ON MAPS RELATED TO σ-LOCALLY FINITE AND σ-DISCRETE COLLECTIONS OF SETS

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Over the past decade or so, two closely related classes of maps have been introduced and studied independently: The co-σ-discrete maps of R. W. Hansell and the σ-locally finite maps of the author. The principal purpose of this note is, first, to study the relationship between these maps and introduce consistent terminology (which Hansell is also adopting), and second, to prove a theorem which relates these maps to quotient s-maps between metric spaces.

The paper is arranged as follows. General properties of our maps, which were previously studied by Hansell and the author in [4], [5], [9], [11], are established in §§ 2-4. After summarizing some (mostly known) results relating our maps to open maps and closed maps in § 5, we shall prove a result (see Theorem 6.1) about quotient s-maps between metric spaces which will be applied by Hansell in [6]. In § 7 we obtain some examples related to results in §§ 5 and 6, and in § 8 we extend a result from [11] and fill a gap in its proof.

No continuity or separation properties are assumed unless explicitly indicated.

2. **Base-σ-locally finite and base-σ-discrete maps.** We begin by establishing some terminology. The collection of subsets of X is denoted by $\mathscr{P}(X)$. If $\mathscr{E}, \mathscr{F} \subset \mathscr{P}(X)$, then $\mathscr{F}$ is a *refinement* of $\mathscr{E}$ if $\bigcup \mathscr{F} = \bigcup \mathscr{E}$ and every $F \in \mathscr{F}$ is a subset of some $E \in \mathscr{E}$, and $\mathscr{F}$ is a *base* for $\mathscr{E}$ if every $E \in \mathscr{E}$ is a union of elements of $\mathscr{F}$.[1] If $\mathscr{E} \subset \mathscr{P}(X)$, then $\mathscr{E}$ is *locally finite* (resp. *discrete*) if every $x \in X$ has a neighborhood intersecting at most finitely many (resp. one) $E \in \mathscr{E}$; we call $\mathscr{E}$ σ-*locally finite* (resp. σ-*discrete*) if $E = \bigcup_{n=1}^{\infty} \mathscr{E}_n$ with each $\mathscr{E}_n$ locally finite (resp. discrete).

DEFINITION 2.1. A map $f: X \to Y$ is *base-σ-locally finite* (resp. *base-σ-discrete*) if, whenever $\mathscr{E} \subset \mathscr{P}(X)$ is locally finite (resp. discrete), then $f(\mathscr{E})$ has a σ-locally finite (resp. σ-discrete) base.[2]

Most of our results for base-σ-locally finite and base-σ-discrete

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[1] This definition of a base for $\mathscr{E}$ is due to Hansell [5]. A base for $\mathscr{E}$ which is also a refinement for $\mathscr{E}$ was called a *baselike refinement* for $\mathscr{E}$ by the author in [11]; clearly every base for $\mathscr{E}$ contains a baselike refinement for $\mathscr{E}$.

[2] Base-σ-discrete maps were called co-σ-discrete by Hansell in [4] and [5]. (Note, however, that base-σ-locally finite maps do *not* coincide with the σ-locally finite maps introduced in [11]; see Footnote 4).

maps are parallel. We shall state and prove these results only for base-σ-locally finite maps, and indicate the validity of the parallel result by Footnote 3.

Before stating the main result of this section, we record the following lemma from [11].

LEMMA 2.2. [11, Lemma 2.1]. *Every locally finite $\mathscr{E} \subset \mathscr{P}(X)$ has a locally finite, disjoint base.*

PROPOSITION 2.3.[3] *The following properties of a map $f: X \to Y$ are equivalent.*

(a) *f is base-σ-locally finite.*

(b) *Every locally finite $\mathscr{E} \subset \mathscr{P}(X)$ has a base $\mathscr{B}$ such that $f(\mathscr{B})$ is σ-locally finite.*

(c) *Every locally finite $\mathscr{E} \subset \mathscr{P}(X)$ has a refinement $\mathscr{F}$ such that $f(\mathscr{F})$ is σ-locally finite.*

(d) *Every locally finite $\mathscr{E} \subset \mathscr{P}(X)$ has a refinement $\mathscr{F}$ such that $f(\mathscr{F})$ has a σ-locally finite base.*

Proof. (a) $\to$ (b). If $\mathscr{E} \subset \mathscr{P}(X)$ is locally finite, then $f(\mathscr{E})$ has a σ-locally finite base $\mathscr{B}$ by (a). Let

$$\mathscr{B}' = \{E \cap f^{-1}(B): E \in \mathscr{E}, \ B \in \mathscr{B}, \ B \subset f(E)\} \ .$$

Then $\mathscr{B}'$ is a base for $\mathscr{E}$, and $f(\mathscr{B}') = \mathscr{B}$ is σ-locally finite.

