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INTRINSICALLY $(n-2)$-DIMENSIONAL CELLULAR DECOMPOSITIONS OF E^n

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Let G be a CE usc decomposition of an n-manifold M. The intrinsic dimension of G is a measure of the minimal dimension of the image of the nondegeneracy set of CE maps from M onto M/G which approximate the natural projection map. Examples of totally noncellular intrinsically n-dimensional decompositions of E^n, $n \geqq 3$, are known to exist. Here it is shown that there also exist cellular decompositions of E^n, $n \geqq 3$, which are intrinsically $(n-2)$-dimensional.

0. Introduction.

Most examples of decompositions presented in the literature are 0-dimensional. Illustrating the extreme alternative, Cannon, Daverman and Walsh have constructed examples of totally noncellular, CE usc decompositions of E^n, $n \geqq 3$ [3] [7]. The fact that these decompositions are totally noncellular (and are known to yield n-dimensional decomposition spaces) makes it clear that they are intrinsically n-dimensional.

Cellular decompositions, however, cannot be quite so complicated. It is not difficult to show that a cellular decomposition of E^n (having finite dimensional decompositson space) is necessarily of intrinsic dimension less than n. For proofs of this fact, see [10, p. 68] or [11, p. 27]. This paper sets forth examples of cellular decompositions of E^n, $n \geqq 3$, that are intrinsically $(n-2)$-dimensional. Such examples were discovered independently by the authors in 1979.

The main point established by these examples is that cellular decompositions form a fairly large and reasonably typical subclass of the total class of CE decompositions. Moreover, the important question of whether $E^n/G \times E^1$ is homeomorphic to E^{n+1} remains open in all dimensions $n \geqq 3$ (even when G is a cellular usc decomposition of E^n and E^n/G is finite dimensional). Whenever G is intrinsically of dimension $\leqq n-3$, $(E^n/G) \times E^1$ is known to be topologically E^{n+1} [6, Theorem 1] [5, Theorem 3.3].

Whether there exist intrinsically $(n-1)$-dimensional cellular decompositions of E^n stands as an unsolved problem.

1. Notation and conventions.

We will be considering cell-like (CE) upper semicontinuous (usc) decompositions of manifolds M without boundary. If G is such a decomposition, H_G represents the set whose elements are the nondegenerate elements of G, and N_G

represents the union of these elements. In general, π or π_G will represent the quotient map from M onto M/G. If p is a CE map from M onto X and H is the decomposition of M with elements $\{p^{-1}(x) \mid x \in X\}$, then $N_p = N_H$. A CE map p from M onto X is said to be 1-1 over A if $A \subset X$ and each $p^{-1}(a)$ for $a \in A$ consists of a single point.

The sup metric ρ on E^n will be used. That is, $\rho(x, y) = \sup_{1 \leqq i \leqq n} |x_i - y_i|$ where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. For maps f and g from X into E^n, $\rho(f, g) \equiv \sup_{x \in X} \rho(f(x), g(x))$. The standard embeddings $[-1, 1] \times \cdots \times [-1, 1] \times \{0\}$ and $[-1, 1] \times \cdots \times [-1, 1] \times \{0\} \times \{0\}$ of the closed $(n-1)$ and $(n-2)$ balls in E^n will be denoted by B^{n-1} and B^{n-2} respectively. Thus, each point y of B^{n-1} can be represented as (x, t) where x is in B^{n-2} and t is in $[-1, 1]$.

2. Preliminaries.

The following definitions and theorem are taken from [3] and provide a general framework for constructing CE usc decompositions.

DEFINITION. Let N be a P.L. n-manifold. A *defining sequence* (in N) is sequence $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \cdots\}$ satisfying the following conditions:

(1) for each i, $\mathscr{M}_i$ is a finite collection $\{M(1), \cdots, M(k_i)\}$ of P.L. n-manifolds with boundary in N such that

$$(\operatorname{Int} M(j)) \cap (\operatorname{Int} M(k)) = \varnothing \quad \text{for} \quad j \neq k\,;$$

(2) for $1 \leqq i < j$ and for each A in $\mathscr{M}_j$, there is a unique element $\operatorname{Pre}^{j-i}(A)$ in $\mathscr{M}_i$ properly containing A; and

(3) for each $i \geqq 1$, each A in $\mathscr{M}_i$, and each pair of points x and y in ∂A, there is an integer $j > i$ such that no element of $\mathscr{M}_j$ contains both x and y.

