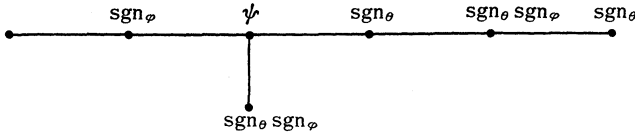


The roots  $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$  span a subsystem of type  $D_6$ , giving an inclusion of Weyl groups  $W(D_6) < W(E_7)$ . Let  $w_0 = -1$  be the longest Weyl element in  $W(E_7)$ . Comparing orders,  $2^9 \cdot 3^2 \cdot 5$  and  $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ ,  $\langle w_0 \rangle x$  a 2-Sylow subgroup of  $W(D_6)$  will be a 2-Sylow subgroup of  $W(E_7)$ . We first classify 2-Sylow subgroups of  $R$ -groups.

Let  $\alpha_7 = e_1 - e_2, \dots, \alpha_3 = e_5 - e_6$ , and  $\alpha_2 = e_5 + e_6$ . Using our  $W(D_6)$  notation and grouping by  $W(E_7)$ -conjugacy classes, elements in  $R$  with order a power of 2 are conjugate to:

- $3A_1$ : (12) (34) (56)  $c_5c_6, w_0c_3c_4c_5c_6$
- $4A_1$ :  $c_3c_4c_5c_6, w_0(12) (34) (56)c_5c_6$
- $5A_1$ : (12)  $c_3c_4c_5c_6, w_0(34) (56), w_0c_5c_6$
- $6A_1$ :  $c_1c_2c_3c_4c_5c_6, w_0(56)$
- $7A_1$ :  $w_0 = -1$
- $D_4(a_1) + 2A_1$ : (12) (34)  $c_2c_4c_5c_6, w_0(12) (34) (56)c_4c_6$
- $2A_3 + A_1$ :  $w_0(12) (34)c_2c_3$ .

Suppose there is an element of order 4 in  $R$ . If it has type  $D_4(\alpha_1) + 2A_1$ , we may assume it is  $(12)(34)c_2c_4c_5c_6$  and  $\lambda$  is given by



with  $\psi^2 = \text{sgn}_\theta \neq \text{sgn}_\varphi$ . Then  $R \cong \langle (13)(24)(56)c_2c_4 \rangle \times \langle (12)(34)c_2c_4c_5c_6 \rangle \cong$  dihedral group  $D_4$ .

If  $\lambda_{\alpha_1}$  is "generic", then  $R \cong D$  is nonabelian with order 8. If  $R$  is larger, a consideration of other possible elements shows that we may assume, by conjugation, that  $R$  contains one of  $w_0c_3c_4c_5c_6$ ,  $w_0c_2c_4c_5c_6$ , or  $w_0c_1c_4c_5c_6$ . Each of these three cases occurs, giving  $R \cong D \times Z_2$  nonabelian of order 16.

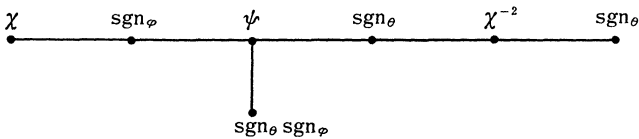
In the first case,  $\lambda_{\alpha_1}^2 = \text{sgn}_\theta \text{sgn}_\varphi$ , which can occur if, and only if  $[k^*: (k^*)^4] \geq 16$ , i.e.,  $p = 2$  or 4 divides  $q - 1$ .

In the second case,  $\lambda_{\alpha_1} \neq \text{sgn}_\theta, \text{sgn}_\varphi, \text{sgn}_\theta \text{sgn}_\varphi$  has order 2, which can occur if, and only if  $p = 2$ .

In the third case,  $\lambda_{\alpha_1}^2 = \text{sgn}_\theta$ , and  $\lambda_{\alpha_1}\psi^{-1} \neq \text{sgn}_\theta, \text{sgn}_\varphi, \text{sgn}_\theta \text{sgn}_\varphi$  has order 2, which occurs if, and only if  $p = 2$ .

We may now suppose that  $R$  contains no elements of type  $D_4(\alpha_1) + 2A_1$ .

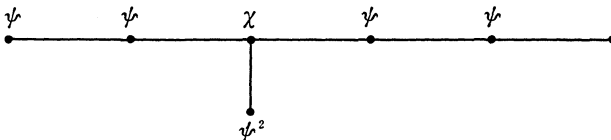
Suppose that  $R$  contains  $w_0(12)(34)c_2c_3$ . Then  $\lambda$  is given by



with  $\psi^2 = \text{sgn}_\theta \neq \text{sgn}_\varphi$ ,  $\chi$  of order 4 and  $\chi^2 \neq \text{sgn}_\theta, \text{sgn}_\varphi$ . If  $\chi^2 = \text{sgn}_\theta \text{sgn}_\varphi$ , then we are in one of the above cases with  $(12)(34)c_2c_4c_5c_6 \in R$ . Otherwise,  $\chi^2 \notin \langle \text{sgn}_\theta, \text{sgn}_\varphi \rangle$ , so  $p = 2$ , and then  $R \cong Z_4$ .

If  $R$  contains no elements of order 4, then a 2-Sylow subgroup of  $R$  is a product of copies of  $Z_2$ . An explicit list shows that  $Z_2$  occurs for any  $k$ , even the reals  $R$ ; that  $Z_2^2$  occurs for any non-Archimedean  $k$  (we will need 2 characters of order 2); that  $Z_2^3$  occurs if, and only if  $p = 2$ ; and  $Z_2^4$  occurs if, and only if  $[k^*: (k^*)^2] \geq 16$ .