(b) $\to$ (a). Clear, for if $\mathscr{B}$ is a base for $\mathscr{E}$ then $f(\mathscr{B})$ is a base for $f(\mathscr{E})$.

(b) $\to$ (c) $\to$ (d). Clear.

(d) $\to$ (b). Let $\mathscr{E} \subset \mathscr{P}(X)$ be locally finite. By Lemma 2.2, $\mathscr{E}$ has a locally finite, disjoint base $\mathscr{D}$. By (d), $\mathscr{D}$ has a refinement $\mathscr{F}$ such that $f(\mathscr{F})$ has a σ-locally finite base, and hence, as in the above proof of (a) $\to$ (b), $\mathscr{F}$ has a base $\mathscr{B}$ with $f(\mathscr{B})$ σ-locally finite. Since $\mathscr{D}$ is disjoint, $\mathscr{F}$ is actually a *base* for $\mathscr{D}$, and hence $\mathscr{B}$ is a base for $\mathscr{E}$.

REMARK. I don't know whether $f: X \to Y$ being base-σ-locally finite is equivalent to the formally weaker requirement that, whenever $\mathscr{E} \subset \mathscr{P}(X)$ is locally finite, then $f(\mathscr{E})$ has a σ-locally finite *refinement*. See, however, Proposition 4.4 and the discussion following it.

We now turn to the relationship between base-σ-locally finite and base-σ-discrete maps. Recall from [1] that a space X is *subparacompact* if every open cover of X has a σ-discrete closed refine-

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[3] This results remains valid with "locally finite" replaced everywhere by "discrete".

ment (or, equivalently by [1, Theorem 1.2], a σ-locally finite closed refinement). Clearly every paracompact space is subparacompact.

LEMMA 2.4. *If Z is subparacompact, then every locally countable $\mathscr{E} \subset \mathscr{P}(Z)$ has a σ-discrete base.*

Proof. Let $\mathscr{U}$ be an open cover of Z such that each $U \in \mathscr{U}$ intersects at most countably many $E \in \mathscr{E}$, and let $\mathscr{A}$ be a σ-discrete refinement of $\mathscr{U}$. Then $\{A \cap E : A \in \mathscr{A}, E \in \mathscr{E}\}$ is a σ-discrete base for $\mathscr{E}$.

COROLLARY 2.5. *If X is subparacompact, then every base-σ-discrete map $f: X \to Y$ is base-σ-locally finite.*

COROLLARY 2.6. *If Y is subparacompact, then every base-σ-locally finite map $f: X \to Y$ is base-σ-discrete.*

It follows from Corollaries 2.5 and 2.6 that base-σ-locally finite and base-σ-discrete maps are identical under the mild restriction that domain and range be subparacompact. I don't know whether they are identical without any restriction, at least for maps between completely regular spaces.

3. **Indexed collections and maps.** This section deals with the indexed analogues of such concepts as discrete, locally finite, and point-countable, and with maps defined in terms of them. All these indexed concepts are pretty self-explanatory; for example, an indexed collection $\{E_\alpha : \alpha \in A\}$ of subsets of X is *index-σ-locally finite* if $A = \bigcup_{n=1}^{\infty} A_n$ such that, for each n, $\{E_\alpha : \alpha \in A_n\}$ is index-locally finite (i.e., each $x \in X$ has a neighborhood U which intersects E_α for only finitely many $\alpha \in A_n$). We shall use the prefix "index-" whenever appropriate, and we make the convention that if $f: X \to Y$ and $\mathscr{E} \subset \mathscr{P}(X)$, then $\{f(E): E \in \mathscr{E}\}$ is to be regarded as being indexed by $\mathscr{E}$.

We begin with the following lemma, whose simple proof is omitted.

LEMMA 3.1.[3)] *The following are equivalent for any indexed $\{E_\alpha\} \subset \mathscr{P}(X)$.*
   (a) *$\{E_\alpha\}$ is index-σ-locally finite.*
   (b) *$\{E_\alpha\}$ is σ-locally finite, and $\{\alpha : E_\alpha = S\}$ is countable for every $S \subset X$.*

Following Hansell [5], we call an indexed collection $\{E_\alpha\} \subset \mathscr{P}(X)$

σ-*discretely decomposable* if $E_\alpha = \bigcup_{n=1}^{\infty} E_{\alpha,n}$ with $\{E_{\alpha,n}\}_\alpha$ index-discrete for all n; σ-*locally finitely decomposable* is defined analogously. We now have the following characterization, where the implication (c) → (a) was kindly communicated to me by R. W. Hansell.