DEFINITION. Let $\mathscr{S}$ be a defining sequence in an n-manifold N. Then

$$st(x, \mathscr{M}_j) = st_1(x, \mathscr{M}_j) = \{x\} \cup \bigcup \{A \in \mathscr{M}_j \mid x \in A\} \quad \text{and}$$
$$st_k(x, \mathscr{M}_j) = \bigcup \{st(y, \mathscr{M}_j) \mid y \in st_{k-1}(x, \mathscr{M}_j)\} \quad \text{when} \quad k \geqq 2\,.$$

DEFINITION. The *decomposition G of N associated with a defining sequence $\mathscr{S}$* in N is described as follows. Distinct points x and y of N are in the same element of G if there is an integer r, depending only on x and y, such that for each j, $y \in st_r(x, \mathscr{M}_j)$.

THEOREM 1 [3, § 3]. *The decomposition G of N associated with* *a defining sequence $\mathscr{S}$ in N is* usc. *If, in addition, each A in $\mathscr{M}_j$ is null homotopic in* $\mathrm{Pre}^1(A)$ *for all $j \geqq 2$, then G is CE.*

In general, each x in N has the property that $\pi^{-1}\circ\pi(x) = \bigcap_{j=1}^{\infty} st_2(x, \mathscr{M}_j)$. Let $B = \bigcup\{\partial A \mid A$ is an element of some $\mathscr{M}_j\}$. If $x \in g \in G$ and either $x \in B$ or $g \cap B = \varnothing$, then $\pi^{-1}\circ\pi(x) = \bigcap_{j=1}^{\infty} st(x, \mathscr{M}_j)$.

3. Measuring intrinsic dimension. This section sets the stage for the construction of the next section. Methods for determining the intrinsic dimension of certain decompositions are set forth.

DEFINITIONS. Let G be a CE usc decomposition of an n-manifold M. Then G is said to be:

(i) *d-dimensional* if $\pi(N_G)$ has dimension d;

(ii) *closed d-dimensional* if the closure of $\pi(N_G)$ has dimension d;

(iii) *secretly d-dimensional* if π is arbitrarily closely approximable by CE maps p from M onto M/G with $p(N_p)$ of dimension less than or equal to d; and

(iv) *intrinsically d-dimensional* if it is secretly d-dimensional, but not secretly $(d-1)$-dimensional.

For a defining sequence $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \cdots\}$ in E^n consider the following *Special Hypothesis*:

(SH*) There exist maps F_1 and F_2 from B^2 into E^n and $\varepsilon > 0$ so that $F_1(B^2) \cap F_2(B^2) = \varnothing$ and $\rho(F_e(\partial B^2), \bigcup \mathscr{M}_1) > \varepsilon$ for $e = 1, 2$.

(SH_i) (a) R_i is the subdivision of B^{n-2} into $2^{(i-1)(n-2)}$ $(n-2)$-cells obtained by dividing each $[-1, 1]$ factor into 2^{i-1} equal subintervals.

S_i is a triangulation of $[-1, 1]$ with S_{i+1} refining S_i.

T_i is the subdivision of B^{n-1} obtained by taking $R_i \times S_i$.

T_i has mesh less than or equal to 2^{2-i}.

(b) For each element A of $\mathscr{M}_i$, $A \cap \{B^{n-1} \times [-1/i, 1/i]\} = C \times [-1/i, 1/i]$ where C is an $(n-1)$-cell of T_i.

(c) For distinct elements A and $\tilde{A}$ of $\mathscr{M}_i$, $A \cap \tilde{A}$ is contained in $\partial C \times [-1/i, 1/i]$ where C is an $(n-1)$-cell of T_i.

