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**AN AFFIRMATIVE ANSWER TO GLAUBERMAN'S  
CONJECTURE**

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Let  $G$  be a finite group and  $P$  be a Sylow  $p$ -subgroup of  $G$  for a prime  $p$ . The following question is raised by G. Glauberman.

**Question 16.8.** Does there exist a function  $f$  from the positive integers  $i$  to the positive integers such that

$$H^i(G, F_p) \cong H^i(N_G(K_\infty(P)), F_p) \quad \text{whenever } p \geq f(i)?$$

Here  $K_\infty$  denotes the section conjugacy functor constructed by G. Glauberman and  $F_p$  denotes the finite field consisting of  $p$  elements and by forgetting its multiplicative structure, we consider it as a trivial  $G$ -module.

In relation to the above conjecture, he proved the case  $i = 1$  and D. F. Holt has recently proved  $f(2) \leq 11$ . The purpose of this paper is to provide an affirmative answer to the question.

**THEOREM C.** If  $p \geq 12 \times 6^{m-2} + 3$ , the  $H^m(G, F_p) \cong H^m(N_G(K_\infty(P)), F_p)$  for all integers  $m \geq 2$ .

This theorem is a consequence of the following more detailed theorem since  $K_\infty$  has degree 4.

**THEOREM B.** Let  $W$  be a section conjugacy functor of  $G$  of degree  $t$ . If  $p \geq 4 \times 6^{m-2} \times (t-1) + 3$ , then the restriction map induces  $H^m(G, V) \cong H^m(N_G(W(P)), V)$  for a trivial  $p$ -primary  $G$ -module  $V$  and all integers  $m \geq 2$ .

**COROLLARY 1.** Let  $V$  be a faithful  $p$ -primary  $G$ -module,  $n$  an integer greater than 0, and  $r$  be the first integer such that  $p \leq 4 \times 6^{n-1} \times (r+1) + 2$ . Set  $A = \langle a \in P : [V, a; r+1] = 1, a^p = 1 \rangle$ . Then we have  $H^m(G, V) \cong H^m(N_G(A), V)$  for all integers  $m$  with  $1 \leq m \leq n$ .

**2. Notations and preparations.** All groups considered in this paper will be finite and we treat only finite modules by the same argument in [6]. In particular, we always assume that  $G$  is a finite group and  $P$  is a Sylow  $p$ -subgroup of  $G$ . Most of our notations are standard and taken from [1] and [2], and we adopt notations from [7] about cohomology functor and  $G$ -functors. In addition, we will use the following:

\* A module  $V$  is said to be an  $F_p[G]$ -module if  $V$  is a  $G$ -module and an elementary Abelian  $p$ -group as group.

\* For a subset  $\mathcal{A}$  and a subgroup  $H$  of  $G$ , set  $\mathcal{A}(H) = \mathcal{A} \cap H$ .

\* If  $T$  is a finite group acting on a solvable finite group  $S$ , then we denote by  $ir_T(S)$  the direct product of all the composition factors of  $S$  under  $T$ .

We sometimes use the following lemma.

**LEMMA 2.1.** *Let  $H$  be a finite  $p$ -group acting on an  $F_p[G]$ -module  $A$ . If an element  $a$  acts on the semiproduct  $HA$  and satisfies the following:  $[H, a; 2] = 1$  and  $[A, a; t] = 1$ , then we have  $[H^n(H, A), a; (n+1)t] = 1$  for all integers  $n \geq 0$ .*

*Proof.* Since  $[H, a]$  is a normal subgroup of  $H\langle a \rangle$ , the assertion follows from the Lyndon-Hochschild-Serre spectral sequence.

**3. Cohomological  $G$ -functors.** In many parts of this paper, we will use cohomological  $G$ -functors, which are generalizations of cohomology group. The concept of  $G$ -functors were introduced by Green [3] during the study of modular representation theory and slightly changed by Yoshida [7]. We will adopt the definition from [7].

**DEFINITION 3.1.** A  $G$ -functor (into an Abelian category  $\mathcal{C}$ ) is a quadruple  $A = (a, \tau, \rho, \sigma)$ , where  $a, \tau, \rho, \sigma$  are families of the following kind;

$a = (a(H))$  assigns an object  $a(H)$  of  $\mathcal{C}$  for each subgroup  $H$  of  $G$ ;

$\tau = (\tau_H^K)$  assigns a morphism  $\tau_H^K = \tau^K: a(H) \rightarrow a(K)$  for each pair  $(H, K)$  such that  $H \leq K \leq G$ , we simply write  $(\alpha)\tau^K = \alpha^K$ ;

$\rho = (\rho_H^K)$  assigns a morphism  $\rho_H^K = \rho_H: a(K) \rightarrow a(H)$  for each pair  $(H, K)$  such that  $H \leq K \leq G$ , we simply write  $(\alpha)\rho_H = \alpha_H$ ;

$\sigma = (\sigma_H^g)$  assigns a morphism  $\sigma_H^g = \sigma^g: a(H) \rightarrow a(H^g)$  for each subgroup  $H$  of  $G$  and each element  $g$  of  $G$ , we write  $(\alpha)\sigma^g = \alpha^g$ . These families of objects and morphisms must satisfy the following:

Axioms for  $G$ -functors. (In these axioms,  $D, H, K, L$  are any subgroups of  $G$ ;  $g, g'$  are elements of  $G$ .)

$$(G.1) \quad \tau_H^H = 1_{a(H)}, \tau_H^K \cdot \tau_K^L = \tau_H^L \text{ if } H \leq K \leq L;$$

$$(G.2) \quad \rho_H^H = 1_{a(H)}, \rho_H^K \cdot \rho_K^D = \rho_D^K \text{ if } K \geq H \geq D;$$

$$(G.3) \quad \sigma_H^g \cdot \sigma^{g'} = \sigma_H^{gg'}, \sigma_H^h = 1_{a(H)} \text{ if } h \in H;$$

$$(G.4) \quad \tau_H^K \cdot \sigma^g = \sigma_H^g \cdot \tau_K^g, \rho_H^K \cdot \sigma^g = \sigma_K^g \cdot \rho_{H^g};$$

$$(G.5) \quad (\text{ackey axiom}) \text{ If } H \text{ and } K \text{ are subgroup of } L, \text{ then } \tau_H^L \cdot \rho_K^L = \sum_g \{ \sigma_H^g \cdot \rho_{H^g \cap K} \cdot \tau^K: g \in H \backslash L / K \text{ a double coset representation} \}.$$

**DEFINITION 3.2.** A  $G$ -functor  $A = (a, \tau, \rho, \sigma)$  is called cohomological

logical if it satisfies the following Axiom C:

(C) Whenever  $H \leq K \leq G$ ,  $\rho_H^K \cdot \tau_H^K = |K:H| 1_{\alpha(K)}$ .

For example, let  $V$  be a  $G$ -module and  $n$  be an integer ( $\geq 0$ ). For subgroups  $H, K$  of  $G$  with  $H \leq K$ , set  $\alpha(H) = H^n(H, V)$ , then the restriction map  $\rho_H^K: H^n(K, V) \rightarrow H^n(H, V)$ , the transfer map  $\tau_H^K: H^n(H, V) \rightarrow H^n(K, V)$ , and the conjugation map  $\sigma^g: H^n(H, V) \rightarrow H^n(H^g, V)$  makes a cohomological  $G$ -functor, we call this a cohomology functor.

Although we explained the definition of  $G$ -functors, we will really use only  $G$ -functors which are induced from cohomology functors. Therefore, readers may regard all  $G$ -functor in this paper as cohomology functors. Next, we will show a construction of  $G$ -functor induced from the given  $G$ -functor. All  $G$ -functors considered in this paper will be cohomological.

3.3. A quotient  $G$ -functor. Let  $A = (a, \tau, \rho, \sigma)$  be a  $G$ -functor over a field  $k$ . A  $G$ -functor  $B = (a', \tau', \rho', \sigma')$  is called a sub- $G$ -functor of  $A$  if  $B$  satisfies the following properties:

- (i)  $a'(H) \subseteq a(H)$  for all  $H \leq G$ ; and
- (ii)  $\tau' = \tau|_{a'(H)}$ ,  $\rho' = \rho|_{a'(H)}$ , and  $\sigma' = \sigma|_{a'(H)}$ .