LEMMA 3.2.[3] *The following are equivalent for any indexed* $\{E_\alpha\} \subset \mathscr{P}(X)$.

(a) $\{E_\alpha\}$ *is* σ-*locally finitely decomposable.*

(b) $\{E_\alpha\}$ *is index-point-countable and has a* σ-*locally finite base.*

(c) $\{E_\alpha\}$ *is index-point-countable, and if* $E_\alpha' \subset E_\alpha$ *for all* α, *then* $\{E_\alpha'\}$ *has a* σ-*locally finite refinement.*

Proof. (a) → (b) and (b) → (c). Clear.

(b) → (a). Let $\mathscr{B}$ be a σ-locally finite base for $\{E_\alpha\}$, with $B \neq \varnothing$ for all $B \in \mathscr{B}$. For each $B \in \mathscr{B}$, let $A(B) = \{\alpha: B \subset E_\alpha\}$. Then $A(B)$ is countable (since $\{E_\alpha\}$ is index-point-countable), so we can write $A(B) = \{\alpha_m(B): m \in N\}$. Write $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$, with each $\mathscr{B}_n$ locally finite. For each α, let

$$E_{\alpha,m,n} = \bigcup \{B \in \mathscr{B}_n: \alpha_m(B) = \alpha\} \, .$$

Then $\{E_{\alpha,m,n}\}$ is a σ-locally finite decomposition of $\{E_\alpha\}$.

(c) → (b). For each x, write $\{\alpha: x \in E_\alpha\}$ as $\{\alpha_n(x): n \in N\}$. For each α and n, let $E_{\alpha,n} = \{x \in E_\alpha: \alpha_n(x) = \alpha\}$. Then $E_\alpha = \bigcup_n E_{\alpha,n}$ for all α, and $\{E_{\alpha,n}\}_\alpha$ is disjoint for all n. By (c), $\{E_{\alpha,n}\}_\alpha$ has a σ-locally finite refinement $\mathscr{S}_n$; since $\{E_{\alpha,n}\}_\alpha$ is disjoint, $\mathscr{S}_n$ is actually a base $\{E_{\alpha,n}\}_\alpha$. Hence $\mathscr{S} = \bigcup_n \mathscr{S}_n$ is a σ-locally finite base for $\{E_\alpha\}$.

DEFINITION 3.3. A map $f: X \to Y$ is *index-σ-locally finite* (resp. *index-σ-discrete*) if, whenever $\{E_\alpha\} \subset \mathscr{P}(X)$ is index-locally finite (resp. index-discrete), then $\{f(E_\alpha)\}$ is σ-locally finitely decomposable (resp. σ-discretely decomposable).[4]

Analogously to Proposition 2.3, we now have the following result.

PROPOSITION 3.4.[3] *The following properties of a map* $f: X \to Y$ *are equivalent.*

(a) f *is index-σ-locally finite.*

(b) *Every locally finite* $\mathscr{E} \subset \mathscr{P}(X)$ *has a base* $\mathscr{B}$ *such that* $\{f(B): B \in \mathscr{B}\}$ *is index-σ-locally finite.*

(c) *Every locally finite* $\mathscr{E} \subset \mathscr{P}(X)$ *has a refinement* $\mathscr{F}$ *such that* $\{f(F): F \in \mathscr{F}\}$ *is index-σ-locally finite.*

(d) *Every locally finite $\mathscr{E} \subset \mathscr{P}(X)$ has a refinement $\mathscr{F}$ such that $\{f(F): F \in \mathscr{F}\}$ is σ-locally finitely decomposable.*

Proof. The implications (a) → (b) and (d) → (b) are proved as for Proposition 2.3, and (b) → (c) → (d) is clear. It remains to prove (b) → (a).

Let $\{E_\alpha\} \subset \mathscr{P}(X)$ be index-σ-locally finite. By (b), $\{E_\alpha\}$ has a base $\mathscr{B}$ such that $\{f(B): B \in \mathscr{B}\}$ is index-σ-locally finite. This implies that $\{f(E_\alpha)\}$ is index-point-countable and that $f(\mathscr{B})$ is a base for $\{f(E_\alpha)\}$. Hence $\{f(E_\alpha)\}$ is σ-locally finitely decomposable by Lemma 3.2 (b) → (a). That completes the proof.

Analogously to Lemma 2.4 and Corollaries 2.5 and 2.6, we have the following results.

LEMMA 3.5. *If Z is subparacompact, every index-locally countable $\{E_\alpha\} \subset \mathscr{P}(Z)$ is σ-discretely decomposable.*

Proof. By Lemmas 2.4 and 3.2 (with Footnote 3).