(d) If $x \in \partial A$ for A in $\mathscr{M}_{i-1}$, either $x \notin \bigcup \mathscr{M}_i$ or $x \in \partial C \times [-1/i, 1/i]$ for some $(n-1)$-cell C of T_i.

DEFINITION. Fix t in $[-1, 1]$. Maps f_1 and f_2 from B^2 into E^n are *$(t, \mathscr{S})$ slice* maps if for all x in B^{n-2}, $\pi(x, t) \cap \pi(f_1(B^2)) \cap \pi(f_2(B^2)) \neq \varnothing$. Assume SH_i holds. Then f_1 and f_2 are *$(A, \mathscr{M}_i)$ slice* maps (A an interval of S_i) if $P \times A$ is contained in an element of $\mathscr{M}_i$ that intersects both $f_1(B^2)$ and $f_2(B^2)$ for every P in R_i.

The next two lemmas are technical and will guide the construction in the following section.

LEMMA 1. *Assume that* SH^* *holds, and that*:

(i) $\pi|B^{n-1}$ *is homeomorphism*;

(ii) $\pi(N_\pi) \subset \pi(B^{n-1})$;

(iii) *if* f_1 *and* f_2 *are maps from* B^2 *into* E^n, *with* $\rho(f_e|\partial B^2, F_e|\partial B^2) < \varepsilon/2$ *for* $e = 1, 2$, *then for some* t *in* $[-1, 1]$, f_1 *and* f_2 *are* $(t, \mathscr{S})$ *slice maps*; *and*

(iv) *the decomposition* G *of* E^n *associated with* $\mathscr{S}$ *is cellular. Then* G *is intrinsically* $(n-2)$-*dimensional*.

Proof. First, it will be shown that G is secretly $(n-2)$-dimensional. Note that $Q = E^n/G - \pi(B^{n-1})$ is an F_σ set and that π is already 1-1 over Q. Choose a countable dense subset $\{x_i\}$ of B^{n-1} so that $O = B^{n-1} - \bigcup_{i=1}^\infty \{x_i\}$ is $(n-2)$-dimensional. Since G is cellular, $\pi: E^n \to E^n/G$ can be closely approximated by a CE map $p_i: E^n \to E^n/G$ that is 1-1 over $\pi(x_i)$. It follows from [9, p. 15] that the map π from E^n onto E^n/G can be closely approximated by a CE map p from E^n onto E^n/G with $p(N_p) \subset O$. This implies G is secretly $(n-2)$-dimensional.

Next, it will be shown that G is not secretly $(n-3)$-dimensional. Assume the contrary. Then π can be approximated by a CE map q so that $q(N_q)$ has dimension less than or equal to $(n-3)$. Since $F_1(B^2) \cap F_2(B^2) = \varnothing$, it follows that $h_1 = q \circ F_1$ and $h_2 = q \circ F_2$ have the property that $h_1(B^2) \cap h_2(B^2)$ has dimension less than or equal to $n-3$. By [8, p. 80], there exists a path α from $B^{n-2} \times \{1\}$ to $B^{n-2} \times \{-1\}$ in B^{n-1} so that $\pi(\alpha) \cap h_1(B^2) \cap h_2(B^2) = \varnothing$.

By choosing q close enough to π, it is possible to find approximate lifts f_1 and f_2 to h_1 and h_2 so that $f_1(B^2) \cap f_2(B^2) \cap \alpha = \varnothing$, and so that $\rho(f_e|\partial B^2, F_e|\partial B^2) < \varepsilon/2$. This contradicts hypothesis (iii) of the lemma and implies that G cannot be secretly $(n-3)$-dimensional.