An easy calculation shows that  $R$  can contain no elements of order 5 or 7. Of elements of order a power of 3, only conjugates of  $w_{\alpha_1}w_{\alpha_3}w_{\alpha_6}w_{\alpha_5}w_{\alpha_2}w_{123210}(\mathfrak{3A}_2)$  can be in an  $R$ -group. If this element is in  $R$ , then  $\lambda$  is given by



with  $\psi \neq \chi^{\pm 1}$  of order 3. There are no other elements of order 3 in  $R$  with this one, besides its inverse. Since we may specify only one character of order 2, there can be at most one element of order 2 in this  $R$ . Thus,  $R \cong Z_6$ . This does occur, with  $R$  generated by an element of type  $A_5 + A_2$ .

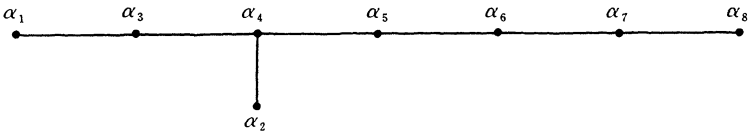
7. Type  $E_8$ .

**THEOREM  $E_8$ .** *A nonabelian  $R$ -group will occur if and only if  $[k^* : (k^*)^2] \geq 16$ . All nonabelian  $R$  are conjugate to  $\langle (12)(34)(56)(78)C_7C_8, (13)(24)(57)(68)C_6C_8, (15)(26)(37)(48)C_4C_8 \rangle \times \langle C_1C_3C_5C_7, C_2C_4C_6C_8, C_1C_2C_3C_4, C_8C_4C_5C_6 \rangle$ .*

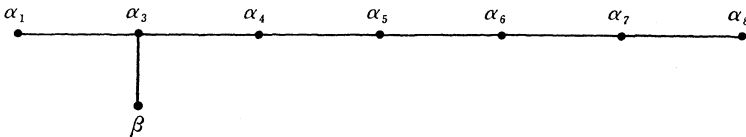
*If  $R$  is abelian, then  $R \cong Z_2^n$  with  $0 \leq n \leq 4$ ,  $Z_4$ ,  $Z_4 \times Z_2$ ,  $Z_3$ ,  $Z_3 \times Z_3$ , or  $Z_5$ .*

*$Z_2^n$  occurs if and only if  $[k^* : (k^*)^2] \geq 2^{n+1}$ ,  $0 \leq n \leq 4$ .  $Z_4$  occurs if and only if  $p = 2$  or 4 divides  $q - 1$ .  $Z_4 \times Z_2$  occurs if and only if  $p = 2$ .  $Z_3$  occurs if and only if  $[k^* : (k^*)^3] \geq 9$  and  $Z_3 \times Z_3$  occurs if and only if  $[k^* : (k^*)^3] \geq 27$ .  $Z_5$  occurs if and only if  $[k^* : (k^*)^5] \geq 25$ , i.e.,  $p = 5$  or 5 divides  $q - 1$ .*

Arrange the simple roots in the diagram



Letting  $\beta = -\binom{2454321}{2}$ ,  $\Phi$  contains a subsystem of type  $D_8$  spanned by



giving an inclusion of Weyl groups  $W(D_8) < W(E_8)$ . Comparing orders,  $2^{14} \cdot 3^2 \cdot 5 \cdot 7$  and  $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ , we see that a 2-Sylow subgroup of  $W(D_8)$  is also a 2-Sylow subgroup of  $W(E_8)$ . Thus we may assume that a 2-Sylow subgroup of  $R$  is contained in  $W(D_8)$ ; we first classify these groups.

In this realization, the orderings determined by the positive roots are not compatible between the  $D_8$  and  $E_8$  root systems. However, easy modifications of the proofs show that Lemmas 1 and 3 of § $D_n$  hold. Adding the condition  $w\alpha_2 \equiv \alpha_2$  in  $E_8$ , we see that possible

elements in  $W(D_8) \cap R$ , grouped by  $W(E_8)$ -conjugacy classes, are conjugate to

$$4A_1: C_5 C_6 C_7 C_8, (12) (34) (56) (78) C_7 C_8,$$

$$6A_1: C_3 C_4 C_5 C_6 C_7 C_8, (12) (34) C_5 C_6 C_7 C_8,$$

$$7A_1: (12) C_3 C_4 C_5 C_6 C_7 C_8,$$

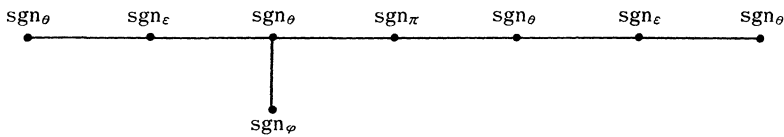
$$8A_1: w_0 = -1,$$

$$2D_4(a_1): (12) (34) (56) (78) C_2 C_4 C_6 C_7,$$

$$D_4(a_1) + 3A_1: (12) (34) (56) C_4 C_6 C_7 C_8, \text{ or}$$

$$D_4(a_1) + 4A_1: (12) (34) C_2 C_4 C_5 C_6 C_7 C_8.$$

Suppose there is an element of order 4 in  $R$ . If there is one of type  $2D_4(a_1)$ , we may assume it is  $(12) (34) (56) (78) C_2 C_4 C_6 C_7$ . Then  $\lambda$  is given by



with  $|\langle \text{sgn}_\theta, \text{sgn}_\varphi, \text{sgn}_\epsilon, \text{sgn}_\pi \rangle| = 16$ . This can occur if and only if  $[k^*: (k^*)^2] \geq 16$ , and in this case,  $\mathcal{A}' = \phi$ .