COROLLARY 3.6. *If X is subparacompact, every index-σ-discrete map $f: X \to Y$ is index-σ-locally finite.*

COROLLARY 3.7. *If Y is subparacompact, every index-σ-locally finite map $f: X \to Y$ in index-σ-discrete.*

4. Maps with $\aleph_1$-compact fibers: The relation between indexed and non-indexed maps. Recall that a space X is called $\aleph_1$-*compact* if every closed discrete subset is countable. Clearly every Lindelöf space X is $\aleph_1$-compact, and the following lemma, whose proof we omit, implies that the converse is true if X is paracompact (or even merely subparacompact).

LEMMA 4.1.[3] *A space X is $\aleph_1$-compact if and only if every locally finite collection of subsets is countable.*

COROLLARY 4.2. *If $f: X \to Y$ has $\aleph_1$-compact fibers, and if $\{E_\alpha\} \subset \mathscr{P}(X)$ is index-σ-locally finite, then $\{f(E_\alpha)\}$ is index-point-countable.*

PROPOSITION 4.3.[3] *If $f: X \to Y$ has closed fibers,[5] then the following are equivalent.*

(a) *f is index-σ-locally finite.*

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[5] This assumption is needed only for (a) → (b).

(b)   *f is base-σ-locally finite and has $\aleph_1$-compact fibers.*

Proof. (a) → (b). Assume (a). Then f is surely base-σ-locally finite. To see that each $f^{-1}(y)$ is $\aleph_1$-compact, let $T \subset f^{-1}(y)$ be closed and discrete. Then (a) implies that $\{f(x): x \in T\}$ is index-point-countable, so T must be countable.

(b) → (a). This follows from Corollary 4.2 and Lemma 3.2.

The following result, which was kindly called to my attention by R. W. Hansell, is related to Proposition 4.3.

PROPOSITION 4.4.[3]   *If $f: X \to Y$ has $\aleph_1$-compact fibers, the following are equivalent.*

(a)   *f is base-σ-locally finite.*

(b)   *If $\mathscr{E} \subset \mathscr{P}(X)$ is locally finite, then $f(\mathscr{E})$ has a σ-locally finite refinement.*

Proof. (a) → (b). Clear.

(b) → (a). Assume (b). Let $\mathscr{E} \subset \mathscr{P}(X)$ be locally finite. Then $\{f(E): E \in \mathscr{E}\}$ is index-point-countable by Corollary 4.2. Now if $E' \subset E$ for each $E \in \mathscr{E}$, then $\{E': E \in \mathscr{E}\}$ is also locally finite, so $\{f(E'): E \in \mathscr{E}\}$ has a σ-locally finite refinement by (b). Hence $\{f(E): E \in \mathscr{E}\}$ has a σ-locally finite base by Lemma 3.2 (c) → (b), and that completes the proof.

REMARK. I don't know whether the hypothesis that f has $\aleph_1$-compact fibers can be omitted from Proposition 4.4. It can, however, be shown that this hypothesis on the fibers can be replaced by the following hypothesis on Y:

(*)   Every $S \subset Y$ which is not σ-discrete in Y has a subset S' of cardinality $\aleph_1$ which is not σ-discrete in Y.

Condition (*) is clearly satisfied if card $Y \leqq \aleph_1$ or if Y is Lindelöf (more generally, $\aleph_1$-compact). It is also satisfied if Y is a metric space of weight $\leqq \aleph_1$ (see [14, Corollary 1]) or if Y is a complete (more generally, an absolutely analytic) metric space [3]. On the other hand, there exist paracompact spaces (such as the space Y of Example 7.6) in which (*) is false, and a result of W. R. Fleissner [2] implies that it is consistent with (ZFC) for (*) to be false even in metric spaces.[6]

Our next result is related to [5, Proposition 3.7 (ii)] and [11, Proposition 2.2 (a)].

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[6] I am grateful to R. W. Hansell for calling my attention to the results and references in this paragraph.

PROPOSITION 4.5.[7] *Let $f: X \to Y$ have $\aleph_1$-compact fibers, with X paracompact. Then the following are equivalent.*

(a) *f is base-σ-locally finite.*

(b) *If $\mathscr{U}$ is a locally finite collection of open subsets of X, then $f(\mathscr{U})$ has a σ-locally finite base.*

Proof. (a) → (b). Clear.