LEMMA 2. *Assume that* SH_* *and* SH_i *hold for* $1 \leqq i < \infty$, *that the decomposition* G *associated with* $\mathscr{S}$ *is cellular, and that for* $1 \leqq i < \infty$ *the following condition holds*:

(a_i) *whenever* f_1, f_2 *are maps of* B^2 *into* E^n *in general position with respect to all the elements of* $\mathscr{M}_k$, $k \leqq i$, *and for which* $\rho(f_e|\partial B^2, F_e|\partial B^2) < \varepsilon/2$ *for* $e = 1, 2$, *then there exists* $A_i \in S_i$ *such that* f_1 *and* f_2 *are* $(A_i, \mathscr{M}_i)$ *slice maps. Moreover, in case* $i \geqq 2$, *the choice of* A_i *can be made so that* $A_i \subseteqq A_{i-1}$.
Then G *is intrinsically* $(n-2)$-*dimensional*.

Proof. It follows from SH_i that each nondegenerate element of G intersects B^{n-1} and that, for $x \in B^{n-1}$, $B^{n-1} \cap st_2(x, \mathscr{M}_i)$ has diameter less than 2^{4-i}. By Theorem 1, $\pi|B^{n-1}$ is an embedding and $\pi(N_\pi) = \pi(N_G) \subset \pi(B^{n-1})$. Moreover, Conditions ($a_i$), $1 \leqq i < \infty$, imply that hypothesis (iii) of Lemma 1 holds. Thus, all the hypotheses of that lemma are satisfied, and G must be intrinsically $(n-2)$-dimensional.

4. The construction. Lemma 2 indicates how the construction will proceed. A defining sequence $\mathscr{S}$ for a cellular decomposition G will be constructed in E^n so that SH* is satisfied. At each stage i, SH_i will be satisfied, as will Condition a_i from Lemma 2. The construction will complete the proof of the following theorem.

THEOREM 2. *For $n \geqq 3$, there exist intrinsically $(n-2)$-dimensional cellular* usc *decompositions of E^n.*

The following definition and lemma from [4] will be used in the course of the construction. Anyone familiar with the examples of wild Cantor sets in E^n constructed by Antoine [1] or Blankinship [2] may prefer to use the appropriate manifolds from their specific examples in place of the more general construction procedure used below.

DEFINITION. Let M be a manifold with boundary, H a disc with holes and f a map from H into M with $f(\partial H) \subset \partial M$. Then f is said to be *I-inessential* if there exists a map $\tilde{f}$ from H into ∂M with $f \mid \partial H = \tilde{f} \mid \partial H$. Otherwise, f is said to be *I-essential*.

LEMMA 3 [4, p. 147]. *Let S denote a closed* P.L. *$(n-2)$-manifold and $M = S \times B^2$. Choose $\varepsilon > 0$. Then there exists a finite collection $\{M_i\}$ of pairwise disjoint, locally flat manifolds in* Int (M) *such that*:

(i) *each M_i is homeomorphic to the product of B^2 and a closed* P.L. *$(n-2)$-manifold*;

(ii) *the diameter of M_i is less than ε; and*

(iii) *whenever H is a disc with holes and $g: H \to M$ is an I-essential map, then $g(H) \cap (\bigcup M_i) \neq \varnothing$.*

Stage 1. T_1: Let R_1 be as in SH 1 and S_1 be the trivial triangulation of $[-1, 1]$. Let $T_1 = R_1 \times S_1$.

$\mathscr{M}_1$: Let V be a P.L. embedded copy of

$$T^n \equiv B^2 \times \underbrace{S^1 \times \cdots \times S^1}_{n-2 \text{ copies}}$$

in $B^{n-1} \times [3, 4]$ and W a P.L. embedded copy of T^n in $B^{n-1} \times [-4, -3]$. $\mathscr{M}_1$ will have one element, $M(1)$, consisting of $B^{n-1} \times [-1, 1]$, V, W, and P.L. n-tubes joining $B^{n-1} \times \{1\}$ to V and $B^{n-1} \times \{-1\}$ to W.

Figure 1 shows $M(1)$ in the case $n = 3$.