We write  $A \geq B$ . Then we can make a new  $G$ -functor called the quotient (or section)  $G$ -functor  $A/B = (a_0, \tau_0, \rho, \sigma_0)$  as follows: Set  $a_0(H) = a(H)/a'(H)$  for  $H \leq G$ . Since the above inclusion are commutative with  $\tau$ ,  $\rho$ , and  $\sigma$ , we can define morphisms  $\tau_0$ ,  $\rho_0$ ,  $\sigma_0$ , naturally.

In connection with the above notions, we define the following:

DEFINITION 3.4. A  $G$ -functor  $A$  is said to be irreducible if  $A$  has no nontrivial proper sub- $G$ -functors.

DEFINITION 3.5. A chain of sub- $G$ -functors  $A = A_0 \geq A_1 \geq \dots \geq A_n = 0$  is called a composition series if each  $A_i/A_{i+1}$  is irreducible. Then the factors  $A_i/A_{i+1}$  are called its composition factors.

Then we have a proposition of Jordan-Hölder type.

PROPOSITION 3.6. Any two composition series of a  $G$ -functor have the same length and, with respect to a suitable reordering of the composition factors, the corresponding factors are isomorphic.

The proof is similar to that of Jordan-Hölder theorem. Thus the composition factors of a  $G$ -functor  $A$  are completely determined up to isomorphism (and ordering) by any one composition series. Especially, we have the following:

LEMMA 3.7. *The  $n$ th cohomology functor  $H^n(, V)$  of  $G$  with coefficient in a finite  $G$ -module  $V$  has a composition series.*

Next lemma about cohomological  $G$ -functors will be useful.

LEMMA 3.8. (*Factorization lemma*). *Assume that  $G$  has two subgroups  $N_1$  and  $N_2$  containing  $P$  such that  $G = N_1 N_2$ . Let  $A = (a, \tau, \rho, \sigma)$  be a cohomological  $G$ -functor. If an element  $\alpha$  in  $a(P)$  is stable in both  $N_1$  and  $N_2$ , then  $\alpha$  is stable in  $G$  itself.*

*Proof.* Suppose false, that is, there is an element  $g$  in  $G$  such that  $(\alpha_F)^g \neq \alpha_{F^g}$  and  $F = P^{g^{-1}} \cap P$  by the definition. Since  $g$  is an element of  $G = N_1 N_2$ , there are elements  $g_1, g_2$  in  $N_1, N_2$ , respectively, such that  $g = g_1 g_2$ . We here assert that we can choose  $g_1$  and  $g_2$  such that  $F^{g_1} \subseteq P$ . To see this, take an arbitrary representation  $g = g_1 g_2$ . Then we get that  $F^{g_1} \subseteq N_1$  and  $F^{g_1} = F^{g_2 g_2^{-1}} \subseteq N_2$  since  $F, F^g \subseteq P \subseteq N_1 \cap N_2$ . Combining these,  $F^{g_1}$  is contained in a Sylow  $p$ -subgroup  $P_0$  of  $N_1 \cap N_2$ , which is conjugate to  $P$  in  $N_1 \cap N_2$ . Therefore, there is an element  $k$  in  $N_1 \cap N_2$  such that  $F^{g_1 k} \subseteq P$ . Then we have a desired representation  $g = (g_1 k)(k^{-1} g_2)$ . Since  $\alpha$  is stable in both  $N_1$  and  $N_2$ , we observe that  $(\alpha_F)^{g_1} = (\alpha_{P^{g_1} \cap P} - 1)_{F^{g_1}}^{g_1} = (\alpha_{P \cap P^{g_1}})_{F^{g_1}}^{g_1} = \alpha_{F^{g_1}}$  and  $(\alpha_F)^g = ((\alpha_F)^{g_1})^{g_2} = (\alpha_{F^{g_1}})^{g_2} = (\alpha_{P^{g_1} \cap P}^{g_2})_{F^g}^{g_2} = \alpha_{F^g}$ . This contradicts the choice of  $g$ .

In association with composition factors, we will use the following:

DEFINITION 3.9. Let  $A = (a, \tau, \rho, \sigma)$  be a  $G$ -functor and  $\{B_i; i \in I\}$  be a set of  $G$ -functors. We shall say that the  $G$ -functor  $A$  is covered by the set  $\{B_i; i \in I\}$  or the set  $\{B_i; i \in I\}$  is a covering of  $A$  if each composition factor of  $A$  is isomorphic to a composition factor of one of  $B_i$ .

For making research on a stability, we define the following map.

DEFINITION 3.10. Let  $A = (a, \tau, \rho, \sigma)$  be a cohomological  $G$ -functor over  $F_p$ . We will define a map  $q_A: \text{Image}(\rho_F^{N_G(P)}: a(N_G(P)) \rightarrow a(P)) \rightarrow a(P)$  by  $q_A(\alpha) = \alpha - \beta_P^g$  where  $\beta \in a(N_G(P))$  with  $(\beta)_P = \alpha$ .

LEMMA 3.11. (*Properties of  $q_A$* ). *We have the following:*

- (a) *If  $\alpha$  is stable in  $G$ , then  $q_A(\alpha) = 0$ .*
- (b) *If  $\alpha^g = 0$ , then  $q_A(\alpha) = \alpha$ .*
- (c)  *$q_A(\alpha) \in \text{Ker}(\tau_P^g) \cap \text{Image}(\rho_F^{N_G(P)})$ .*
- (d)  *$q_A \circ q_A = q_A$ .*

(e) If  $\alpha \in \text{Image}(\rho_P^L)$  for  $L \geq N_G(P)$ , then  $q_A(\alpha) \in \text{Image}(\rho_P^L)$ .

*Proof.* All results are immediate consequences of the definition of the map  $q_A$  and Lemma 4.4 in [7].

Using the above map  $q_A$ , we get a few lemmas about covering.

**LEMMA 3.12.** *Let  $\{B_i; i \in I\}$  be a covering of a cohomological  $G$ -functor  $A = (a, \tau, \rho, \sigma)$ . Suppose that a subgroup  $N$  of  $G$  containing  $N_G(P)$  controls every composition factor of  $B_i$  for each  $i \in I$ . Then  $N$  controls  $A$  itself.*

Here, the statement that  $N$  controls  $A$  means that if an element  $\alpha$  of  $a(P)$  is stable in  $N$  then  $\alpha$  is stable in  $G$ .

*Proof.* Let  $A = A_0 \not\geq A_1 \not\geq \dots \not\geq A_n = 0$  be a composition series of  $A$ . Suppose that  $N$  does not control  $A$ , then there is an element  $\alpha \neq 0$  in  $a(P) \cap \text{Image}(\rho_P^N) \text{Ker}(\tau_P^a)$  by Lemma 4.4 in [7]. By the choice of  $\alpha$  and (b) of Lemma 3.11, we have  $q_A(\alpha) = \alpha$ . Let  $A_i = (a_i, \tau_i, \rho_i, \sigma_i)$  be the quadruples of the  $G$ -functors  $A_i$ . Since  $q_A((a_i(N_G(P)))\rho_P^{N_G(P)})$  is contained in  $a_i(P)$ ,  $q_A$  defines the map  $q_{\bar{A}_i}$  for each section  $G$ -functor  $\bar{A}_i = A_i/A_{i+1}$  which has the same properties as  $q_A$ . Since every composition factor  $A_i/A_{i+1}$  of  $A$  is isomorphic to a composition factor of one of  $B_j$  and so is controlled by  $N$ , we have  $q_{\bar{A}_0}(\alpha + a_1(P)/a_1(P)) = 0$ . We thus get  $\alpha \in a_1(P)$ . By iteration, we finally obtain  $\alpha = 0$ , a contradiction.

**4. Main result.** In this section, we will get the result which will be useful in the next section, where we will prove theorems. Namely, we will investigate properties of the minimal counterexample on the assumption that Theorem B is false. We will divide this section into three parts. In Part 1 and 3, we will assume Hypothesis I and treat Proposition A. In Part 2, we will assume Hypothesis II and prepare results which will be used in the last part.

