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ON THE ZEROS OF COMPOSITE POLYNOMIALS

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# ON THE ZEROS OF COMPOSITE POLYNOMIALS

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Let  $P(z) = \sum_{j=0}^{n} C(n,j) A_j z^j$  and  $Q(z) = \sum_{j=0}^{n} C(n,j) B_j z^j$ ,  $A_n B_n \rightleftharpoons 0$ , be two polynomials of the same degree n. If P(z) and Q(z) are apolar and if one of them has all its zeros in a circular region C, then according to a famous result known as Grace's theorem, the other will have at least one zero in C. In this paper we propose to relax the condition that P(z) and Q(z) are of the same degree. Instead, we will assume P(z) and Q(z) to be the polynomials of arbitrary degree n and m respectively,  $m \le n$ , with their coefficients satisfying an apolar type relation and obtain certain generalizations of Grace's theorem for the case when the circular region C is a circle |z| = r. As an application of these results, we also generalize some results of Szegö, Cohn and Egerváry.

Two polynomials

$$P(z)=\sum\limits_{j=0}^{n}C(n,\,j)A_{j}z^{j}$$
 and  $Q(z)=\sum\limits_{j=0}^{n}C(n,\,j)B_{j}z^{j}$  ,  $A_{n}B_{n}
eq 0$  ,

of the same degree n are said to be apolar if their coefficients satisfy the relation

$$(1) \quad A_0B_n-C(n,1)A_1B_{n-1}+C(n,2)A_2B_{n-2}+\cdots+(-1)^nA_nB_0=0$$
.

As to the relative location of the zeros of P(z) and Q(z), we have the following fundamental result due to Grace [1, p. 61].

THEOREM A. If P(z) and Q(z) are apolar polynomials and if one of them has all its zeros in a circlar region C, then the other will have at least one zero in C.

Here we propose to relax the condition that the polynomials P(z) and Q(z) are of the same degree and prove

THEOREM 1. If  $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$  and  $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$  are two polynomials of degree n and m respectively,  $m \leq n$ , such that

(2) 
$$C(m, 0)A_0B_m - C(m, 1)A_1B_{m-1} + \cdots + (-1)^mC(m, m)A_mB_0 = 0$$
, then the following holds.

(i) If Q(z) has all its zeros in the circle  $|z| \leq r$ , then P(z) has at least one zero in  $|z| \leq r$ .

(ii) If P(z) has all its zeros in the region  $|z| \ge r$ , then Q(z) has at least one zero in  $|z| \ge r$ .

REMARK 1. If in Theorem 1, the polynomial P(z) has all its zeros in a circle  $|z| \le r$  and m < n, then the polynomial Q(z) need not have any in zero in  $|z| \le r$ . For consider the polynomials

$$P(z) = 1 + z + z^2 + \cdots + z^n \equiv \sum_{j=0}^n C(n, j) A_j z^j$$
 ,  $n > 1$ 

and

$$Q(z) = n + z ,$$

then m = 1 < n and the relation (2) is satisfied. But P(z) has all its zeros in the circle  $|z| \le 1$ , whereas the only zero of Q(z) lies in |z| > 1.

In case m = n, Theorem 1 reduces to Theorem A when the circular region C is the circle |z| = r.

For the proof of Theorem 1, we need the following lemmas.

LEMMA 1. If all the zeros of a polynomial P(z) of degree n lie in |z| > r, r > 0 and  $|w| \le r$ , then the polynomial

$$P_1(z) = nP(z) + (w-z)P'(z)$$

has all its zeros in |z| > r.

This result is, essentially, due to Szegö [2]. For the sake of completeness we shall present an independent proof of this lemma.

*Proof of Lemma* 1. The polynomial P(z) has all its zeros in |z| > r > 0, therefore, if  $z_1, z_2, \dots, z_n$  are the zeros of P(z), then  $|z_j| > r$  for  $j = 1, 2, \dots, n$  and

$$\frac{zP'(z)}{P(z)} = \sum_{j=1}^n \frac{z}{z-z_j}.$$

Now if  $z = re^{i\theta}$ ,  $0 \le \theta < 2\pi$ , then we have

$$ext{Re}\,rac{re^{i heta}P'(re^{i heta})}{P(re^{i heta})}=\sum_{j=1}^n ext{Re}\,rac{re^{i heta}}{re^{i heta}-z_i}<\sum_{j=1}^nrac{1}{2}=rac{n}{2}\;.$$

This implies

$$\operatorname{Re}rac{zP'(z)}{nP(z)}<rac{1}{2} \qquad ext{for} \quad |z|=r \; .$$

Equivalently

$$\left|rac{zP'(z)}{nP(z)}
ight|<\left|1-rac{zP'(z)}{nP(z)}
ight| \qquad ext{for} \quad |z|=r \; .$$

Since  $P(z) \neq 0$  for |z| = r, it follows that

(3) 
$$|zP'(z)| < |nP(z) - zP'(z)|$$
 for  $|z| = r$ .