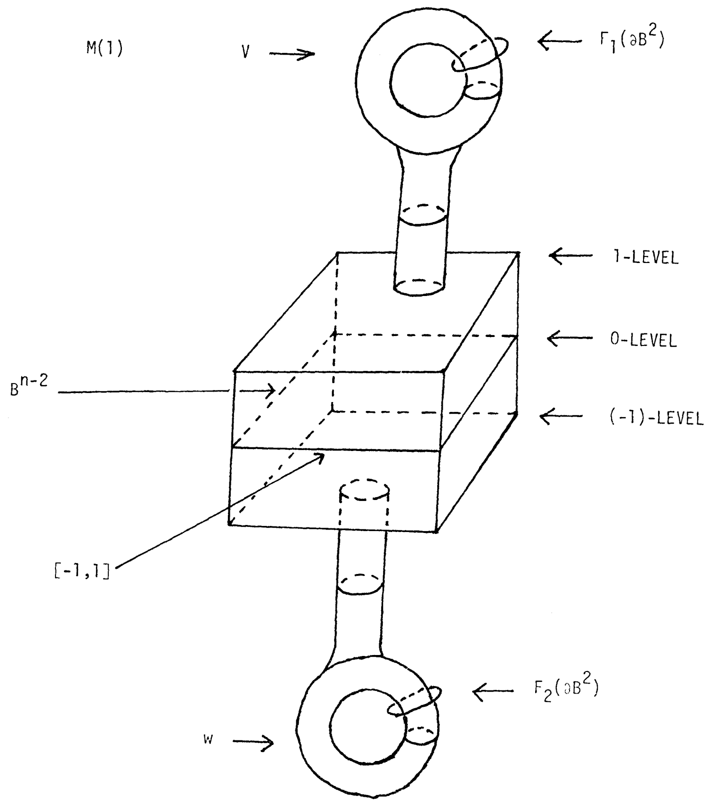

FIGURE 1.

SH 1: The choice of T_1 and $\mathscr{M}_1$ allows one to verify that SH 1 is satisfied.

Note 1. The construction allows one to choose $\varepsilon > 0$ and maps F_1, F_2 from B^2 into E^n so that

(i) $F_1(B^2) \cap F_2(B^2) = \emptyset$;

(ii) $\rho(F_e(\partial B^2), M(1)) > \varepsilon$ for $e = 1, 2$; and

(iii) whenever f_1 and f_2 are maps from B^2 into E^n in general position with respect to $M(1)$, and with $\rho(f_e | \partial B^2, F_e | \partial B^2) < \varepsilon/2$, $e = 1, 2$, then there exists a disc with holes H_1 (resp. L_1) so that $f_1 | H_1$ (resp. $f_2 | L_1$) is I-essential in V (resp. W).

To find F_1 (F_2) choose any embedding of B^2 in $E^{n-1} \times (0, \infty)$ (in $E^{n-1} \times (-\infty, 0)$) satisfying condition (ii) above and such $F_1(B^2) \cap V$ ($F_2(B^2) \cap W$) equals the image in V (W) of $B^2 \times pt. \times \cdots \times pt. \subset T^n$.

The above note yields immediately the fact that SH_* and Condition (a_1) of Lemma 2 are are satisfied.

Stage i. Assume that $\mathscr{M}_{i-1}$ has been constructed so that the following inductive hypotheses are true for $j = i - 1$.

IH I. SH_j and Condition a_j from Lemma 2 hold.

IH II. $\mathscr{V}_j$ ($\mathscr{W}_j$) is a collection of pairwise disjoint, connected, locally flat n-manifolds with boundary in V (W) of diameter less than $1/j$, and of the form $B^2 \times$ (closed $(n-2)$-manifold).