*Part 1.* At first, we will consider the following: Hypothesis I. Assume that:

- (a)  $G$  a group,  $P$  a Sylow  $p$ -subgroup of  $G$ ,  $n$  an integer  $\geq 1$ ;
  - (b)  $W$  a section conjugacy functor of  $G$ ;
  - (c)  $W$  controls all composition factors of the  $G^*$ -functor  $H^m(, V^*)$  in a trivial  $F_p[G^*]$ -module  $V^*$  in every section  $G^*$  of  $G$  for all integers  $m$  with  $0 \leq m \leq n$ ;
  - (d)  $V$  an  $F_p[G]$ -module;
  - (e)  $r_n$  the first integer such that  $p \leq 4 \times 6^{n-1} \times (r+1) + 2$ ;
- and

(f)  $\mathcal{A} = \{a \in P: a^p = 1, [ir_G(V), a; r_n + 1] = 1, \text{ and } a \text{ satisfies } \text{Con}(r_n) \text{ in } G\}$  and set  $B = \langle \mathcal{A} \rangle$ . Here we explain the notation.

DEFINITION 4.1. We shall say that an element  $a$  of  $G$  satisfies  $\text{Con}(s)$  in  $G$  for some integer  $s (\geq 0)$  if  $a$  satisfies the following property; whenever  $a$  normalizes a  $p$ -subgroup  $T$  of  $G$ ,  $[ir_{N_G(T)}(T), a; 2s + 1] = 1$ .

Our final purpose in this section is to get the following:

PROPOSITION A. *Under Hypothesis I, we have that  $N_G(B)$  controls all composition factors of the  $n$ th cohomology functor  $H^n(\cdot, V)$*

Let us begin by listing up some properties of the hypothesis.

LEMMA 4.2.  *$B$  is weakly closed in  $P$  with respect to  $G$ .*

LEMMA 4.3. *Let  $N$  be a subgroup of  $G$  and  $K$  be a normal subgroup of  $N$ . If an element  $a$  of  $N$  satisfies  $\text{Con}(s)$  in  $G$ , then  $a$  satisfies  $\text{Con}(s)$  in  $N$  and the image  $aK/K$  of  $a$  in  $N/K$  satisfies  $\text{Con}(s)$  in  $N/K$ .*

Now we start the proof of Proposition A. Suppose that Proposition A is false and let  $\mathcal{T} = \{(n, G, V)\}$  be the set of counterexamples. We introduce an order in  $\mathcal{T}$  by setting  $(n, G, V) \gg (n', G', V')$  if one of the following conditions holds:

(i)  $n \not\leq n'$ ; (ii)  $n = n', |G| \not\leq |G'|$ ; (iii)  $n = n', |G| = |G'|, |V| \not\leq |V'|$ . Let  $(n, G, V)$  be a minimal element in  $\mathcal{T}$  with respect to the above order. Then we have the following lemmas.

LEMMA 4.4.  $O_p(G) = 1$ .

*Proof.* Suppose false and set  $H = O_p(G)$ . By Lemma 4.2, there is an element  $a$  in  $\mathcal{A} \cap H$  such that  $[ir_G(H), a; 2r_n + 1] = 1$  by the property  $\text{Con}(r_n)$ . By Proposition 1 in [6], we then obtain

$$[ir_G(H^i(H, V)), a; (2i + 1)r_n + 1] = 1$$

for every integer  $i \geq 0$ . Especially, since  $[ir_G(H^n(H, V)), a; p - 1] = 1$  by the choice of  $r_n$ , it follows that  $X^G = X^{N_G(B)}$  ( $X^G = \{x \in X: gx = x \text{ for all } g \in G\}$ ) for every composition factor  $X$  of  $H^n(H, V)$  under  $G$  by Theorem A1.4 in [1], which implies that  $N_G(B)$  controls all composition factors of the  $G$ -functor  $H^0(\cdot, H^n(H, V))$ . Since the  $G$ -functor  $H^n(\cdot, V)$  is covered by the set of  $G$ -functors  $H^i(\cdot/H, H^{n-i}(H, V))$ :  $i = 0, \dots, n$  by Hochschild-Serre spectral sequence, there is an integer

$i \geq 1$  such that  $N_G(B)$  does not control a composition factor of the  $G$ -functor  $H^i(\quad/H, V^i)$  where  $V^i$  is a composition factor of  $H^{n-i}(H, V)$  under  $G$ . Summarizing the above argument, we have got the following situation:

(a)  $G/H$  acts on an  $F_p[G/H]$ -module  $V^i$ ;

(b)  $r_i$  is the first integer such that  $p \leq 4 \times 6^{i-1} \times (r_i + 1) + 2$ ; and

(c)  $\mathcal{S}^i = \{\bar{a} \in P/H: \bar{a}^p = 1, [i r_{G/H}(V^i), \bar{a}; r_i + 1] = 1, \text{ and } \bar{a} \text{ satisfies } \text{Con}(r_i) \text{ in } G/H\}$  contains the image of  $\mathcal{S}$  in  $G/H$ . By the minimality of  $G, n, N_{G/H}(\langle \mathcal{S}^i \rangle)$  controls all composition factors of the  $G/H$ -functor  $H^i(\quad/H, V^i)$ , which contradicts Lemma 4.2.

LEMMA 4.5.  $V$  is an irreducible  $G$ -module.

*Proof.* Suppose false and let  $\{V_i: i \in I\}$  be the set of composition factors of  $V$  under  $G$ . For each  $i$ , set

$$\mathcal{S}^i = \{a \in P: a^p = 1, [V_i, a; r_n + 1] = 1,$$

and  $a$  satisfies  $\text{Con}(r_n)$  in  $G\}$ . We then get that  $N_G(\langle \mathcal{S}^i \rangle)$  controls all composition factors of the  $G$ -functor  $H^n(\quad, V^i)$  for each  $i$  and  $\mathcal{S}^i \supseteq \mathcal{S}$ , which is a contradiction.

LEMMA 4.6.  $C_G(V) \subseteq O_p(G)$ .

*Proof.* Suppose false and set  $H = C_G(V)$  and  $S$  is a nontrivial Sylow  $p$ -subgroup of  $H$ . The Frattini argument yields that  $G = N_G(S)H$ . It thus follows from Lemma 4.4 that  $N_G(S) \leq G$  and so  $N_{N_G(S)}(B)$  controls all composition factors of the  $N_G(S)$ -functor  $H^n(\quad, V)$ . Therefore,  $N_{PH}(B)$  does not control all composition factors of the  $PH$ -functor  $H^n(\quad, V)$  by Lemma 3.8 (Factorization lemma) and Lemma 3.12. It thus follows from the choice of  $G$  that  $G = PH$ . Since  $H$  centralizes  $V$  and a  $p$ -group  $PH/H$  acts on the irreducible  $F_p[G]$ -module  $V, G = H$ . By the condition (c) in Hypothesis I,  $N_G(W(P))$  controls all composition factors of the  $G$ -functor  $H^n(\quad, V)$ . However, since  $G \neq N_G(W(P))$ ,  $N_{N_G(W(P))}(B)$  controls all composition factors of the  $N_G(W(P))$ -functor  $H^n(\quad, V)$ , a contradiction.

LEMMA 4.7. Let  $H$  be a finite group and  $X$  be a faithful  $H$ -module (or  $C_H(X) \subseteq O_p(H)$ ). If an element  $a$  of  $H$  satisfies  $[X, a; s + 1] = 1$  for an integer  $s \geq 1$ , then  $a$  satisfies  $\text{Con}(s)$  in  $H$ .

*Proof.* We get the conclusion by the same way as Theorem A2.4 and Lemma A2.3 in [1].



LEMMA 4.8.  $\mathcal{A}$  is equal to the set  $\{a \in P: a^p = 1, [V, a; r_n + 1] = 1\}$ .

Because we have supposed that Proposition A is false,  $N_G(B)$  does not control a composition factor  $A = (a, \tau, \rho, \sigma)$  of the  $n$ th cohomology functor  $H^n(\cdot, V)$ . According to the definition of control, we have  $\text{Image}(\rho_P^{N_G(B)}) \not\subseteq \text{Image}(\rho_P^G)$ . By Lemma 4.4 in [7], there exists a nontrivial element  $\alpha$  in  $\text{Image}(\rho_P^{N_G(B)}) \cap \text{Ker}(\tau_P^G)$ . The next lemma follows from the choice of  $\alpha$  and Lemma 3.11.

LEMMA 4.9.  $q_A(\alpha) = \alpha$ .