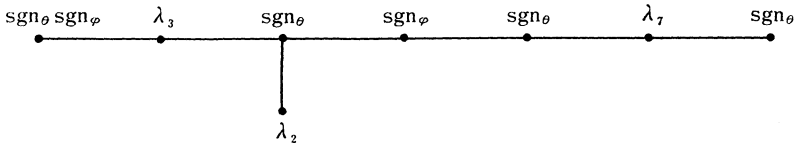
Then

$$R = W_\lambda$$

$$\begin{aligned} &= \langle (12) (34) (56) (78) C_7 C_8, (13) (24) (57) (68) C_6 C_8, (15) (26) (37) (48) C_4 C_8 \rangle \\ &\quad \times \langle C_1 C_3 C_5 C_7, C_2 C_4 C_6 C_8, C_1 C_2 C_3 C_4, C_3 C_4 C_5 C_6 \rangle \end{aligned}$$

is nonabelian of order 128 and has 65 conjugacy classes.  $R \text{ odm } \langle w_0 \rangle$  is abelian of order 64.

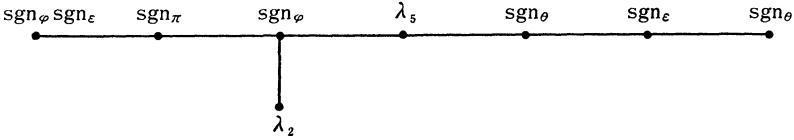
Now suppose that  $R$  does not contain an element of type  $2D_4(a_1)$ , but does contain  $(12) (34) (56) C_4 C_6 C_7 C_8$  of type  $D_4(a_1) + 3A_1$ . Then  $\lambda$  is given by



with  $\lambda_3^2 = \text{sgn}_\theta$  and  $\lambda_2^2 \lambda_7^2 = \text{sgn}_\theta \text{sgn}_\varphi$ . This can occur if and only if  $p = 2$  or 4 divides  $q - 1$ . An examination of other possible elements in  $R$  with this one shows that  $R \cong \mathbf{Z}_4$  if  $\lambda_2$  and  $\lambda_7$  satisfy no additional conditions. If  $\lambda_7$  has order 2,  $\lambda_7 \notin \langle \text{sgn}_\theta, \text{sgn}_\varphi \rangle$ , then  $C_1 C_2 C_3 C_4 \in R \cong \mathbf{Z}_4 \times \mathbf{Z}_2$ . If instead  $\lambda_7^2 = \text{sgn}_\varphi$ ,  $\lambda_2 \lambda_3^{-1}$  has order 2,  $\lambda_2 \lambda_3^{-1} \notin \langle \text{sgn}_\theta, \text{sgn}_\varphi \rangle$ , then  $(12) (35) (46) (78) C_1 C_2 \in R \cong \mathbf{Z}_4 \times \mathbf{Z}_2$ . These two cases occur if and only if  $p = 2$ .

Next, assume that  $R$  does not contain elements of types  $2D_4(a_1)$

or  $D_4(\alpha_1) + 3A_1$ , but does contain  $(12)(34)C_2C_4C_5C_6C_7C_8$  of type  $D_4(\alpha_1) + 4A_1$ . Then  $\lambda$  is given by



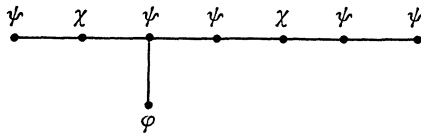
with  $\lambda_5^2 = \text{sgn}_\theta$  and  $\lambda_2^2 = \text{sgn}_\theta \text{sgn}_\epsilon$ , and  $R \cong Z_4$ . This case occurs if and only if  $[k^*: (k^*)^2] \geq 16$ .

Now assume that  $R$  contains no elements of order 4. An explicit list shows that a 2-Sylow subgroup of  $R$  is then  $Z_2^n$  with  $0 \leq n \leq 4$ . Further,  $Z_2^n$  occurs if and only if  $[k^*: (k^*)^2] \geq 2^{n+1}$ ,  $0 \leq n \leq 4$ .

Using the fact that no elements of order 6 can be in an  $E_8$   $R$ -group, it is easy to see that the other  $R$ -groups which occur are isomorphic to  $Z_3$ ,  $Z_3 \times Z_3$  or  $Z_5$ .

$R \cong Z_3$  may be generated by an element of type  $3A_2$  or  $4A_2$ .

To construct  $R \cong Z_3 \times Z_3$ , note that  $\Phi$  contains a subsystem of type  $A_3$  spanned by  $\{\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ , where  $\alpha_0 = \begin{smallmatrix} 1354321 \\ 3 \end{smallmatrix}$ . Letting  $\alpha_0 = e_1 - e_2$ ,  $\alpha_1 = e_2 - e_3, \dots, \alpha_8 = e_8 - e_9$ ,  $R = \langle (123)(456)(789), (147)(258)(369) \rangle$  occurs for the character



with  $\psi, \chi$  and  $\varphi$  of order 3 and  $|\langle \psi, \chi, \varphi \rangle| = 27$ .

$Z_5$  will occur as an  $R$ -group, generated by an element of type  $2A_4$ , if and only if  $[k^*: (k^*)^3] \geq 25$ .

8. *Type  $F_4$ .*  $\Phi^v = \{\pm 2e_k, \pm e_i \pm e_j, \pm e_1 \pm e_2 \pm e_3 \pm e_4 \mid 1 \leq k \leq 4, 1 \leq i < j \leq 4\}$  is of type  $F_4$ . A base for  $\Phi^v$  is given by  $\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = 2e_4$ , and  $\alpha_4 = e_1 - e_2 - e_3 - e_4$ .