(b) → (a). Let $\mathscr{E} \subset \mathscr{P}(X)$ be locally finite. Pick a locally finite cover $\mathscr{U}$ of X such that each $U \in \mathscr{U}$ intersects only finitely many $E \in \mathscr{E}$. Then $f(\mathscr{U})$ has a σ-locally finite base $\mathscr{V}$ by (b), and if $V \in \mathscr{V}$ then $V \subset f(U)$ for only countably many $U \in \mathscr{U}$ by Corollary 4.2. Hence

$$\{V \cap f(U \cap E): \ V \in \mathscr{V}, \ U \in \mathscr{U}, \ E \in \mathscr{E}, \ V \subset f(U)\}$$

is the required σ-locally finite base for $f(\mathscr{E})$.

COROLLARY 4.6.[3] *If $f: X \to Y$ has $\aleph_1$-compact fibers, with X metrizable, then the following are equivalent.*

(a) *f is base-σ-locally finite.*

(b) *$f(\{U \subset X: U \text{ open in } X\})$ has a σ-locally finite base.*

REMARK. Example 7.3 shows that Proposition 4.5 and Corollary 4.6 become false if one does not assume that f has $\aleph_1$-compact fibers, even when X and Y are both metrizable.

5. Open and closed maps. The following result summarizes some sufficient conditions, mostly obtained in [5] and [11], for a map to be base-σ-locally finite or base-σ-discrete.

PROPOSITION 5.1. *Among the following conditions on a continuous $f: X \to Y$, (a)-(d) and (f) imply that f is base-σ-locally finite, while (a) and (c)-(e) imply that f is base-σ-discrete.*[8]

(a) *X is $\aleph_1$-compact.*

(b) *f is perfect.*

(c) *f is closed, f has Lindelöf fibers, and Y is subparacompact.*

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[7] This results remains true if "locally finite" is everywhere replaced by "discrete", and "X paracompact" can then even be weakened to "X collectionwise normal". The proof of (b) → (a) need only be modified by taking $\mathscr{U}$ to be a discrete family $\{U_E: E \in \mathscr{E}\}$ of open sets in X with $E \subset U_E$ for all $E \subset \mathscr{E}$. Alternatively, this modification of Proposition 4.5 follows from Hansell's [5, Proposition 3.7 (ii)], Corollary 4.2, and Lemma 3.2 (b) → (a) (with Footnote 3).

[8] In all but (d), f is actually *index-σ-locally finite* or *index-σ-discrete*; see Proposition 4.3.

(d) *f is closed and Y is metrizable.*[9]

(e) *f is open, f has $\aleph_1$-compact fibers, X is collectionwise normal, and Y is a σ-space.*[10]

(f) *Same as (e), but with X paracompact.*

Proof. (a) By Lemma 4.1.

(b) Perfect maps preserve locally finite collections. (See [11, Proposition 1.1(c)].)

(c) That f is base-σ-locally finite was proved in [11, Proposition 1.1(c)]. Hence f is also base-σ-discrete by Corollary 2.6.

(d) This was proved in [5, Proposition 3.10], where the proof shows that f is both base-σ-discrete and base-σ-locally finite.

(e) See [5, Proposition 3.11]. (While that result assumed that Y is metrizable, the proof remains valid if Y is only a σ-space.) Alternatively, (e) follows immediately from Proposition 4.5 and Footnote 7.

(f) This follows from Proposition 4.5. (Alternatively, it follows from (e) and Corollary 2.5.)

6. Quotient s-maps. The principal purpose of this section is to prove the following theorem, which will be applied by Hansell in [6].

THEOREM 6.1. *Let $f\colon X \to Y$ be a quotient s-map,[11] with X and Y metrizable. Then there exists a G_δ-subset X' of X such that $f(X') = Y$ and $f\,|\,X'$ in index-σ-locally finite and index-σ-discrete.[12]*

Proof. Let $\mathscr{B}$ be a σ-discrete base for X, and let $\mathscr{V}$ be a σ-discrete base for Y. For each $y \in Y$, let us say that $\mathscr{F} \subset \mathscr{B}$ is y-*minimal* if $\mathscr{F}$ is finite and $f(\mathscr{F})$ is a minimal cover of some $V \in \mathscr{V}$ which contains y. Let X' be the set of all $x \in X$ such that, whenever $x \in U$ with $U \in \mathscr{B}$, then $U \in \mathscr{F}$ for some $f(x)$-minimal $\mathscr{F} \subset \mathscr{B}$. Let us check that this X' has the required properties.

(a) $f(X') = Y$: Suppose not. Then there exist a $y \in Y$ and a

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[9] As the proof of this result in [5] shows, the assumption that Y is metrizable can be weakened to assuming only that every open subset of Y is an F_σ and that every non-closed subset of Y has a countable nonclosed subset. Since every continuous, closed image of a metrizable space has these properties, the metrizability of Y may be replaced by the metrizability of X. That some restriction on X or Y is needed is shown by Example 7.6.

