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# BEST POSSIBLE RESULTS IN A CLASS OF INEQUALITIES

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**In this paper we shall prove the following theorem.**

THEOREM. **Suppose** $1 < p \leqq \infty$, **and** $rp > 1$ **if** $p < \infty$, $r > 0$ **if** $p = \infty$. **Suppose the matrix** $A = (a_{nk})$ **with** $a_{nk} = n^{-r}$ $(k \leqq n)$, $a_{nk} = 0$ $(k > n)$. **Suppose** $\tilde{w}$ **be the subset of** $w$ **consisting of nonnegative, monotone sequences. Then** $\{n^{r-1}\}_n$ **is maximum, with respect to** $<$, **in** $I$ **where**

$$I = \{b \in \tilde{w}: \text{ for some } K > 0, \ \| A \,|\, bx \|\|_p \leqq K \| x \|_p$$

**for all** $x \in l_p\}$.

1. Let $w$ be the space of all real or complex sequences. For $x = \{x_n\} \in w$, $y = \{y_n\} \in w$, $xy$ is the sequence $\{x_n y_n\}$, and $|x| = \{|x_n|\}$. Let $A = (a_{nk})$ $(n, k = 1, 2, \cdots)$ be a lower triangular matrix with nonnegative entries. $Ax$, the $A$ transform of a sequence $x \in w$, is given by

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k \, .$$

$l_p$ space will have its usual meaning and $\| \cdot \|_p$ will denote the usual norm, if $1 \leqq p \leqq \infty$, or quasinorm, if $0 < p < 1$.

Let $C = (c_{nk})$ be the Cesàro matrix that is $c_{nk} = n^{-1}$ $(k = 1, 2, \cdots, n)$ and $c_{nk} = 0$ $(k > n)$. Hardy [2, Theorem 215] proved the following inequality:

THEOREM A. *If* $p > 1$, $x_n \geqq 0$ $(n = 1, 2, \cdots)$, *then*

$$\| Cx \|_p \leqq \left( \frac{p}{p-1} \right) \| x \|_p \, . \tag{1}$$

Subsequently Petersen [4] and Davies and Petersen [1] generalized Theorem A by replacing the Cesàro matrix $C$ by a lower triangular matrix $A$ satisfying certain conditions. Their results were of the form

$$\| Ax \|_p \leqq C(p) \| dx \|_p$$

where $d = \{d_n\}$ is a suitable sequence and $C(p)$ is a positive constant which depends upon $p$.

In [3] we had tried to systematize and unify inequality results of the following form:

There exists $K > 0$ such that for all $x \in \mu$,

$$\|A|bx|\|_\lambda \leqq K\|x\|_\mu , \tag{2}$$

where $(\lambda, \|\cdot\|_\lambda)$, $(\mu, \|\cdot\|_\mu)$ are normal, normally quasinormed $FK$ spaces, $b$ is a sequence in $\lambda$, and $A$ is a lower triangular matrix with nonnegative entries.

In [3] we ordered the sequences in $w$ by defining that for $b, c \in w$, $b < c$ ($b$ is less than $c$) if and only if, for some $M > 0$, $|b_n| \leqq M|c_n|$ for all $n$.

Now one can observe that an inequality of the form (2) is better the smaller the $K$, or the larger the sequence $b$ with the notion of largeness of sequences defined above.

It is useful to note that (see [3], Proposition 3.1) if a set $S$ of sequences is closed under addition, and has the property that $x \in S$ implies $|x| \in S$ (which is satisfied if every sequence in $S$ is nonnegative), then a sequence $b \in S$ is maximal in $S$ with respect to $<$(meaning that $x \in S$ and $b < x$ imply that $x < b$) if and only if $b$ is maximum in $S$ (meaning $x < b$ for every $x \in S$).

In [3, remarks at the end of §8] we have proved the following:

THEOREM B. *Suppose* $1 < p \leqq \infty$, *and* $rp > 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^{rp}} \left| \sum_{k=1}^{n} k^{r-1} x_k \right|^p \leqq K(r, p) \sum_{k=1}^{\infty} |x_k|^p , \tag{3}$$

*where* $K(r, p)$ *is a positive constant which depends upon* $r$ *and* $p$.