$\Phi' = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}$  forms a sub-root system of type  $D_4$  with Weyl group  $W(\Phi') \cong S_4 \times Z_2^3$  acting as permutations and even sign changes on the  $e_i$ . The Weyl group for  $\Phi$  and  $\Phi^v$  of type  $F_4$  is  $S_3 \times W(\Phi') \cong S_3 \times (S_4 \times Z_2^3)$ , where  $S_3$  acts as permutations of  $e_1 - e_2, e_3 - e_4$ , and  $e_3 + e_4$ .

If  $w_\beta \alpha = \alpha - n(\alpha, \beta) \beta$ , the Cartan matrix  $[n(\alpha, \beta)]$  of  $\Phi^v$  is

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

The reader is referred to Bourbaki [2] for more details.

**THEOREM F<sub>4</sub>.**  $R \cong 1, \mathbf{Z}_2$  or  $\mathbf{Z}_3$ .  $\mathbf{Z}_3$  can occur as an  $R$ -group if and only if  $p = 3$  or 3 divides  $q - 1$ .

**LEMMA 1.** Suppose  $w = sd \in R$  with  $s \in S_3$  and  $d \in S_4 \times \mathbf{Z}_2^3$ . Then  $s$  has order 1 or 3.

*Proof.*  $s \in S_3$  has order 1, 2, or 3, so that  $w = sd, w^2$ , or  $w^3$  is in the normal subgroup  $S_4 \times \mathbf{Z}_2^3$ . Further, this element must be able to give reducibility for  $D_4$ , so that  $w, w^2$ , or  $w^3$  is conjugate to one of 1,  $c_3c_4, c_1c_2c_3c_4, (12)c_3c_4, (12)(34),$  or  $(12)(34)c_2c_4$ .

But of these, only 1,  $c_1c_2c_3c_4$ , and  $(12)c_3c_4$  can be in an  $R$ -group for  $\Phi$  of type  $F_4$ . Thus  $w, w^2$ , or  $w^3$  is conjugate to one of 1,  $c_1c_2c_3c_4$ , or  $(12)c_3c_4$ .

Suppose that  $s$  has order 2, so that  $w^2 = 1, c_1c_2c_3c_4,$  or  $(12)c_3c_4$ . We may assume that  $s = w_{\alpha_3} = c_4 = (e_3 - e_4, e_3 + e_4)$ . Then if  $d = \sigma c$  with  $\sigma \in S_4$  and  $c \in \mathbf{Z}_2^3, w^2 = c_4(\sigma c c_4 \sigma^{-1})(\sigma^2 c \sigma^{-2})\sigma^2$ . Since  $(12) \neq \sigma^2$  for any  $\sigma$ , we must have  $\sigma^2 = 1$  and thus  $c c_4(\sigma c c_4 \sigma^{-1}) = w^2 = 1$  or  $c_1c_2c_3c_4$ .

By conjugation we may assume that  $\sigma = 1, (12), (34),$  or  $(12)(34)$ . But then  $w^2 \neq c_1c_2c_3c_4$  for any  $c \in \mathbf{Z}_2^3$ , so we have  $w^2 = 1$ . But  $\sigma = (12)(34)$  will not give  $w^2 = 1$  for any  $c$ .

Thus  $\sigma = 1$  and  $w = c_4c, c \in \mathbf{Z}_2^3,$  or  $\sigma = (12)$  and  $c = 1, c_1c_2, c_3c_4,$  or  $c_1c_2c_3c_4,$  or  $\sigma = (34)$  and  $c$  is conjugate to  $c_2c_3$  or  $c_2c_4$ . Then  $w = \sigma c_4c$  is conjugate to one of  $c_4, c_2c_3c_4,$  or  $(12)c_4$ . But none of these can be in an  $R$ -group for  $\Phi$  of type  $F_4$ . Thus  $s$  can not have order 2.

If  $s = 1$ , then  $w = sd \in W(\Phi')$ .

**LEMMA 2.** Suppose  $R \leq W(\Phi')$ . Then  $R \cong \mathbf{Z}_2$ .

*Proof.* Any element of  $R \cap W(\Phi')$  is conjugate to one of 1,  $c_1c_2c_3c_4,$  or  $(12)c_3c_4$ .  $c_1c_2c_3c_4$  can not be in an  $R$ -group with any conjugate of  $(12)c_3c_4$ , so if  $c_1c_2c_3c_4 \in R$ , then  $R = \langle c_1c_2c_3c_4 \rangle \cong \mathbf{Z}_2$ .

Suppose  $(12)c_3c_4 \in R$ . Then  $\lambda$  is given by



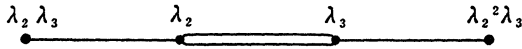
where  $\text{sgn}_\theta \neq \text{sgn}_{\theta\prime}$  are of order 2. If there is another nontrivial element of  $R$ , we may assume by conjugation that it is  $c_1c_2(34)$ . We then would need  $\alpha_1$  to correspond to a character  $\Psi$  with  $\Psi^2 = \text{sgn}_\theta \text{sgn}_{\theta\prime}$ . But then  $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \in \mathcal{L} \cap \Phi^\vee$  and  $c_1c_2(34)$  sends this to a negative root, so it is not in an  $R$ -group with  $(12)c_3c_4$ . Thus  $R = \langle (12)c_3c_4 \rangle \cong \mathbf{Z}_2$ .