IH III. Each element m of $\mathscr{M}_j$ consists of (an $(n-1)$-cell of T_j) $\times [-1/j, 1/j]$ connected by n-tubes to a unique element $v(m)$ of $\mathscr{V}_j$ and also to a unique element $w(m)$ of $\mathscr{W}_j$. Furthermore, when $j > 1$ each $v \in \mathscr{V}_j$ ($w \in \mathscr{W}_j$) is contained in some flat n-cell C_v (C_w) that lies interior to some element of $\mathscr{V}_{j-1}$ ($\mathscr{W}_{j-1}$), and then, for $m \in \mathscr{M}_j$, $m \cup C_{v(m)} \cup C_{w(m)}$ is a flat n-cell Q_m such that

$$Q_m \cap (B^{n-1} \times [-1/j, 1/j]) = (\text{an } (n-1)\text{-cell of } T_j) \times [-1/j, 1/j] .$$

IH IV. Whenever f_1 and f_2 and A_j are as in Condition a_j of Lemma 2, P is an element of R_j and v and w are the elements of $\mathscr{V}_j$ and $\mathscr{W}_j$ associated with $P \times A_j$, there exists a disc with holes H (resp. L) in B^2 so that $f_1|H$ (resp. $f_2|L$) is I-essential in v (resp. w).

Note 2. The above inductive hypotheses are true for $j = 1$.

$\mathscr{M}_i$ will be constructed by considering each "slice" $B^{n-2} \times E$ (E an interval in S_{i-1}) separately. Focus attention on one such slice.

R_i: Let $P(1), \cdots, P(r)$ be the $(n-2)$-cells of R_{i-1}, and $v(1), \cdots, v(r)$, and $w(1), \cdots, w(r)$ the associated elements of $\mathscr{V}_{i-1}$ and $\mathscr{W}_{i-1}$ respectively.

As in SH (i-1), $r = 2^{(i-2)(n-2)}$. R_i is chosen as in SH i so that each $P(j)$, $1 \leqq j < r$, contains $s \equiv 2^{n-2}$ $(n-2)$-cells of R_i.

Finding interior manifolds. Consider a specific $P(j) \times E$, $1 \leqq j \leqq r$. Use Lemma 3, with $\varepsilon = 1/i$, to obtain a collection of n-manifolds with boundary satisfying the conclusions of Lemma 3 in the interior of $v(j)$ and $w(j)$.

Without loss of generality, the same number l of *interior manifolds* can be chosen in each $v(j)$ and $w(j)$ so that each interior manifold in $v(j)$ (resp. $w(j)$) is contained in a P.L. n-cell interior to $v(j)$ (resp. $w(j)$).

Note 3. There are l^{2r} distinct ways of choosing exactly one interior manifold from each $v(j)$ and $w(j)$, $1 \leqq j \leqq r$.

Ramifying the interior manifolds. Each *interior manifold* M is of the form $B^2 \times N$ for N a closed $(n-2)$-manifold. Choose $m \equiv s \cdot l^{(2r-1)}$ pairwise disjoint subdiscs $D_1, \cdots, D_m$ of B^2, and form m

"parallel interior" copies of $B^2 \times N$ by taking $D_1 \times N, \cdots, D_m \times N$.

$\mathscr{V}_i$, $\mathscr{W}_i$: The part of $\mathscr{V}_i$ (resp. $\mathscr{W}_i$) associated with the slice $B^2 \times E$ consists of the union of all the "parallel interior" manifolds constructed in $v(j)$ (resp. $w(j)$), $1 \leqq j \leqq r$.

Note 4. There are a total of $r \cdot s \cdot l^{2r}$ components of $\mathscr{V}_i$ (resp. $\mathscr{W}_i$) associated with the slice $B^{n-2} \times E$.

S_i, T_i: Subdivide E into l^{2r} equal subintervals, so that T_i has $r \cdot s \cdot l^{2r}$ $(n-1)$-cells in $B^{n-2} \times E$.

$\mathscr{M}_i$: For each of the l^{2r} choices mentioned in Note 3, choose a distinct slice $B^{n-2} \times \tilde{E}$ for $\tilde{E}$ in S_i. Thus, associated with $B^{n-2} \times \tilde{E}$, we have one of the original *interior manifolds* from each of $v(j)$ and $w(j)$, $1 \leqq j \leqq r$.