4.10. Since  $q_A(\alpha) = \alpha \neq 0$ , we get  $((\alpha_{P \cap P^g} - 1)^g)\tau^P \neq 0$  for some element  $g \notin N_G(P)$  by the definition of  $q_A$  and Mackey axiom. Especially, there is a subgroup  $H_0$  in  $P$  such that  $H_0^g \subseteq P$  and  $((\alpha_{H_0})^g)\tau^P \neq 0$ . From now on, let  $(\alpha, g, H_0)$  be such a triple set.

LEMMA 4.11.  $B \not\subseteq H_0^g$ .

*Proof.* Suppose false. By Lemma 4.2,  $g \in N_G(B)$ . In this case, it can be seen that  $(\alpha_{H_0})^g = \alpha_{H_0^g}$  and  $(\alpha_{H_0^g})\tau^p = 0$ , a contradiction.

LEMMA 4.12.  $B$  is a non-Abelian subgroup.

*Proof.* Suppose false. By Lemma 4.11, we can choose an element  $a$  in  $\mathcal{A} - H_0^g$ . Take a subgroup  $K$  of  $P$  for which  $K$  contains  $H_0^g$ ,  $a$  normalizes  $K$ , and  $K$  does not contain  $a$ . Since  $B$  is Abelian,  $a$  stabilizes  $K \geq K \cap B \geq 1$ . Furthermore, it follows from the choice of  $a$  that  $[V, a; r_n + 1] = 1$ . Combining these, we obtain  $[H^n(K, V), a; (n + 1)(r_n + 1)] = 1$  by Lemma 2.1. Since  $(n + 1)(r_n + 1) \leq p - 1$ , we get  $((a(K))\tau_K^{K\langle a \rangle})_K = 0$  by the same way as Lemma A1.8 in [1]. Furthermore, since  $a^p = 1$ , we obtain  $a(K)\tau_K^{K\langle a \rangle} = 0$ , which contradicts  $(a(H_0^g))\tau^P \ni ((\alpha_{H_0})^g)\tau^p \neq 0$ .

Since  $B$  is not Abelian,  $B$  has a nontrivial subgroup  $B_1 = [B, Z_2(B)]$ .

LEMMA 4.13.  $[V, a^{-1}a^k; 2r_n + 1] = 1$  for all  $a \in \mathcal{A}$  and  $k \in Z_2(B)$ .

*Proof.* It follows from the definition of  $\mathcal{A}$  that  $a^{-1}, a^k$  are elements of  $\mathcal{A}$ . Since they are commutative together, we get  $(a^{-1}a^k - 1)^{2r_n+1} \cdot V \subseteq \sum_{i+j=2r_n+1} (a^{-1} - 1)^i (a^k - 1)^j \cdot V$ . Since one of  $i, j$  exceeds  $r_n + 1$  in  $i + j = 2r_n + 1$ , we have the desired conclusion.

We now interest in the new set.

DEFINITION 4.14. Set  $\mathcal{A}_1 = \{a \in P: a^p = 1, [V, a; 2r_n + 1] = 1\}$ .

Then the above lemma means  $\langle \mathcal{A}_1(B_1) \rangle = B_1$ . Clearly,  $\mathcal{A}_1$  contains  $\mathcal{A}$ . Thus  $N_G(\langle \mathcal{A}_1 \rangle)$  does not control all composition factors of the  $G$ -functor  $H^n(\cdot, V)$ .

LEMMA 4.15.  $B_1 \subseteq H_0^g$ .

*Proof.* Suppose false and choose  $a$  in  $\mathcal{A}_1(B_1) - H_0^g$ . Taking a subgroup  $K$  of  $P$  as well as Lemma 4.12, we also get a contradiction since  $(n+1)(2r_n+1) \leq p-1$  and  $a^p = 1$ .

In order to continue the proof, we need a few results. The remaining proofs will be completed in the last part.

*Part 2.* In this part, we will assume Hypothesis II but not Hypothesis I.

Hypothesis II. Assume that:

- (a)  $L$  a group,  $S$  a Sylow  $p$ -subgroup of  $L$ ,  $n$  as defined in Part 1;
- (b) the condition (b) and (c) of Hypothesis I hold in  $L$ ;
- (c)  $X$  a faithful  $L$ -module (or  $C_L(X) \subseteq O_{p'}(L)$ ); and
- (d)  $\mathcal{A}^* = \{a \in S : a^p = 1, [X, a; 2r_n + 1] = 1\}$  and  $B^* = \langle \mathcal{A}^* \rangle$ .

Furthermore, we assume the following:  $G_0 = L_1/L_2$  is a section of  $L$  (that is,  $L \geq L_1 \supseteq L_2$ ) and  $P_0$  is a Sylow  $p$ -subgroup of  $G_0$ , and each  $i$ ,  $S_i = S \cap L_i$  is a Sylow  $p$ -subgroup of  $L_i$  so that  $P_0 = S_1 L_2 / L_2$ . Moreover, we denote  $\mathcal{A}^*(P_0) = \{a L_2 / L_2 : a \in \mathcal{A}^*(S_1)\}$  and  $B_0 = \langle \mathcal{A}^*(P_0) \rangle$ . Then it is clear that  $B_0$  is weakly closed in  $P_0$  with respect to  $G_0$ .

Under the above hypotheses, the following lemmas hold.

LEMMA 4.16. *Let  $V_0$  be an  $F_p[G_0]$ -module and  $[V_0, a; r_n + 1] = 1$  for all  $a \in \mathcal{A}^*(P_0)$ . Then  $N_{G_0}(B_0)$  controls all composition factors of the  $G_0$ -functor  $H^m(\cdot, V_0)$  for all  $m$  with  $0 \leq m \leq n$ .*

*Proof.* Since  $C_L(X) \subseteq O_{p'}(L)$ , it follows from Lemma 4.7 that all elements of  $\mathcal{A}^*$  satisfy  $\text{Con}(2r_n)$  in  $L$  and so all elements of  $\mathcal{A}^*(P_0)$  satisfy  $\text{Con}(2r_n)$  in  $G_0$ . Since  $n \not\geq m$ , we can see  $r_m \geq 2r_n$  and we thus have that all elements of  $\mathcal{A}^*(P_0)$  satisfy  $\text{Con}(r_m)$  in  $G_0$ . Then the minimality choice of  $n$  yields the desired assertion.

LEMMA 4.17. *Let  $V_0$  be a trivial  $F_p[G_0]$ -module. Then  $N_{G_0}(B_0)$  controls all composition factors of the  $G_0$ -functor  $H^n(\cdot, V_0)$ .*

*Proof.* Suppose false and let  $G_0$  be a minimal counterexample section of  $L$ . By the condition (b) of Hypothesis II,  $N_{G_0}(W(P_0))$  controls all composition factors of the  $G_0$ -functor  $H^n(\cdot, V_0)$ . It thus follows from the choice of  $G_0$  that  $G_0 = N_{G_0}(W(P_0))$ . Set  $H = O_p(G_0)$

and then the  $G_0$ -functor  $H^n(, V_0)$  is covered by the set

$$\{H^i(, /H, H^{n-i}(H, V_0)): i = 0, \dots, n\}$$

by Hochschild-Serre spectral sequence. Choose an element  $a$  in  $\mathcal{A}^*(P_0) - H$ . An application of Proposition 1 in [6] yields that  $[ir_{G_0}(H^{n-i}(H, V_0)), a; 4r_n(n-i)+1] = 1$ . Furthermore, since  $4r_n(n-i) \leq r_i$  for  $i \leq n$ , it follows from Lemma 4.16 that  $N_{G_0}(B_0)$  controls all composition factors of the  $G_0$ -functor  $H^i(, /H, H^{n-i}(H, V_0))$  for each  $i \leq n$ . While, for  $i = n$ , the minimality of  $G_0$  yields that  $N_{G_0/H}(B_0H/H)$  controls all composition factors of the  $G_0/H$ -functor  $H^n(, /H, V_0)$ . Summarizing, we get a contradiction.

REMARK. It should be note that we can get the same conclusion on the assumption that  $G_0$  stabilized  $V_0$ .