Now suppose there exists an element  $w = sd \in R$  with  $s \neq 1$ ,



$s \in S_3$  and  $d \in W(\Phi')$ . Then  $s$  has order 3 by Lemma 1, and  $w^3 \in W(\Phi')$  must be conjugate to 1,  $c_1c_2c_3c_4$ , or  $(12)c_3c_4$ . Thus  $w$  has order 3 or 6.

Consider the elements  $w_{\alpha_3}w_{\alpha_4}$ ,  $w_{\alpha_1}w_{\alpha_2}$ ,  $(w_{\alpha_2}w_{\alpha_3}w_{\alpha_4})^2 = w_{2\alpha_2+\alpha_3}w_{\alpha_3+\alpha_4}$ , and  $(w_{\alpha_1}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4})^2$  of order 3. The first 3 elements can not give reducibility. The last gives reducibility if  $\lambda$  is given by



where  $\lambda_2 \neq \lambda_3^{\pm 1}$  are characters of order 3.

The above 4 elements are pairwise nonconjugate. Further, none is conjugate to the inverse of another. Since the order of the Weyl group  $W$  is  $3^3 \cdot 2^7$ , we see that any 3-Sylow subgroup of  $W$  consists of 1 and conjugates of the above four elements and their inverses. Thus there is a unique subgroup of order 3 in any 3-Sylow subgroup which can be part of an  $R$ -group.

Thus any element of order 3 in  $R$  is conjugate to  $(w_{\alpha_1}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4})^4$  or its inverse. In this case all  $\alpha$  correspond to characters of order 3, and thus  $R$  can not contain an element of order 2, which would have to be conjugate to  $c_1c_2c_3c_4$  or  $(12)c_3c_4$ . Thus an element of order 6 can not occur, and we have shown that  $R \cong \{1\}$ ,  $Z_2$ , or  $Z_3$ .

Explicitly, if  $R \neq \{1\}$ , then  $R$  is conjugate to one of  $\langle c_1c_2c_3c_4 \rangle \cong Z_2$  with all  $\lambda_\alpha$  of order 2, or  $\langle (12)c_3c_4 \rangle \cong Z_3$  with  $\lambda$  given by



with  $\text{sgn}_\theta \neq \text{sgn}_{\theta'}$ , or  $\langle (w_{\alpha_1}w_{\alpha_2}w_{\alpha_3}w_{\alpha_4})^4 \rangle \cong Z_3$  with  $\lambda$  given by



where  $\lambda_2 \neq \lambda_3^{\pm 1}$  are of order 3.

We note that if  $\mathfrak{k} = R$ , then  $R = \{1\}$ , and thus  $\text{Ind}_B^G \lambda$  is irreducible if  $G$  is a Chevalley group of type  $F_4$  over the reals.

9. *Type  $G_2$ .* Let  $\{\alpha, \beta\}$  be a base for  $\Phi^v$  with Cartan matrix  $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ . The positive roots in  $\Phi^v$  are  $\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta$ , and  $3\alpha + 2\beta$ . The Weyl group  $W$  is a dihedral group of order 12.

**THEOREM  $G_2$ .**  $R = \{1\}$  or  $R = \langle w_0 \rangle \cong Z_2$ , where  $w_0$  is the Weyl group element of maximal length.

One checks that the element  $w_0$  of maximal length is the only Weyl group element giving reducibility.  $R = \langle w_0 \rangle$  if and only if  $\alpha$  and  $\beta$  correspond to distinct characters of order 2.

If  $\mathfrak{k}$  is nonArchimedean,  $[\mathfrak{k}^*: (\mathfrak{k}^*)^2] \geq 4$ , and such characters exist. If  $\mathfrak{k} = R$ , then  $R = \{1\}$  and  $\text{Ind}_B^G \lambda$  will be irreducible.

CHAPTER III  
ON THE DECOMPOSITION OF  $\text{Ind}_B^G \lambda$

1. **Multiplicities of the irreducible components.** If  $R$  is abelian, then there are  $|R|$  irreducible components, each occurring with multiplicity 1.

Write  $C[R] = M_{m_1}(C) \oplus \cdots \oplus M_{m_k}(C)$ . Then  $m_1, m_2, \dots, m_k$  are the multiplicities of the  $k$  inequivalent irreducible components of  $\text{Ind}_B^G \lambda$ .  $k$  is equal to the dimension of the center of  $C[R]$ , which equals the number of conjugacy classes in  $R$ . Further, the  $m_i$  are the degrees of the irreducible representations of the group  $R$ . We note that if  $R$  has a normal abelian subgroup  $R'$ , then the degrees  $m_i$  divide the index of  $R'$  in  $R$ , by Ito's Theorem.

Suppose  $R$  is non-abelian. Then  $G$  is of type  $D_n$ ,  $E_7$  or  $E_8$ . Suppose  $G$  is type  $D_n$ , with  $n$  odd. Then  $R \cong \mathbf{Z}_4 \times R'$  contains a normal abelian subgroup  $R'$  of index 2, so  $m_i = 1$  or 2. If  $p$  is odd, then  $R \cong \mathbf{Z}_4 \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$  has order 16 and there are 10 conjugacy classes in  $R$ . Thus we have the decomposition  $16 = 2 \cdot 2^2 + 8 \cdot 1^2$ , and  $\text{Ind}_B^G \lambda$  decomposes into 2 irreducible components of multiplicity 2, and 8 irreducible components of multiplicity 1.

If  $p = 2$ , there may be more factors of  $\mathbf{Z}_2$  in  $R$ . We note that  $R \cong \mathbf{Z}_4 \times (\mathbf{Z}_2^4)$  has 28 conjugacy classes, giving the decomposition  $64 = 12 \cdot 2^2 + 16 \cdot 1^2$ , and  $R \cong \mathbf{Z}_4 \times (\mathbf{Z}_2^6)$  has 88 conjugacy classes, giving the decomposition  $256 = 56 \cdot 2^2 + 32 \cdot 1^2$ .