[10] A space Y is a σ-*space* if it has a σ-locally finite closed network (equivalently: a σ-discrete closed network). Here a collection $\mathscr{S} \subset \mathscr{P}(Y)$ is called a *network* for Y if it is a base for the collection of open subsets of Y in the sense of this paper.

[11] An *s-map* is a map with separable fibers.

[12] If f is open or closed, one can take $X' = X$ (see Proposition 5.1). In general, however, Example 7.1 shows that one cannot take $X' = X$.

$\mathscr{U} \subset \mathscr{B}$ covering $f^{-1}(y)$ such that, if $U \in \mathscr{U}$, then $U \notin \mathscr{F}$ for any y-minimal $\mathscr{F} \subset \mathscr{B}$. Since f is an s-map and Y is first-countable, f must be bi-quotient by [8, Proposition 3.3], so $y \in \operatorname{Int} f(\mathscr{F})$ for some finite $\mathscr{F} \subset \mathscr{U}$; we may suppose that $\mathscr{F}$ is minimal for this property and hence y-minimal. But then any $U \in \mathscr{F}$ violates our assumption about $\mathscr{U}$.

(b) $f \mid X'$ is index-σ-locally finite and index-σ-discrete: By Corollaries 3.6 and 3.7 and Proposition 4.3, both these properties of $f \mid X'$ are equivalent to $f \mid X'$ being base-σ-locally finite. Let $\mathscr{B}' = \{B \cap X' : B \in \mathscr{B}\}$. Then $\mathscr{B}'$ is a base for X', so by Proposition 4.5 it will suffice to show that $f(\mathscr{B}')$ has a σ-locally finite base.

Since $\{f(B) : B \in \mathscr{B}\}$ is index-point-countable (see Corollary 4.2), a result of A. S. Miščenko [12] implies that for each $S \subset Y$ there are only countably many finite collections $\mathscr{F}_n(S) \subset \mathscr{B}$ ($n = 1, 2, \cdots$) such that $f(\mathscr{F}_n(S))$ is a minimal cover of S. Let

$$\mathscr{W} = \{V \cap f(B \cap X') : V \in \mathscr{V}, \; B \in \bigcup_{n=1}^{\infty} \mathscr{F}_n(V)\} .$$

Then $\mathscr{W}$ is σ-discrete (because $\mathscr{V}$ is σ-discrete and $\bigcup_{n=1}^{\infty} \mathscr{F}_n(V)$ is countable for each $V \in \mathscr{V}$), so we need only show that $\mathscr{W}$ is a base for $f(\mathscr{B}')$.

Suppose that $B \in \mathscr{B}$ and that $y \in f(B \cap X')$. From the definition of X', we have $B \in \mathscr{F}$ for some y-minimal $\mathscr{F} \subset \mathscr{B}$. Hence $f(\mathscr{F})$ is a minimal cover of some $V \in \mathscr{V}$ containing y, so $\mathscr{F} = \mathscr{F}_n(V)$ for some n. Let $W = V \cap f(B \cap X')$. Then $W \in \mathscr{W}$ and $y \in W \subset f(B \cap X')$. Hence $\mathscr{W}$ is a base for $f(\mathscr{B}')$.

(c) X' is a G_δ in X: For each $U \in \mathscr{B}$, let U^* be the set of all $x \in U$ such that $U \in \mathscr{F}$ for some $f(x)$-minimal $\mathscr{F} \subset \mathscr{B}$. Then U^* is open in X, so $U - U^*$ is an F_σ. Let $E = \bigcup \{U - U^* : U \in \mathscr{B}\}$. Then E is the union of a σ-discrete collection of F_σ's in X, so E is also an F_σ in X. But it is easily checked that $X' = X - E$, so X' is a G_δ in X.

That completes the proof.

REMARK. As the proof of Theorem 6.1 shows, $f^{-1}(y) \cap X'$ is closed in X for every $y \in Y$. I don't know whether X' itself can be chosen to be closed in X. I also don't know whether X' can be chosen so that $f \mid X'$ is again a quotient map.

As Examples 7.2 and 7.5 show, the assumption that f is an s-map cannot be omitted from Theorem 6.1, even if X is locally compact or if f is open. The situation changes, however, if both of these additional hypotheses are satisfied simultaneously, as the following result shows.

PROPOSITION 6.2. *If* $f: X \to Y$ *is an open, continuous map from a locally separable, metrizable space* X *onto a paracompact space* Y, *then there exists an open* $X' \subset X$ *such that* $f(X') = Y$ *and* $f \mid X'$ *is index-σ-locally finite and index-σ-discrete.*

Proof. Let $\{U_\alpha\}$ be an open cover of X by separable sets. Then $\{f(U_\alpha)\}$ is an open cover of Y, so it has an open, locally finite refinement $\{V_\alpha\}$ with $V_\alpha \subset f(U_\alpha)$ for all α. Let $W_\alpha = U_\alpha \cap f^{-1}(V_\alpha)$, and let $X' = \bigcup_\alpha W_\alpha$.