Next, we assume that  $G_0$  is a subgroup of  $L$ . Then  $G_0$  acts on  $X$ . From now on, let  $A = (a, \tau, \rho, \sigma)$  be a section of the  $L$ -functor  $H^n(, X)$  and  $A = A_1/A_2$  for sub- $L$ -functors  $A_i = (a_i, \tau_i, \rho_i, \sigma_i)$  of  $H^n(, X)$   $i = 1, 2$ . For each element  $\alpha \in a(P_0)$ , we set  $\alpha^*$  to be an element of  $a_1(P_0)$  such that  $\alpha = (\alpha^* + a_2(P)/a_2(P))$ . Moreover, let  $\text{Inf} \cdot H^n(P_0/P_1)$  denote the image of the inflation map  $\text{Inf}: H^n(P_0/P_1, X^{P_1}) \rightarrow H^n(P_0, X)$  for  $p$ -subgroups  $P_0 \supseteq P_1$ . And  $\text{Inf} \cdot a(P_0/P_1)$  denotes the image of  $\text{Inf} \cdot H^n(P_0/P_1) \cap a_1(P_0)$  in  $a(P_0)$ .

LEMMA 4.18. Assume that  $H_0 = O_p(G_0) \neq 1$  and an element  $\alpha$  of  $a(P_0)$  is stable in  $N_{G_0}(B_0)$ . Then we have an representation  $\alpha = \alpha_1 + \alpha_2$  such that  $\alpha_1 \in \text{Int} \cdot a(P_0/C_{P_0}(X^{A^*(H_0)}))$  and  $\alpha_2$  is stable in  $G_0$ .

Proof. We first note that the set of  $G_0$ -functors  $\{H^i(, /H_0, H^{n-i}(H_0, X)): i = 0, \dots, n\}$  is a covering of the  $G_0$ -functor  $H^n(, X)$ . Suppose that Lemma 4.18 is false, especially,  $\alpha$  is not stable in  $G_0$ , which implies that  $N_{G_0}(B_0)$  does not control the section  $A(G_0)$  (the restriction of  $A$  on  $G_0$ ). Since all elements  $a$  of  $\mathcal{A}^*(P_0)$  satisfy  $[ir_{G_0}(H^{n-j}(H_0, X)), a; 4r_n(n-j)+2r_n+1] = 1$  and  $4r_n(n-j)+2r_n \leq r_j$  for each  $j \leq n$ , it follows from Lemma 4.16 that  $N_{G_0}(B_0)$  controls all composition factors of the  $G_0$ -functor  $H^j(, /H_0, H^{n-j}(H_0, X))$  for all  $j \leq n$ . We thus have that every composition factor of the  $G_0$ -functor  $H^n(, X)$  which is not controlled by  $N_{G_0}(B_0)$  is that of  $H^n(, /H_0, X^{H_0})$ . Therefore, by Lemmas 3.11 and 3.12, we get  $q_{A_1}(\alpha^*) \in \text{Inf} \cdot H^n(P_0/H_0) + a_2(P_0)$  for  $\alpha^* \in a_1(P_0)$  with  $\alpha = \alpha^* + a_2(P_0)/a_2(P_0)$ . Since  $\alpha^* - q_{A_1}(\alpha^*)$  is stable in  $G_0$ , it is sufficient to treat  $q_A(\alpha) = q_{A_1}(\alpha^*) + a_2(P_0)/a_2(P_0)$  and so we can reset  $\alpha = q_A(\alpha)$  and  $\alpha^* = q_{A_1}(\alpha^*)$  for the convenience of notations. If  $C_{P_0}(X^{H_0}) \subseteq H_0$ , then we have already obtained the desired assertion. Set  $C = C_{G_0}(X^{H_0})$  and  $P_1 = P_0 \cap C(\not\subseteq H_0)$ . Then the Frattini argument yields  $G_0 = N_{G_0}(P_1)C$ .

Since  $P_0C$  stabilizes  $X^{H_0}$ ,  $N_{P_0C}(B_0)$  controls all composition factors of the  $P_0C$ -functor  $H^n(\ /H_0, X^{H_0})$  by Lemma 4.17. Especially,  $\alpha$  is stable in  $P_0C$ . It thus follows from Lemma 3.8 that  $\alpha$  is not stable in  $N_{G_0}(P_1) = N$ . Then the  $N$ -functor  $H^n(\ /H_0, X^{H_0})$  is covered by the set of  $N$ -functors  $\{H^i(\ /P_1, H^{n-i}(P_1/H_0, X^{H_0}))\}$  by Lemma 4.16. Therefore, we get  $q_{A_1(N)}(\alpha^*) \in \text{Inf} \cdot H^n(P_0/P_1, X^{P_1}) + a_2(P_0)$ . Since  $C \cap P_0 = P_1$ , it follows from the structure of  $G_0$  that every element of  $\text{Inf} \cdot H^n(P_0/P_1)$  is stable in  $P_0C$ . Summarizing the above statements, we have that  $\alpha - q_{A_1(N)}(\alpha)$  is stable in both  $P_0C$  and  $N$ , and so in  $G_0$ . We finally have a desired representation  $\alpha = q_{A(N)}(\alpha) + \alpha - q_{A(N)}(\alpha)$  such that  $q_{A(N)}(\alpha) \in \text{Inf} \cdot a(P_0/P_1)$  and  $\alpha - q_{A(N)}(\alpha)$  is stable in  $G_0$ , where  $A(N)$  is the restriction of  $A$  on  $N$ , a contradiction.

Next, we consider the set  $\mathscr{A}^* = \{a \in S : a^p = 1, [X, a; 2r_n + 1] = 1\}$ . In relation to this set, we define the following group.

**DEFINITION 4.19.** A subgroup  $H$  of  $L$  is called to be a  $C$ -group of depth 1 if  $H$  is generated by elements of  $\mathscr{A}^*(H)(=H \cap \mathscr{A}^*)$ .

**LEMMA 4.20.** Let  $H$  be a  $C$ -group in  $S$  of depth 1. Assume that  $\alpha \in a(S)$  is stable in  $N_L(B^*)$ . Then we have a representation  $\alpha_{N_S(H)} = \alpha_1 + \alpha_2$  such that  $\alpha_1 \in \text{Inf} \cdot a(N_S(H)/N_S(H) \cap C(X^H))$  and  $\alpha_2$  is stable in  $N_L(H)$ .

*Proof.* Suppose false and let  $H$  be a maximal counterexample. Furthermore, choose  $H$  such that  $|N_S(H)|$  is maximal subject to the maximality of  $H$ . We first assert that  $N_S(H)$  is a Sylow  $p$ -subgroup of  $N_L(H)$ . To see this, we follow the proof of [5]. Suppose false and let  $F$  be a conjugate subgroup of  $H$  contained in  $S$  such that  $N_S(F)$  is a Sylow  $p$ -subgroup of  $N_L(F)$ . Then we have that for some element  $f$  of  $L$ ,  $N_S(H)^f \subseteq N_S(F)$  and  $H^f = F$ . By Alperin's theorem, there is a set of pair  $\{(K_i, g_i) : g_i \in N_L(K_i), K_i \leq S; i = 1, \dots, m\}$  such that it satisfies the following:

$$N_S(H) \subseteq K_1, \dots, N_S(H)^{g_1 \cdots g_{i-1}} \subseteq K_i, \dots, N_S(H)^{g_1 \cdots g_{m-1}} \subseteq K_m, \\ \text{and } g_1 \cdots g_m = f.$$

Since all conjugate subgroups of  $H$  in  $S$  are  $C$ -groups of depth 1, in order to get a contradiction, it suffices to treat only one step. We therefore assume that  $H^{g_1}$  satisfies the assertion of Lemma 4.20. It will be convenient to reset  $K = K_1$ ,  $g = g_1$ . Set  $K^* = \langle \mathscr{A}^*(K) \rangle$ , then  $K^* \supseteq \langle H, H^g \rangle$ . By the maximality of  $H$  and the choice of  $H^g$ , we have two representations:

(i)  $\alpha_{N_S(K^*)} = \beta_1 + \beta_2$  such that  $\beta_1 \in \text{Inf} \cdot a(N_S(K^*)/C_{N_S(K^*)}(X^{K^*}))$  and  $\beta_2$  is stable in  $N_L(K^*)$ ;