Suppose  $G$  is type  $D_n$  with  $n$  even. Then any non-abelian  $R$  is isomorphic to  $(\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2) \times R'$ . If  $p$  is odd,  $R'$  is the group of even sign changes on  $\{e_n, e_{n-1}, e_{n-2}, e_{n-3}\}$  and the first factor is  $\langle (12)(34) \cdots (n-1 n) \rangle$  or  $\langle (12)(34) \cdots (n-1 n), (13)(24) \cdots (n-2 n) \rangle$ .

In the first case,  $m_i = 1$  or 2,  $|R| = 16$  and there are 10 conjugacy classes in  $R$ .  $16 = 2 \cdot 2^2 + 8 \cdot 1^2$  gives the decomposition into 2 irreducible components of multiplicity 2, and 8 of multiplicity 1.

In the second case,  $|R| = 32$  and there are 17 conjugacy classes. The two possible decompositions are  $32 = 4^2 + 16 \cdot 1^2 = 5 \cdot 2^2 + 12 \cdot 1^2$ . But since  $R/\langle c_n c_{n-1} c_{n-2} c_{n-3} \rangle$  is abelian, there are at least 16 one-dimensional representations of  $R$ , so the decomposition must be  $32 = 4^2 + 16 \cdot 1^2$ . Thus  $\text{Ind}_B^G \lambda$  decomposes into 16 irreducible components of multiplicity 1, and 1 component of multiplicity 4.

If  $G$  is type  $E_7$ , the nonabelian  $R$ -groups are the dihedral group  $D$  of order 8 and  $D \times \mathbf{Z}_2$ .  $R \cong D$  gives the decomposition  $8 = 1 \cdot 2^2 + 4 \cdot 1^2$  and  $R \cong D \times \mathbf{Z}_2$  gives the decomposition  $16 = 2 \cdot 2^2 + 8 \cdot 1^2$ .

If  $G$  is type  $E_8$ , the nonabelian  $R$ -group has order 128, 65 conjugacy classes, and  $R/\langle w_0 \rangle$  is abelian of order 64. This gives the decomposition  $128 = 1 \cdot 8^2 + 64 \cdot 1^2$ , so  $\text{Ind}_B^G \lambda$  decomposes into 1 irre-

ducible components with multiplicity 8, and 64 irreducible components each with multiplicity 1.

2. **Some analysis on  $L^2(V)$ .** In this section we realize the operators  $\alpha(w, \lambda)$  on  $L^2(V)$  via a Fourier transform, where  $V$  is the unipotent radical of the Borel subgroup opposed to  $B$ . We find a class of functions in  $L^2(V)$  on which  $\hat{\alpha}(w, \lambda)$  acts as multiplication by a bounded function  $M(w, \lambda)$ . This class has nonzero intersection with each invariant subspace for groups of type  $A_n$  and  $B_n$ .

Write  $\varphi_\delta(y)$  for  $\varphi_\delta \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$  in  $U_\delta$  and let  $n_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  for  $\alpha$  simple, where  $\varphi_\delta: \text{SL}(2) \rightarrow G$  is the canonical homomorphism corresponding to the root  $\delta$ .

Write  $V = \prod_{\delta < 0} U_\delta$  in some fixed order. Since each  $U_\delta$  is isomorphic to  $\mathfrak{k}$ , this gives a topological isomorphism of  $V$  with the product of  $|\Phi^-|$  copies of  $\mathfrak{k}$ . We then define a Fourier transform on  $L^2(V)$  by  $\hat{f}(\prod_{\delta < 0} \varphi_\delta(c_\delta)) = \int f(\prod_{\delta < 0} \varphi_\delta(y_\delta)) \bar{\chi}(\sum_{\delta < 0} c_\delta y_\delta) \prod dy_\delta$ , where  $\chi$  is a fixed additive character of  $\mathfrak{k}$  with conductor the ring of integers.

Fix a simple root  $\alpha > 0$ . Then

$$\begin{aligned} A(w_\alpha, \lambda)f(g) &= A(n_\alpha, \lambda)f(g) \\ &= \int_{U_\alpha} f\left(g\varphi_\alpha\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right) du \\ &= \int f\left(g\varphi_\alpha\begin{pmatrix} 1 & 0 \\ 1/u & 1 \end{pmatrix}\varphi_\alpha\begin{pmatrix} -u & 1 \\ 0 & -1/u \end{pmatrix}\right) du \\ &= \int f\left(g\varphi_\alpha\begin{pmatrix} 1 & 0 \\ 1/u & 1 \end{pmatrix}\right) \lambda_\alpha^{-1}(-u) \frac{du}{|u|} \\ &= \int f\left(g\varphi_\alpha\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) \lambda_\alpha(-u) \frac{du}{|u|} . \end{aligned}$$

Let  $g \in V$ ,  $g = \prod_{\delta < 0} \varphi_\delta(y_\delta)$ . Then

$$g\varphi_\alpha\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \prod \varphi_\delta(y_\delta) \cdot \varphi_{-\alpha}(u) = \prod_{\delta < 0} \varphi_\delta(y_\delta + P_\delta(y_\beta, u)) ,$$

where the  $P_\delta = P_{\delta, \alpha}$  are polynomials arising from Chevalley's commutation relations. Make the change of variables  $y_{-\alpha} \mapsto y_{-\alpha} - u$  to define polynomials  $Q_\delta(y_\beta, u)$ . Then  $Q_\delta \equiv 0$  if  $\delta$  is simple.