Clearly X' is open in X and $f(X') = Y$. If $\mathscr{B}_\alpha$ is a countable base for W_α, and if $\mathscr{B} = \bigcup_\alpha \mathscr{B}_\alpha$, then $\mathscr{B}$ is a base for X' and $f(\mathscr{B})$ is σ-locally finite. Since $f \mid X'$ has separable fibers, it follows from Corollary 4.6 and Proposition 4.3 that f is index-σ-locally finite. Hence f is also index-σ-discrete by Corollary 3.7.

REMARK. Example 7.3 shows that one cannot always take $X' = X$ in Proposition 6.2, and Example 7.5 shows that one cannot omit the assumption in Proposition 6.2 that X is locally separable.

7. Examples. To state our examples as simply and sharply as possible, we follow Hansell in saying that a map $f: X \to Y$ is *σ-discrete preserving* if $f(S)$ is σ-discrete in Y whenever S is σ-discrete in X. It is easy to see that every base-σ-discrete or base-σ-locally finite map (more generally, any map satisfying 4.4 (b)) is σ-discrete-preserving, but Example 7.6 implies that the converse is false.[13] Call a map $f: X \to Y$ *inductively σ-discrete preserving* if there exists an $X' \subset X$ such that $f(X') = Y$ and $f \mid X'$ is σ-discrete-preserving.

In Examples 7.1–7.3 and 7.5, our proofs that certain maps are not σ-discrete-preserving are based on the simple observation that every σ-discrete subset of a Lindelöf space is countable. We denote the closed unit interval by I.

EXAMPLE 7.1. A two-to-one quotient map $f: X \to Y$, with X locally compact metric and $Y = I$, which is not σ-discrete preserving.

Proof. Let $Y = I$ and let $X = X_1 + X_2$ (topological sum), where $X_1 = I$ and X_2 is I with the discrete topology. Let $f: X \to Y$ be the obvious map. Clearly f is a quotient map. But $X_2 \subset X$ is discrete while $f(X_2) = Y$ is not σ-discrete, so f is not σ-discrete-preserving.

EXAMPLE 7.2. A quotient map $f: X \to Y$, with X locally compact metric and $Y = I$, which is not inductively σ-discrete-preserving.

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[13] R. W. Hansell has kindly informed me that, by an example of R. Pol [**15**, p. 141], it is consistent with the usual axioms of set theory for the converse to be false even for continuous *s*-maps between metric spaces.

Proof. Let $Y = I$, let X be the topological sum of all convergent sequences in Y, and let $f: X \to Y$ be the obvious map. Then f is surely a quotient map. Now suppose $X' \subset X$ and $f(X') = Y$. Then one can find an uncountable $S \subset X'$ such that distinct elements of S belong to distinct summands of X. Then $S \subset X'$ is discrete while $f(S)$ is uncountable and hence not σ-discrete, so f is not inductively σ-discrete-preserving.

EXAMPLE 7.3. A continuous open surjection $f: X \to Y$, with X locally compact metric and $Y = I$, which is not σ-discrete-preserving.

Proof. See [5, Example 3.12].

Before giving our next example, we need the following lemma. I am grateful to R. Pol for observing that the space X in this lemma is σ-discrete, and that the lemma therefore immediately yields Example 7.5.

LEMMA 7.4. *If Y is any (metrizable) space, then there exists a continuous, open map $f: X \to Y$ from a σ-discrete (metrizable) space X onto Y.*[14]

Proof. Such a map is constructed in [10, Lemma 4.2], where all the required properties except the σ-discreteness of X are established. To see that X is σ-discrete, it suffices to show that the subset $E^* = \bigcup_{\beta \in B} E_\beta$ of E in [10, Lemma 4.1] is σ-discrete. To see that, we let $E(n)$ (where $n \in N$) be the set of all $x \in E$ such that all but at most n coordinates of x are the same. Then $E^* = \bigcup_{n \in N} E(n)$, and it is easy to check that each $E(n)$ is discrete; hence E^* is σ-discrete, and that completes the proof.

The following example combines all the features of Examples 7.2 and 7.3 except for the local compactness of X.

EXAMPLE 7.5. A continuous open surjection $f: X \to Y$, with X a metric space and $Y = I$, which is not inductively σ-discrete-preserving.