(ii)  $\alpha_{N_S(H^g)} = \gamma_1 + \gamma_2$  such that  $\gamma_1 \in \text{Inf} \cdot a(N_S(H^g)/C_{N_S(H^g)}(X^{H^g}))$  and  $\gamma_2$  is stable in  $N_L(H^g)$ . Since  $N_L(K) \subseteq N_L(K^*)$ , combining the two representations and resetting  $R = N_S(H)$ , we get  $((\beta_2)_R)^g = ((\beta_2)_K)^g R^g = ((\beta_2)_{N_S(H^g)}) R^g = (\gamma_2 + \gamma_1 - (\beta_1)_{N_S(H^g)}) R^g = (\gamma_2 + (\gamma_1 - (\beta_1)_{N_S(H^g)})) R^g$ . We thus have  $(\beta_2)_R = (\gamma_2)_R^{g^{-1}} + (\gamma_1 - (\beta_1)_{N_S(H^g)})_R^{g^{-1}}$  such that  $(\gamma_1 - (\beta_1)_{N_S(H^g)})_R^{g^{-1}}$  is contained in  $\text{Inf} \cdot a(R/C_R(X^H))$  and  $(\gamma_2)_R^{g^{-1}}$  is stable in  $N_L(H)$ , since  $(\gamma_2)_{R^g}$  is stable in  $N_L(H^g)$  and  $\gamma_1 - (\beta_1)_{N_S(H^g)}$  is an element of  $\text{Inf} \cdot a(N_S(H^g)/C_{N_S(H^g)}(X^{H^g}))$  by the choice of  $H^g$ . Finally we obtain  $\alpha_R = (\beta_1 + \beta_2)_R = (\beta_1)_R + ((\gamma_1 - (\beta_1)_{N_S(H^g)})_R^{g^{-1}})_R + ((\gamma_2)_R^{g^{-1}})_R$  such that  $(\beta_1)_R + ((\gamma_1 - (\beta_1)_{N_S(H^g)})_R^{g^{-1}})_R \in \text{Int} \cdot a(R/C_R(X^H))$  and  $(\gamma_2)_R^{g^{-1}}$  is stable in  $N_L(H)$ , as desired, which is a contradiction. So we have proved that  $N_S(H)$  is a Sylow  $p$ -subgroup of  $N_L(H)$ . Since  $H \not\cong B^*$ , we easily check that  $\langle \mathcal{S}^*(N_S(H)) \rangle \cong H$ . Set  $H_1 = \langle \mathcal{S}^*(N_S(H)) \rangle$ , then  $H_1$  is a  $C$ -group of depth 1 containing  $H$  properly. It follows from the maximality of  $H$  that we have a representation  $\alpha_{N_S(H_1)} = \delta_1 + \delta_2$  such that  $\delta_1 \in \text{Int} \cdot a(N_S(H_1)/C_{N_S(H_1)}(X^{H_1}))$  and  $\delta_2$  is stable in  $N_L(H_1)$ . For  $\delta_1$  we get  $(\delta_1)_R \in \text{Int} \cdot a(R/C_R(X^H))$  where  $R = N_S(H)$ , while for  $\delta_2$  we have  $(\delta_2)_R = \xi_1 + \xi_2$  such that  $\xi_1 \in \text{Inf} \cdot a(R/C_R(X^H))$  and  $\xi_2$  is stable in  $N_L(H)$  by Lemma 4.18. We therefore obtain a desired representation  $\alpha_R = ((\delta_1)_R + \xi_1) + \xi_2$ , a contradiction.

We next define more generalized  $C$ -groups.

**DEFINITION 4.21.** A subgroup  $H$  of  $S$  is called to be a  $C$ -group (or depth  $t$ ) if  $H$  has a series  $H = H_t \not\cong H_{t-1} \not\cong \cdots \not\cong H_2 \not\cong H_1$  such that  $H_1 = \langle \mathcal{S}^*(H) \rangle$  and  $H_{i+1} = \langle a \in H: a^p \in H_i, [X^{H_i}, a; 2r+1] = 1 \rangle$  for  $i = 1, \dots, t-1$ . Then we call  $H_i$  to be the  $C$ -subgroup of  $H$  of depth  $i$ .

**LEMMA 4.22.** Let  $H$  and  $K$  be  $C$ -groups in  $S$  of depth at most  $t$ . Then  $\langle H, K \rangle$  is also a  $C$ -group of depth at most  $t$ .

*Proof.* This follows from the definition.

In association with  $C$ -group of depth  $t$ , we have a similar result.

**LEMMA 4.23.** Let  $H$  be a  $C$ -group in  $S$  of depth  $t$ . Assume that  $\alpha \in a(S)$  is stable in  $N_L(B^*)$ . Then we have a representation  $\alpha_{N_S(H)} = \alpha_1 + \alpha_2$  such that  $\alpha_1 \in \text{Int} \cdot a(N_S(H)/C_{N_S(H)}(X^H))$  and  $\alpha_2$  is stable in  $N_L(H)$ .

*Proof.* Suppose false and let  $t$  be a minimal counterexample. We already got  $t \neq 1$  by Lemma 4.20. Furthermore let  $H$  be a maximal counterexample subject to the minimality of  $t$ . Let  $F$  be the  $C$ -subgroup of  $H$  of depth  $t-1$ . It follows from the same argument in Lemma 4.20 that we may assume that  $N_S(F)$  is a Sylow  $p$ -subgroup

of  $N_L(F)$ . We then have a representation  $(\alpha)_{N_S(F)} = \beta_1 + \beta_2$  such that  $\beta_1 \in \text{Inf} \cdot a(N_S(F)/C_{N_S(F)}(X^F))$  and  $\beta_2$  is stable in  $N_L(F)$  by the minimality of  $t$ . Since  $N_L(H)$  is contained in  $N_L(F)$ ,  $(\beta_2)_{N_S(H)}$  is stable in  $N_L(H)$ . Thus it is sufficient to treat  $\beta_1$ . Set  $K = \langle a \in N_S(F): a^p \in F, [X^F, a; 2r_n + 1] = 1 \rangle$ . Then  $K$  is a  $C$ -group of depth at most  $t$ . Suppose first  $K \cong H$ , then we have a representation  $\alpha_{N_S(K)} = \gamma_1 + \gamma_2$  such that  $\gamma_1 \in \text{Int} a(N_S(K)/C_{N_S(K)}(X^K))$  and  $\gamma_2$  is stable in  $N_L(K)$  by the maximality of  $H$ . By combining the two representations, we can see  $\alpha_{N_S(F)} = \beta_1 + \beta_2 = (\gamma_1)_{N_S(F)} + (\gamma_2)_{N_S(F)}$ . Thus  $\xi = (\gamma_2)_{N_S(F)} - \beta_2 = \beta_1 - (\gamma_1)_{N_S(F)}$  is stable in  $N_L(K) \cap N_L(F)$  and contained in

$$\text{Inf} \cdot a(N_S(F)/C_{N_S(F)}(X^F)).$$

Summarizing, we have the following situation:

- (i)  $\overline{N_L(F)} = N_L(F)/C(X^F) \cap N_L(F)$  acts on  $X^F$  faithfully;
- (ii)  $\bar{K}$  is generated by the set of elements  $\bar{a}$  of  $\overline{N_S(F)}$  with  $[X^F, \bar{a}; 2r_n + 1] = 1$  and  $\bar{a}^p = \bar{1}$ ;

(iii)  $\overline{N_S(F)}$  is a Sylow  $p$ -subgroup of  $\overline{N_L(F)}$ ; and