Consider the operator  $A(w_\alpha, \lambda)$  under the Fourier transform. Let  $\hat{g} = \prod \varphi_\delta(c_\delta)$ . Then

$$\hat{A}(w_\alpha, \lambda)\hat{f}(\hat{g}) = \iint f(\prod \varphi_\delta(y_\delta + P(y_\beta, u))) \lambda_\alpha(-u) \frac{du}{|u|} \bar{\chi}(\sum c_\delta y_\delta) \prod dy_\delta$$

$$= \iint f(\prod \varphi_\delta(y_\delta + Q_\delta(y_\beta, u))) \lambda_\alpha(-u) \bar{\chi}(-c_{-\alpha}u) \frac{du}{|u|} \\ \times \bar{\chi}(\sum c_\delta y_\delta) \prod dy_\delta .$$

We define a function  $f \in C_c^\infty(V)$  as follows. For  $\delta < 0$ ,  $\delta$  simple, let  $\hat{f}_\delta$  be any function in  $C_c^\infty(\mathfrak{k}^*)$ , i.e., such that the support of  $\hat{f}_\delta$  avoids zero. Let  $S_\delta$  be the support of  $\hat{f}_\delta$ . If  $\delta < 0$  is nonsimple with  $Q_\delta \equiv 0$ , take any  $f_\delta \in C_c^\infty(U_\delta)$ , and let  $S_\delta$  be its support.

Define the other  $S_\delta$  inductively from right to left in the product  $\prod_{\delta < 0} U_\delta$ . If  $S_\beta$  is defined for all  $\beta$  to the right of  $\delta$  in the product, let  $S_\delta$  be the fractional ideal generated by  $\{Q_{\delta,\alpha}(y_\beta, uc_{-\alpha}^{-1}) | y_\beta \in S_\beta, u \in \mathcal{O} \text{ if } \lambda_\alpha \text{ is unramified and } |u| = q^h \text{ if } \lambda_\alpha \text{ is ramified of degree } h, \text{ and } c_{-\alpha} \in \text{supp } \hat{f}_{-\alpha} \text{ for } \alpha \text{ simple}\}$ . Define  $f_\delta$  to be the characteristic function of  $S_\delta$ .

For root systems of type  $A_n, B_n, C_n, D_n$  and  $G_2$ , we may arrange the negative roots such that  $Q_{\delta,\alpha}(y_\beta, u) \neq 0$  implies  $Q_{\beta,\alpha} \equiv 0, Q_{\delta,\alpha}(y_\beta, u) \neq 0$  implies  $\beta \neq -\alpha$ , and  $Q_{\delta,\alpha}(y_\beta, u) = Q_{\delta,\alpha}(y_\beta)u$ . Then if  $f = \prod f_\delta$ ,

$$\hat{A}(w_\alpha, \lambda) \hat{f}(\hat{g}) = \iiint \prod_{\substack{\beta \\ Q_\beta \equiv 0}} f_\beta(y_\beta) \bar{\chi}(\sum c_\beta y_\beta) \prod_{\substack{\delta \\ Q_\delta \neq 0}} f_\delta(y_\delta + Q_\delta(y_\beta)u) \bar{\chi}(\sum c_\delta y_\delta) \\ \times \lambda_\alpha(-u) \bar{\chi}(-c_{-\alpha}u) \frac{du}{|u|} \prod dy_\delta \prod dy_\beta = 0$$

unless  $y_\beta \in S_\beta$  for all  $\beta$  with  $Q_\beta \equiv 0$ . Fix  $y_\beta \in S_\beta$  and consider

$$\iint \prod_{\substack{\delta \\ Q_\delta \neq 0}} f_\delta(y_\delta - Q_\delta(y_\beta)u) \bar{\chi}(\sum c_\delta y_\delta) \lambda_\alpha(u) \bar{\chi}(c_{-\alpha}u) \frac{du}{|u|} \prod dy_\delta .$$

This will be zero unless  $y_\delta - Q_\delta(y_\beta)u \in S_\delta$  for all  $\delta$ . Thus we need only integrate  $u$  over the intersection  $\bigcap_\delta (1/Q_\delta(y_\beta))(y_\delta - S_\delta) = (1/Q_{\delta_0}(y_\beta))(y_{\delta_0} - S_{\delta_0})$ , for some  $\delta_0$ , and integrate  $y_{\delta_0}$  over the coset

$$\frac{Q_\delta(y_\beta)}{Q_{\delta_0}(y_\beta)} y_{\delta_0} + S_\delta .$$

Write

$$\int_{u_{\delta_0}} = \int_{S_{\delta_0}} + \sum \int_{\mathcal{S}}$$

where the sum is over shells  $\mathcal{S}$  consisting of nonzero cosets of  $S_{\delta_0}$ . The above integral becomes

$$\int_{\mathfrak{k}^*} \int_{S_{\delta_0}} \int_{\prod U_\delta, \delta \neq \delta_0} \prod f_\delta(y_\delta - Q_\delta(y_\beta)u) \bar{\chi}(\sum c_\delta y_\delta) \lambda_\alpha(u) \bar{\chi}(c_{-\alpha}u) \frac{du}{|u|} \prod dy_\delta \\ + \sum \iiint_{\mathcal{S}} \prod f_\delta(y_\delta - Q_\delta(y_\beta)u) \bar{\chi}(\sum c_\delta y_\delta) \lambda_\alpha(u) \bar{\chi}(c_{-\alpha}u) \frac{du}{|u|} \prod dy_\delta .$$

In each term in the sum, we are integrating  $\lambda_\alpha(u) \bar{\chi}(c_{-\alpha}u)$  over

a shell  $(1/Q_{\delta_0}(y_\beta))\mathcal{S}$  which is disjoint from  $S_{\delta_0}/Q_{\delta_0}(y_\beta)$ . Thus we are integrating  $\lambda_\alpha(u)\bar{\chi}(u)$  over a shell disjoint from  $(1/Q_{\delta_0}(y_\beta))c_{-\alpha}S_{\delta_0}$ , which gives zero, by the definition of  $S_{\delta_0}$  and properties of the gamma function [29, 35].