Applying Rouche's theorem and noting that  $P(z) \neq 0$  for  $|z| \leq r$ , we conclude that the polynomial nP(z) - zP'(z) has no zero in |z| < r. If now w is any complex number such that  $|w| \leq r$ , then from (3) we have

$$|wP'(z)| \le r|P'(z)| = |zP'(z)| < |nP(z) - zP'(z)|$$
 for  $|z| = r$ .

This implies according to Rouche's theorem again, that the polynomials nP(z)-zP'(z) and nP(z)+(w-z)P'(z) have the same number of zeros in |z| < r. Consequently, the polynomial nP(z)+(w-z)P'(z) has no zero in |z| < r. This polynomial does not vanish for |z|=r either. Because, if for some  $z=z_0$ , with  $|z_0|=r$ 

$$nP(z_0) + (w - z_0)P'(z_0) = 0$$
,

then

$$|nP(z_0)-z_0P'(z_0)|=|wP'(z_0)| \leq r|P'(z_0)|=|z_0P'(z_0)|.$$

But this is a contradiction to (3). Hence we conclude that the polynomial nP(z) + (w-z)P'(z) has no zero in  $|z| \le r$  and this proves the lemma.

An immediate consequence of Lemma 1 is the following

LEMMA 2. If all the zeros of a polynomial P(z) of degree n lie in  $|z| \ge r$ , r > 0 and |w| < r, then the polynomial

$$P_{\scriptscriptstyle 1}(z) = nP(z) + (w-z)P'(z)$$

has all its zeros in  $|z| \ge r$ .

We also need

LEMMA 3 [1, p. 52, Eq. (13, 9)]. If  $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$  is a polynomial of degree n and  $w_1, w_2, \dots, w_m$  are  $m, m \leq n$ , arbitrary real or complex numbers, then the kth polar derivative

$$P_{k}(z)=(n-k+1)P_{k-1}(z)+(w_{k}-z)P_{k-1}'(z)$$
 ,  $k=1,\,2,\,\cdots,\,m$  of  $P(z)$  with  $P_{0}(z)=P(z)$ , can be written in the form

$$P_k(z) = \sum_{j=0}^{n-k} C(n-k, j) A_j^{(k)} z^j$$
,

where

$$A_{j}^{(k)} = n(n-1) \cdot \cdot \cdot (n-k+1) \sum_{i=0}^{k} S(k,i) A_{i+j}$$
,

and S(k, i) being the symmetric function consisting of the sum of all possible products of  $w_1, w_2, \dots, w_k$  taken i at a time.

*Proof of Theorem* 1. Let  $w_1, w_2, \dots, w_m$  be the zeros of Q(z), so that we have

$$(\ 4\ ) \qquad \sum_{j=0}^m C(m,\ j) B_j z^j = B_m (z-w_1) (z-w_2) \, \cdots \, (z-w_m) \; .$$

Equating the coefficients of the like powers of z on the two sides of (4), we get

(5) 
$$C(m, j)B_{m-j} = C(m, m-j)B_{m-j} = (-1)^{j}S(m, j)B_{m}$$

where S(m, j) is the symmetric function consisting of the sum of all possible products of  $w_1, w_2, \dots, w_m$  taken j at a time.