(iv)  $\bar{\xi}$  is an inverse element of  $\xi$  in  $a'(\overline{N_S(F)})$  which is stable in  $\overline{N(K)} \cap \overline{N(F)}$ , that is,  $\text{Inf}(H^n(\overline{N_L(F)}, X^F) \rightarrow H^n(N_L(F), X)): \xi \rightarrow \bar{\xi}$ . Here  $(-)$  denotes the image of the natural homomorphism:  $N_L(F) \rightarrow \overline{N_L(F)}$  and  $A' = (a', \tau', \rho', \sigma')$  is the section of  $\overline{N(F)}$ -functor  $H^n(\cdot, X^F)$  which is isomorphic to the image of the section  $N(F)$ -functor  $\text{Inf} \cdot (H^n(\cdot / C_{N(F)}(X^F), X^F))$ . By taking  $\overline{N_L(F)}, \overline{N_S(F)}, \bar{K}$  in place of  $L, S, B^*$  of Hypothesis II, respectively, we can see that they satisfy the all conditions of Hypothesis II. In this case, since  $\bar{H}$  is a  $C$ -group of  $\overline{N_S(F)}$  of depth 1 and  $\bar{\xi}$  is stable in  $N_{\overline{N(F)}}(\bar{K})$ , by Lemma 4.22 applied to  $\bar{\xi}$  and  $\overline{N_L(F)}$  instead of  $\alpha$  and  $L$ , respectively, we have a representation  $\xi_{N_S(H)} = \xi_1 + \xi_2$  such that  $\xi_1 \in \text{Inf} \cdot a(N_S(H)/C_{N_S(H)}(X^H))$  and  $\xi_2$  is stable in  $N_L(H) \subseteq N_L(F)$ . Consequently, we have a representation  $\alpha_R = (\gamma_1)_R + (\gamma_2)_R - (\beta_2)_R + (\beta_2)_R = (\gamma_1)_R + \xi_R + (\beta_2)_R = ((\gamma_1)_R + \xi_1) + (\xi_2 + (\beta_2)_R)$  such that  $(\gamma_1)_R + \xi_1 \in \text{Inf} \cdot a(R/C_R(X^H))$  and  $\xi_2 + (\beta_2)_R$  is stable in  $N_L(H)$ , as desired, where  $R = N_S(H)$ , a contradiction. we therefore have that  $H = K$ . In this case we assert that  $F$  is the unique maximal  $C$ -group of  $N_S(F)$  of depth  $t - 1$ . To see this, let  $T$  be the unique maximal  $C$ -group of  $N_S(F)$  of depth  $t - 1$  and  $T = T_{t-1} \supseteq T_{t-2} \supseteq \cdots \supseteq T_1$  be the chain of  $C$ -subgroups of  $T$ . Then since  $T_1$  is a  $C$ -group of depth 1,  $K \supseteq T_1$  by the definition of  $K$ . Since  $K = H$  and  $F$  is the unique maximal  $C$ -subgroup of  $H$  of depth  $t - 1$ ,  $F \supseteq T_1$ . By iteration, we obtain  $T \subseteq K$  and  $T \subseteq F$ , as desired. The result that  $F$  is the unique maximal  $C$ -group of  $N_S(F)$  of depth  $t - 1$  implies that  $S = N_S(F)$ , which means  $F \supseteq B^*$  and  $N_L(H) \subseteq N_L(F) \subseteq N_L(B^*)$ , a contradiction.

*Part 3.* Now we can return to the proof of Proposition A. In this part, we take the proof of Proposition A up again. This is a continuation of Part 1. We adopt same notations of Part 1, such as  $G$ ,  $V$ ,  $\mathcal{A}_1$ ,  $B$  and the like.

LEMMA 4.24.  $G$ ,  $V$ ,  $\mathcal{A}_1$  satisfy the conditions of Hypothesis II.

*Proof.* By taking  $G$ ,  $V$ ,  $\mathcal{A}_1$  in place of  $L$ ,  $X$ ,  $\mathcal{A}^*$ , respectively, the assertion follows from Lemmas 4.6 and 4.8.

In the comments following Lemma 4.12, we defined the subgroup  $B_1 = [B, Z_2(B)]$ . We here define the following subgroups  $B_i$ .

DEFINITION 4.27.  $B_i$  is the inverse image of  $[B/B_{i-1}, Z_2(B/B_{i-1})]$  in  $B$  for  $i = 1, \dots$ .

Then clearly, the chain  $B_1 \subseteq B_2 \subseteq \dots$  is ascendent and there is an integer  $k$  such that  $B_{k-1} \neq B_k = B_{k+1}$ . It follows from the definition of  $B$  that  $B/B_k, B_i/B_{i-1}$  are all elementary Abelian. Set  $B_{k+1} = B$ . Moreover the following is clear.

LEMMA 4.26.  $B_i$  is a  $C$ -group of depth at most  $i$  for  $i \leq k$ .

Now we recall the triple  $(H_0 \leq P, g \in G, \alpha \in a(P))$  defined in the statement (4.10). Since  $H_0, H_0^g \subseteq P$ , there is a set of pair  $\{(K_i, g_i) : K_i \leq P, g_i \in N(K_i); i = 1, \dots, m\}$  such that  $H_0$  is conjugate to  $H_0^g$  via  $g$  by this set, namely, this set satisfies the following:

$$H_0 \subseteq K_1, \dots, H_0^{g_1 \cdots g_{i-1}} \subseteq K_i, \dots, H_0^{g_1 \cdots g_{m-1}} \subseteq K_m \quad \text{and} \quad g_1 \cdots g_m = g.$$

Since we choose  $\{g, H_0\}$  on the only assumption that  $((\alpha_{H_0})^g)^P \neq 0$ , we can rechoose  $g$  and  $H_0$  such that  $((\alpha_{H_0} g_1)^{g_1^{-1}g})^P = 0$ . Then we get  $((\alpha_{K_1})^{g_1} - \alpha_{K_1})_{H_0} g_1^{-1}g)^P \neq 0$ . Set  $\beta = ((\alpha_{K_1})^{g_1} - \alpha_{K_1})_{H_0} g_1^{g_1^{-1}g}$ . As we showed,  $\beta^P \neq 0$ , especially,  $a(H_0^g)\tau^p \neq 0$ . In Part 1, we already got the following:

$$(4.27) \quad B \not\subseteq H_0^g \quad \text{and} \quad B_1 \subseteq H_0^g.$$

We next show that  $B_2 \subseteq H_0^g$ . To see this, we need some arguments. Since  $B_1 \subseteq H_0^g$  and  $\langle \mathcal{A}_1(B_1) \rangle = B_1$ , we have  $\langle \mathcal{A}_1(K_1) \rangle \supseteq (\mathcal{A}_1(B_1))^{g^{-1}} \neq \{1\}$ . Let  $L = \langle \mathcal{A}_1(K_1) \rangle$ , then  $L$  is a normal  $C$ -group of  $K_1$  of depth 1. We thus have a representation  $\alpha_{N_P(L)} = \alpha_1 + \alpha_2$  such that  $\alpha_1 \in \text{Inf} \cdot a(N_P(L)/C_{N_P(L)}(V^L))$  and  $\alpha_2$  is stable in  $N_G(L)$  by Lemma 4.22. It should be noted that we can choose the set  $\{(K_i, g_i) : i \in I\}$  such that  $N_P(\langle \mathcal{A}_1(K_i) \rangle)$  is a Sylow  $p$ -subgroup of  $N_G(\langle \mathcal{A}_1(K_i) \rangle)$  for each  $i \in I$ , by the same argument in Lemma 4.20. To simplify the

notation, we set  $f = k_1$  and  $F = K_1$ . We then have  $(\alpha_F)^f - \alpha_F = (((\alpha_1)_F)^f + ((\alpha_2)_F)^f) - ((\alpha_1)_F + (\alpha_2)_F) = ((\alpha_1)_F)^f - (\alpha_1)_F$  and so we get  $\beta = ((\alpha_F^f - \alpha_F)_{H_0 f})^{f^{-1}g} = (((\alpha_1)_F^f - (\alpha_1)_F)_{H_0 f})^{f^{-1}g}$  is contained in  $\text{Inf} \cdot a(H_0^g / \langle \cdot \mathcal{A}_1(H_0^g) \rangle)$ . By these arguments, the following holds.

LEMMA 4.28.  $B_2 \subseteq H_0^g$ .

*Proof.* Suppose false and then there is an element  $a$  in  $B_2 - H_0^g$  such that  $[V^{B_1}, a; 2r_n + 1] = 1$  and  $a^p \in B_1$  by the construction of  $B_2$ . Taking  $K$  as Lemma 4.12, we have that  $a$  stabilizes  $K \geq K \cap B_2 \geq B_1$ . We thus get  $[H^*(K/B_1, V^{B_1}), a; (n+1)(2r_n+1)] = 1$  by Lemma 2.1. Since  $a^p \in B_1$  and  $(n+1)(2r_n+1) \leq p-1$ , we have

$$(\text{Inf} \cdot a(K/B_1))\tau^{K\langle a \rangle} = 0,$$

which contradicts  $\beta \in \text{Inf} \cdot a(H_0^g/B_1)$ .