We are left with only the first term. Note that

$$\int_{(1/Q_{\delta_0}(y_\beta))S_{\delta_0}} \lambda_\alpha(u)\bar{\chi}(c_{-\alpha}u)\frac{du}{|u|} = \lambda_\alpha^{-1}(c_{-\alpha})\cdot\Gamma(\lambda_\alpha).$$

We get that

$$\begin{aligned} \hat{A}(w_\alpha\lambda)\hat{f}(\hat{g}) &= \int_{\prod U_\delta; \delta \neq \delta_0} \int_{S_{\delta_0}} \int_{\mathfrak{t}^*} \prod f_\beta(y_\beta)\bar{\chi}(c_\beta y_\beta) \prod f_\delta(y_\delta - Q_\delta(y_\beta)u) \\ &\quad \times \bar{\chi}(\sum c_\delta y_\delta)\lambda_\alpha(u)\bar{\chi}(c_{-\alpha}u)\frac{du}{|u|} \prod dy_\delta \prod dy_\beta. \end{aligned}$$

This is zero unless  $y_\beta \in S_\beta$ ,  $c_{-\alpha} \in \text{supp } \hat{f}_{-\alpha}$  and  $u \in 1/c_{-\alpha} \times (\mathfrak{p}^{-h} \setminus \mathfrak{p}^{-h+1})$  for  $\lambda_\alpha$  ramified of degree  $h$ , or  $u \in 1/c_{-\alpha} \times \mathcal{O}$  for  $\lambda_\alpha$  unramified. But then  $Q_\delta(y_\beta)u \in S_\delta$  and  $f(y_\delta - Q_\delta(y_\beta)u) = f(y_\delta)$ . Thus for such  $\hat{f}$ ,

$$\hat{A}(w_\alpha, \lambda)\hat{f}(\hat{g}) = \lambda_\alpha^{-1}(c_{-\alpha})\Gamma(\lambda_\alpha) \prod_{\delta < 0} \hat{f}_\delta(c_\delta) = \lambda_\alpha^{-1}(c_{-\alpha})\Gamma(\lambda_\alpha)\hat{f}(\hat{g}).$$

Thus  $\hat{a}(w_\alpha, \lambda) = (1/\Gamma(\lambda_\alpha))A(w_\alpha, \lambda)$  acts on such  $\hat{f}$  as multiplication by  $M(w_\alpha, \lambda) = \lambda_\alpha^{-1}(c_{-\alpha})$ . Then if  $w = w_{\alpha_1}w_{\alpha_2} \cdots w_{\alpha_l}$ ,  $\hat{a}(w, \lambda)$  acts on such  $\hat{f}$  as multiplication by the function  $M(w, \lambda) = M(w_{\alpha_1}, w_{\alpha_2} \cdots w_{\alpha_l}) \cdots M(w_{\alpha_l}, \lambda)$ , by the cocycle condition.

We note that  $w \mapsto M(w, \lambda)$  is a homomorphism, and further, that we may evaluate  $M(w, \lambda)$  at  $V_{-\alpha}$  for any simple root  $\alpha$  to obtain a homomorphism from  $W_\lambda$  into  $(\mathfrak{k}^*)^\wedge$ . If this homomorphism is injective on  $R$  for some  $\alpha$ , then the linear independence of distinct characters of  $\mathfrak{k}^*$  implies that the operators  $\{\alpha(w, \lambda) | w \in R\}$  are linearly independent. Further, we may write  $|R|$  nonzero projections giving  $\hat{f}$  as above in each invariant subspace.

The homomorphism is injective on  $R$  for groups of type  $A_n$  and  $B_n$ , but is not necessarily injective for groups of type  $C_n$  and  $D_n$ . We may show the linear independence of the operators  $\{\alpha(w, \lambda) | w \in R\}$  for these groups as follows.

As in [37], let  $f_I = f_{I,\lambda}$  be the function in  $H_\lambda$  whose restriction to  $K$  is supported on the Iwahori  $I$  and is constant on  $I \cap V$ . Then  $\alpha(w, \lambda)f_I(w') = 0$  if and only if  $ww' \neq 1$ , provided that  $l(w') \geq l(w)$ , that  $\Gamma_w(\lambda)$  and  $\Gamma_{w'}(\lambda)$  are defined, and the characters  $\lambda_\beta$  are ramified for all  $\beta \in R(w)$ . (The proof is by induction on the length of  $w$ . Write  $w = w_\alpha \bar{w}$  with  $l(\bar{w}) = l(w) - 1$  and use the fact that  $\lambda_{\bar{w}^{-1}\alpha}$  is ramified.)

To show that  $\{\alpha(w, \lambda) | w \in R\}$  are linearly independent, it is enough to find a  $w_0 \in R$  such that  $\alpha(w, \lambda)f_I(w_0) = 0$  if and only if

$ww_0 \neq 1$ . If all  $\lambda_\alpha$  are ramified, use the above. Otherwise, since we know what groups  $R$  can occur, we may check that  $w_0 \in R$  consisting of as many sign changes as possible will work for groups of type  $C_n$  and  $D_n$ .

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