Now suppose that all the zeros of Q(z) lie in  $|z| \leq r$ . We have to show that at least one zero of P(z) lies in  $|z| \leq r$ . Assume the contrary. That is, assume that the polynomial P(z) has all its zeros in |z| > r. Since  $|w_i| \leq r$ ,  $i = 1, 2, \dots, m$  it follows by the repeated applications of Lemma 1 that all the zeros of each polar derivative

(6) 
$$P_k(z) = (n-k+1)P_{k-1}(z) + (w_k-z)P'_{k-1}(z), \quad k=1,2,\cdots,m$$

also lie in |z| > r. Hence in particular all the zeros of  $P_m(z)$  lie in |z| > r. But by Lemma 3,  $P_m(z)$  can be written as

$$(7) P_m(z) = \sum_{j=0}^{n-m} C(n-m, j) A_j^{(m)} z^j,$$

were

$$A_{j}^{(m)} = n(n-1)\cdots(n-m+1)\sum_{i=0}^{m}S(m,i)A_{i+j}$$
  
=  $\frac{n(n-1)\cdots(n-m+1)}{B_{m}}\sum_{i=0}^{m}(-1)^{i}C(m,i)B_{m-i}A_{i+j}$ .

Since by hypothesis

$$A_0^{(m)} = \frac{n(n-1)\cdots(n-m+1)}{B_m} \sum_{i=0}^m (-1)^i C(m,i) B_{m-i} A_i = 0$$
,

therefore, if n > m, we get from (7)  $P_m(0) = 0$ . This shows that z = 0 is a zero of  $P_m(z)$ , which is a contradiction to (6). In case n = m, from (7) we have

$$P_m(z) \equiv A_0^{(m)} = 0.$$

Since

$$P_m(z) = P_{m-1}(z) + (w_m - z)P'_{m-1}(z)$$
,

it follows that

$$P_{\scriptscriptstyle m-1}(w_{\scriptscriptstyle m})=0.$$

But  $|w_m| \le r$ , this contradicts (6) again. Hence in any case we conclude that P(z) must have a zero in  $|z| \le r$ . This completes the proof of the first part of the theorem.

To establish part (ii) of Theorem 1, we suppose that all the zeros of P(z) lie in  $|z| \ge r$ . We have to show that at least one zero of Q(z) lies in  $|z| \ge r$ . Assume that all the zeros of Q(z) lie in |z| < r, so that  $|w_i| < r$ ,  $i = 1, 2, \cdots, m$ . Then it follows by the repeated applications of Lemma 2 that all the zeros of each polar derivative

$$P_k(z) = (n-k+1)P_{k-1}(z) + (w_k-z)P'_{k-1}(z)$$
,  $k=1, 2, \cdots, m$ ,

lie in  $|z| \ge r$ . We shall now proceed similarly as before and complete the proof of the 2nd part of the theorem.

We may apply Theorem 1 to the polynomials  $z^n$  P(1/z) and  $z^mQ(1/z)$  to get the following

COROLLARY 1. If  $P(z) = \sum_{j=0}^n C(n, j) A_j z^j$ ,  $A_0 A_n \neq 0$  and  $Q(z) = \sum_{j=0}^m C(m, j) B_j z^j$ ,  $B_0 B_m \neq 0$  are two polynomials of degree n and m respectively,  $m \leq n$ , such that

- (8)  $C(m, 0)B_0A_n C(m, 1)B_1A_{n-1} + \cdots + (-1)^mC(m, m)B_mA_{n-m} = 0$ , then the following holds.
- (i) If Q(z) has all its zeros in  $|z| \ge r$ , then P(z) has at least one zero in  $|z| \ge r$ .
- (ii) If P(z) has all its zeros in  $|z| \le r$ , then Q(z) has at least one zero in  $|z| \le r$ .

The next corollary is obtained by applying Theorem 1 to the polynomials P(z) and  $z^m$  Q(1/z) with r=1.

COROLLARY 2. If  $P(z)=\sum_{j=0}^n C(n,j)A_jz^j$ ,  $A_n\neq 0$  and  $Q(z)=\sum_{j=0}^m C(m,j)B_jz^j$ ,  $B_0B_m\neq 0$  are two polynomials of degree n and m

respectively,  $m \leq n$ , such that

$$(9) \qquad C(m, 0)A_0B_0 - C(m, 1)A_1B_1 + \cdots + (-1)^mC(m, m)A_mB_m = 0$$
,

then the following holds.

- (i) If Q(z) has all its zeros in  $|z| \ge 1$ , then P(z) has at least one zero in  $|z| \le 1$ .
- (ii) If P(z) has all its zeros in  $|z| \ge 1$ , then Q(z) has at least one zero in  $|z| \le 1$ .