Now we can get a final contradiction. Namely, we will show that  $B_i \subseteq H_0^g$  implies  $B_{i+1} \subseteq H_0^g$ , and  $B_k \not\subseteq H_0^g$  which contradict together. The proof is similar to that of Lemma 4.28. We assume that  $B_i \subseteq H_0^g$ . Let  $L$  be the unique maximal  $C$ -group in  $F = K_1$ . Since  $F$  contains  $B_i^{g^{-1}}$ ,  $L \supseteq B_i^{g^{-1}}$ . Since  $N_G(L) \subseteq N_G(\langle \cdot \mathcal{A}_1(F) \rangle)$ , we can rechoose the set  $\{(K_i, g_i): i \in I, (K_1 = F, g_1 = f)\}$  such that  $N_P(L)$  is also a Sylow  $p$ -subgroup of  $N_G(L)$  by the same argument in Lemma 4.20. Then by Lemma 4.23, we have a representation  $\alpha_{N_P(L)} = \alpha_1 + \alpha_2$  such that  $\alpha_1 \in \text{Inf} \cdot a(N_P(L)/C_{N_P(L)}(V^L))$  and  $\alpha_2$  is stable in  $N_G(L)$ . We thus have  $(\alpha_F)^f - \alpha_F = ((\alpha_1)_F)^f - (\alpha_1)_F \in \text{Inf} \cdot a(F/C_F(V^L))$ . We therefore obtain  $\beta = (((\alpha_1)_F)^f - (\alpha_1)_F)_{H_0 f}^{f^{-1}g} \in \text{Inf} \cdot a(H_0^g/B_i)$ . By this argument, we have the following lemma which contradicts (4.27).

LEMMA 4.29.  $B_i \subseteq H_0^g$  implies  $B_{i+1} \subseteq H_0^g$ , and  $B_k \not\subseteq H_0^g$ .

*Proof.* Suppose that  $B_i \subseteq H_0^g$  and  $B_{i+1} \not\subseteq H_0^g$  for  $i = i, \dots, k$ . Choose an element  $a$  in  $B_{i+1} - H_0^g$  such that  $[V^{B_i}, a; 2r_n + 1] = 1$  and  $a^p \in B_i$ . Taking  $K$  as Lemma 4.12, we get that  $a$  stabilizes  $K \supseteq K \cap B_{i+1} \supseteq B_i$ . We also obtain a contradiction by the same way of Lemma 4.28 since  $\beta \in \text{Int} \cdot a(H_0^g/B_i)$  and  $(n+1)(2r_n+1) \leq p-1$ .

This completes the proof of Proposition A.

**5. Proofs of theorems.** In this section, we will prove Theorem B and Glauberman's conjecture using Proposition A.

**THEOREM B.** *Let  $G$  be a finite group,  $P$  a Sylow  $p$ -subgroup of*



$G$ , and  $W$  a section conjugacy functor of degree  $t$ . Furthermore, let  $V$  be a trivial  $F_p[G]$ -module. If  $p \geq 4 \times 6^{n-2} \times (t-1) + 3$  for some integer  $n \geq 2$ , then the restriction map induces an isomorphism:

$$H^n(G, V) \cong H^n(N_G(W(P)), V).$$

*Proof.* Suppose false and in particular  $N_G(W(P))$  does not control all composition factors of  $G$ -functor  $H^n(G, V)$ . Let  $\{n, G\}$  be a minimal counterexample so that  $N_G(W(P))$  does not controls all composition factors of the  $G$ -functor  $H^n(G, V)$  in a trivial  $F_p[G]$ -module  $V$ . Then by the minimality of  $G$ , we have the following: (a)  $O_p(G) \neq 1$ ; (b)  $W(P) \not\trianglelefteq G$ ; and (c) for all normal subgroups  $K$  of  $P$  containing  $O_p(G)$  properly,  $W$  controls all sections of the  $N_G(K)$ -functor  $H^n(\cdot, V)$  in  $N_G(K)$ . Let  $H = O_p(G)$ . We then get that the set of the  $G$ -functors  $\{H^i(\cdot/H, H^{n-i}(H, V)) : i = 0, \dots, n\}$  is a covering of the  $G$ -functor  $H^n(\cdot, V)$ . Thus  $W$  does not control a composition factor of the  $G$ -functor  $H^i(\cdot/H, X)$  for some  $i$  and an irreducible  $G/H$ -composition factor  $X$  of  $H^{n-i}(H, V)$ . On the other hand, since  $W(P) \not\trianglelefteq G$ , there is an element  $a$  in  $P - H$  such that  $[ir_G(H), a; t] = 1$  by the definition of degree. We thus obtain  $[X, a; (n-i)(t-1) + 1] = 1$  by Proposition 1 in [6]. By the choice of  $p$ , we especially have  $[ir_G(H^{n-i}(H, V)), a; p-1] = 1$ . By Theorem A1.4 in [1], there is a normal subgroup  $\beta$  of  $P$  containing  $\langle H, a \rangle$  such that  $N_G(\beta)$  controls all sections of the  $G$ -functor  $H^0(\cdot, H^{n-i}(H, V))$ . On the other hand, by the minimality of  $G$ ,  $N_{G/H}(W(P/H))$  controls all sections of the  $G/H$ -functor  $H^n(\cdot/H, H^0(H, V))$ . Thus we have  $0 \leq i \leq n$ , that is, we obtain the following situation:

- (a) there is an integer  $i (\geq 0)$  and  $G/H$  acts on an  $F_p[G]$ -module  $X$ ;
- (b)  $N_{G^*}(W(P^*))$  controls all composition factors of the  $G^*$ -functor  $H^j(\cdot, V^*)$  with coefficient in a trivial  $F_p[G^*]$ -module  $V^*$  for  $0 \leq j \leq i$ , where  $G^*$  is a section of  $G/H$  and  $P^*$  is a Sylow  $p$ -subgroup of  $G^*$ ;
- (c)  $C_{G/H}(X) \subseteq O_{p'}(G/H)$  by the minimality of  $G$  and the Frattini argument; and
- (d) there is a nontrivial element  $a$  in  $P/H$  such that  $[X, a; r_i] = 1$  and so  $a$  satisfies  $\text{Con}(r_i)$  in  $G/H$  since  $(n-i)(t-1) \times 4 \times 6^{i-1} \leq p-3$  by the hypothesis of  $p$ , where  $r_i$  is the first integer such that  $p \leq 4 \times 6^{i-1} \times (r_i + 1) + 2$ . Since  $G/H, i, X$  satisfy the conditions of Hypothesis I in §4 in place of  $G, n, V$ , respectively, there is a normal subgroup  $B$  of  $P$  containing  $\langle H, a \rangle$  such that  $N_G(B)$  controls all composition factors of the  $G/H$ -functor  $H^i(\cdot/H, X)$ , a contradiction. This completes the proof of Theorem B.

Especially, since the section conjugacy functor  $K_\infty$  in Glauberman [1] has degree 4, we have an affirmative answer to his question.

**THEOREM C.** *Let  $m$  be an integer with  $m \geq 2$ . If  $p \geq 12 \times 6^{m-2} + 3$ , then we have  $H^m(G, F_p) \cong H^m(N_G(K_\infty(P)), F_p)$ .*

Taking a new look at Hypothesis I, we notice that the conditions (b) and (c) always hold by Theorem C if  $p \geq 12 \times 6^{n-2} + 3$ . We therefore have a new form of Proposition A.

**THEOREM A.** *Let  $V$  be an  $F_p[G]$ -module,  $n$  an integer  $\geq 1$ , and  $r$  be the first integer such that  $p \leq 4 \times 6^{n-1} \times (r+1) + 2$ . Set  $\mathcal{A} = \{a \in P: a^p = 1, [ir_G(V), a; r+1] = 1, \text{ and } a \text{ satisfies } \text{Con}(r) \text{ in } G\}$ . If  $r \geq 1$ , the restriction map induces an isomorphism:*

$$H^m(G, V) \cong H^m(N_G(\langle \mathcal{A} \rangle), V) \text{ for all integers } m \text{ with } 1 \leq m \leq n.$$

As corollary, we have:

**COROLLARY 1.** *Let  $V$  be a faithful  $p$ -primary  $G$ -module (or  $C_G(V) \subseteq O_{p'}(G)$ ) and  $n, r$  be as above. Set  $B = \langle a \in P: a^p = 1 \text{ and } [V, a; r+1] = 1 \rangle$ . Then  $H^m(G, V) \cong H^m(N_G(B), V)$  for all integers  $m$  with  $1 \leq m \leq n$ .*

**REMARK.** It should be noted that all module in theorems are finite, but it is not necessary to be finite. For all  $p$ -primary  $G$ -modules, the same assertions hold by the same argument in [6].

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