For each P in R_i with $P \subset R(j)$, tube $P \times \tilde{E} \times [-1/i, 1/i]$ to a parallel interior copy of the associated *interior manifolds* in $v(j)$ and $w(j)$. Do this by first choosing an n-cell C_v (resp. C_w) containing the target interior manifold in its interior, so that C_v (resp. C_w) is contained in the interior of v_j (resp. w_j). Run the tube from $B^{n-1} \times \{1\}$ (resp. $B^{n-1} \times \{-1\}$) directly to C_v (resp. C_w) and then, once inside that n-cell, threading the tube through it, never leaving the cell, over to the preselected element of $\mathscr{V}_i$ (resp. $\mathscr{W}_i$).

The number of parallel interior manifolds has been chosen so that each will be used exactly once. Then $\mathscr{M}_i$ consists of the manifolds resulting from the above tubing operation.

Note 5. At this point IH II is satisfied for $j = i$. If the tubing operation is done carefully enough, IH III and SH_i will also be true.

IH IV *and Condition* a_i: Condition a_i of Lemma 2 is implied by IH IV. What follows is a verification of IH IV in case $j = i$.

Let f_1, f_2 and A_{i-1} be as in Condition a_{i-1}, and assume, in addition, that f_1 and f_2 are in general position with respect to all of the elements of $\mathscr{M}_i$. By IH IV for $j = i - 1$, for each $P(k)$ of R_{i-1}, corresponding to the manifolds $v(k)$ and $w(k)$ associated with $P(k) \times A_{i-1}$ are discs with holes $H(k)$ and $L(k)$ such that $f_1|H(k)$ is I-essential in $v(k)$ and $f_2|L(k)$ is I-essential in $w(k)$. It follows from Lemma 3 that $v(k)$ (resp. $w(k)$) contains an interior manifold v_k (resp. w_k) such that, modulo another general position adjustment, there exists a disc with holes H_k (resp. L_k) in $H(k)$ (resp. $L(k)$) for which $f_1|H_k$ is I-essential in v_k ($f_2|L_k$ is I-essential in w_k). Then each of the parallel interior copies of v_k (w_k) must be hit in an I-essential way by f_1 (f_2).

Determination of v_k and w_k constitutes a choice as in Note 3. Thus, the construction of $\mathscr{M}_i$ associates a slice $B^{n-2} \times \tilde{E}$ with this choice and guarantees that IH IV holds for $j = i$.

Cellularity of G. This completes the inductive description of the defining sequence $\mathscr{S}$. It remains to be shown that the associated decomposition G is cellular.

Fix $x \in B^{n-1}$. SH_i, $1 \leqq i < \infty$, together with Theorem 1 implies that the element g of G containing x is obtained by taking $\bigcap_{k=1}^{\infty} st(x, \mathscr{M}_k)$. So it suffices to show that $\bigcap_{k=1}^{\infty} st(x, \mathscr{M}_k)$ is cellular. At some index $j = j(x)$ the number of elements of $\mathscr{M}_j$ contained in $st(x, \mathscr{M}_j)$ must stabilize since this number is bounded above by 2^{n-1}. When this occurs, any $m' \in \mathscr{M}_k$ in $st(x, \mathscr{M}_k)$, contains exactly one $m \in \mathscr{M}_{k+1}$ in $st(x, \mathscr{M}_{k+1})$, $k \geqq j$.

Using the notation of IH III, $st(x, \mathscr{M}_{k+1})$ is contained in the union X_{k+1} of all the n-cells Q_m, where $x \in m \in \mathscr{M}_{k+1}$, and X_{k+1} in turn is contained in $st(x, \mathscr{M}_k)$. It is easy to add the n-cells of X_{k+1} together, one at a time, to show that X_{k+1} is also a flat n-cell. If U is any open set containing $st(x, \mathscr{M}_k)$, X_{k+1} (possibly slightly thickened) is thus a flat n-cell with $st(x, \mathscr{M}_{k+1}) \subset \mathrm{Int}\,(X_{k+1}) \subset U$. It follows that $\bigcap_{k=1}^{\infty} st(x, \mathscr{M}_k)$ is cellular and that G is a cellular decomposition of E^n.

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