If we write $A = (a_{nk})$ with $a_{nk} = n^{-r}$ $(k = 1, \cdots, n)$ and $a_{nk} = 0$ $(k > n)$, and $\lambda = \mu = \ell_p$, then we find that Theorem B is a result of the form (2). It was shown in [3, Theorem 9.3] that the inequality (2) with $b = \{n^{r-1}\}$ is not best possible, and that, indeed, there is no best possible inequality (2) for this triple $A$, $\lambda$, $\mu$. (It is interesting to note that if $p = 1$, $r > 1$, the inequality (2) with $b = \{n^{r-1}\}$ holds and is best possible (by [3, Corollary 4.9]).) However, this result was a little unsatisfying because, in the special case $r = 1$, the only sequences $b$ satisfying (2) and strictly greater than $e = \{1, 1, \cdots\}$, which we could actually find were far from monotone. They were like $e$ with a subsequence tending to infinity thrown in, sparsely.

In [3, Problem 9.4], we had wondered if $\{n^{r-1}\}_n$ is maximal (equivalently, maximum) among the monotone nonnegative sequences satisfying (2). The object of this note is to show that the answer to this question is in fact in the affirmative. Thus, in particular, Hardy's inequality is the best in its class of inequalities.

We shall precisely prove the following:

THEOREM. *Suppose* $1 < p \leqq \infty$, *and* $rp > 1$ *if* $p < \infty$, $r > 0$ *if*

*$p = \infty$. Suppose the matrix $A = (a_{nk})$ with $a_{nk} = n^{-r}$ $(k \leqq n)$, $a_{nk} = 0$ $(k > n)$. Suppose $\tilde{w}$ be the subset of $w$ consisting of nonnegative, monotone sequences. Then $\{n^{r-1}\}_n$ is maximum, with respect to $<$, in $I$ where*

$$I = \{b \in \tilde{w}: \text{ for some } K > 0,\ \| A|bx| \|_p \leqq K \| x \|_p$$

*for all $x \in \ell_p$}.*

*Proof.* It has been mentioned earlier that $\{n^{r-1}\} \in I$ (see [3], §8).

Suppose $b \in I$, and $b \not< \{n^{r-1}\}$. Then $\{n^{1-r}b_n\}_n$ is unbounded. We shall first show that

$$\left\{n^{-r} \sum_{k=1}^{n} b_k\right\}_n \notin \ell_\infty \cdots . \tag{4}$$

If $b$ is nonincreasing, then

$$n^{-r} \sum_{k=1}^{n} b_k \geqq n^{1-r} b_n ,$$

and if $b$ is nondecreasing, then

$$(2n)^{-r} \sum_{k=1}^{2n} b_k \geqq (2n)^{-r} \sum_{k=n}^{2n} b_k \geqq 2^{-r} n^{1-r} b_n ,$$

and we see the truth of (4).

Now, let

$$x^{(m)} = \sum_{k=1}^{m} e_k ,$$

where $e_k$ is the usual coordinate sequence with 1 at the $k$th entry and zero elsewhere. The theorem will be proved if we only show that the sequence

$$\{\| x^{(m)} \|_p^{-1} \| A|bx^{(m)}| \|_p\}_m$$

is unbounded.

Let $p = \infty$ we see that

$$\| x^{(m)} \|_p^{-1} \| A|bx^{(m)}| \|_p \geqq m^{-r} \sum_{k=1}^{m} b_k .$$

The theorem follows by (4).

Let $p < \infty$. Then

$$\begin{aligned} \| x^{(m)} \|_p^{-1} \| A|bx^{(m)}| \|_p &\geqq \left(m^{-1} \sum_{n=m}^{\infty} n^{-rp} \left(\sum_{k=1}^{m} b_k\right)^p\right)^{1/p} \\ &= \left(m^{-1} \left(\sum_{k=1}^{m} b_k\right)^p \left(\sum_{n=m}^{\infty} n^{-rp}\right)\right)^{1/p} \end{aligned}$$

$$\geqq M\left[ m^{-1}\left(\sum_{k=1}^{m} b_k\right)^p m^{1-rp} \right]^{1/p}$$

$$= Mm^{-r} \sum_{k=1}^{m} b_k$$

where $M$ is a positive constant independent of $m$. The proof is complete by appealing to (4).

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