If we apply Theorem 1 to the polynomials  $z^n$  P(1/z) and Q(z) with r=1, we get the following

COROLLARY 3. If  $P(z) = \sum_{j=0}^n C(n, j) A_j z^j$ ,  $A_0 A_n \neq 0$  and  $Q(z) = \sum_{j=0}^m C(m, j) B_j z^j$ ,  $B_m \neq 0$  are two polynomials of degree n and m respectively,  $m \leq n$ , such that

(10) 
$$C(m, 0)A_nB_m - C(m, 1)A_{n-1}B_{m-1} + \cdots + (-1)^mC(m, m)A_{n-m}B_0 = 0$$
, then we have the following:

- (i) If Q(z) has all its zeros in  $|z| \le 1$ , then P(z) has at least one zero in  $|z| \ge 1$ .
- (ii) If P(z) has all its zeros in  $|z| \leq 1$ , then Q(z) has at least one zero in  $|z| \geq 1$ .

As an application of Theorem 1, we shall deduce the following partial generalization of a result due to Szegö [1, p. 65].

THEOREM 2. If all the zeros of a polynomial  $P(z) = \sum_{j=0}^{n} C(n, j) A_{j} z^{j}$  of degree n lie in  $|z| \geq r$  and if  $\beta$  is a zero of the polynomial  $Q(z) = \sum_{j=0}^{m} C(m, j) B_{j} z^{j}$ ,  $B_{0} B_{m} \neq 0$  of degree m,  $m \leq n$ , then every zero w of the polynomial  $R(z) = \sum_{j=0}^{m} C(m, j) A_{j} B_{j} z^{j}$  of degree m, has the form  $w = -\alpha \beta$  where  $\alpha$  is a suitably chosen point in  $|z| \geq r$ .

Proof of Theorem 2. If w is a zero of R(z), then

(11) 
$$R(w) = \sum_{i=0}^{m} C(m, j) A_{j} B_{j} w^{j} = 0.$$

Equation (11) shows that the polynomials

$$P(z) = C(n, 0)A_0 + C(n, 1)A_1z + \cdots + C(n, n)A_nz^n$$

and

$$z^{m}Q(-w/z) = C(m, 0)(-1)^{m}B_{m}w^{m} + \cdots - C(m, m-1)B_{1}wz^{m-1} + C(m, m)B_{0}z^{m}$$

satisfy the condition of Theorem 1. Since all the zeros of P(z) lie in  $|z| \ge r$ , it follows from the 2nd part of Theorem 1 that  $z^m Q(-w/z)$  has at least one zero in  $|z| \ge r$ . If  $\beta_1, \beta_2, \dots, \beta_m$  are the zeros of Q(z), then the zeros of  $z^m Q(-w/z)$  are  $-w/\beta_1, -w/\beta_2, \dots, -w/\beta_m$ . One of these zeros must be  $\alpha$  where  $|\alpha| \ge r$ . Therefore, we must have  $w = -\alpha\beta_j$  for some  $j = 1, 2, \dots, m$ . This complete the proof.

Exactly in the same way as Theorem 2, we may deduce the following result from the 2nd part of Corollary 1.

THEOREM 3. If all the zeros of the polynomial  $P(z) = \sum_{j=0}^n C(n,j)A_jz^j$  of degree n lie in  $|z| \leq r$  and if  $\beta$  is a zero of the polynomial  $Q(z) = \sum_{j=0}^m C(m,j)B_jz^j$ ,  $B_0B_m \neq 0$ , then every zero w of the polynomial

$$R(z) = \sum\limits_{i=0}^m C(m,\,j) A_{n-m+j} B_j z^j$$
 ,  $m \leq n$  ,

has the form  $w = -\alpha \beta$  where  $\alpha$  is a suitably chosen point in  $|z| \leq r$ .

From Theorem 3, we immediately deduce the following corollary which presents a generalization of a result due Cohn and Egerváry [1, p. 66, Cor. (16, 1a)].

COROLLARY 4. If all the zeros of  $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$  of degree n lie in  $|z| \leq r$  and if all the zeros of  $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$  of degree m lie in |z| < s,  $m \leq n$ , then all the zeros of the polynomial

$$R(z) = \sum_{j=0}^{m} C(m, j) A_{n-m+j} B_{j} z^{j}$$

of degree m lie in |z| < rs.

This follows from the fact that  $|\alpha| \le r$  and  $|\beta| < s$  implies |w| < rs.

REMARK 2. In very much the same way as above, we can deduce from Theorem 1 and from Corollaries 1-3 many other interesting results.

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