Proof. Let $Y = I$, and let $f: X \to Y$ be as in Lemma 7.4. Since every subset of a σ-discrete space is σ-discrete, this f cannot be inductively σ-discrete-preserving.

EXAMPLE 7.6. A continuous, closed, σ-discrete-preserving map

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[14] R. Pol has recently shown [16] that, if Y is first-countable and T_1, the map f can even be chosen to have discrete fibers.

$f: X \to Y$, with X and Y hereditarily paracompact, which is neither base-σ-locally finite nor base-σ-discrete.[15]

Proof. Let Y be the space of cardinality $\aleph_2$ with only one nonisolated point p in which a set $U \subset Y$ is a neighborhood of p if $p \in U$ and $|Y - U| \leq \aleph_1$. Let $\mathscr{A}$ be a disjoint collection of subsets of $Y - \{p\}$ such that $|\mathscr{A}| = \aleph_1$ and $|A| = \aleph_2$ for all $A \in \mathscr{A}$. For each $A \in \mathscr{A}$, let $A^+ = A \cup \{p\}$. Let $X = \Sigma\{A^+: A \in \mathscr{A}\}$ (the topological sum), and let $f: X \to Y$ be the ovbious map. We will show that f has the required properties.

(a) f is closed: It suffices to show that, if $A' \subset A^+$ is closed for all $A \in \mathscr{A}$, then $E = \bigcup\{A': A \in \mathscr{A}\}$ is closed in Y. If $p \in E$ this is clear. If $p \notin E$, then $|A'| \leq \aleph_1$ for all $A \in \mathscr{A}$, hence $|E| \leq \aleph_1$, so again E is closed in Y.

(b) f is σ-discrete-preserving: Every closed map has this property.

(c) f is neither base-σ-locally finite nor base-σ-discrete: Since $\{A^+: A \in \mathscr{A}\}$ is a discrete collection of subsets of X, it will suffice to show that $\{A^+: A \in \mathscr{A}\}$ has no σ-locally finite refinement in Y.

Suppose $\mathscr{B}$ were such a refinement. Without loss of generality, we may assume that each $B \in \mathscr{B}$ is closed in Y. Since $\mathscr{A}$ is disjoint, each $B \in \mathscr{B}$ intersects at most one $A \in \mathscr{A}$, and hence $|\mathscr{B}| \leq \aleph_1$.

Let $\mathscr{B}' = \{B \in \mathscr{B}: p \in \mathscr{B}\}$. Then $\mathscr{B}'$ is countable, so $\bigcup \mathscr{B}'$ intersects at most countably many $A \in \mathscr{A}$, and hence some $A_0 \in \mathscr{A}$ does not intersect $\bigcup \mathscr{B}'$. Let $\mathscr{B}'' = \mathscr{B} - \mathscr{B}'$. Then $|\mathscr{B}''| \leq |\mathscr{B}| \leq \aleph_1$. Also $|B| \leq \aleph_1$ for all $B \in \mathscr{B}''$ (since B is closed and $p \notin B$), so $|\bigcup \mathscr{B}''| \leq \aleph_1$. Since $|A_0| = \aleph_2$, it follows that $\mathscr{B}''$ does not cover A_0. Hence $\mathscr{B}$ does not cover A_0, which is impossible.

8. Two characterizations. The purpose of this section is to extend [11, Theorem 1.3] and to fill a gap in its proof. For the definition of σ-spaces, see Footnote 10; for strong Σ-spaces, see K. Nagami [13] or [11, §3].

THEOREM 8.1. *The following properties of a regular space Y are equivalent.*

(a) *Y is a σ-space (resp. strong Σ-space).*

(b) *Y is the image under an index-σ-locally finite continuous map of a metrizable space (resp. paracompact M-space).*

(c) *Y is the image under a base-σ-locally finite continuous map of a metrizable space (resp. paracompact M-space).*

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[15] As the proof shows, there is a discrete $\mathscr{C} \subset \mathscr{P}(X)$ such that $f(\mathscr{C})$ does not have a σ-locally finite refinement. In particular, f does not even satisfy condition 4.4(b).

Proof. The equivalence of (a) and (b) is asserted in [11, Theorem 1.3], and the proof given there for (b) → (a) actually establishes the sharper implication (c) → (a). That (b) → (c) is, of course, trivial.

Let us take this opportunity to fill a gap in the proof of the implication (a) → (b) for strong Σ-spaces which was given in [11, § 6]. Adopting the terminology used in that proof, denote $g^{-1}(\mathscr{B})$ by $\mathscr{E}$. After showing that $\mathscr{E}$ is a (mod k)-network for X and that $\{f(E): E \in \mathscr{E}\}$ is σ-locally finite, the proof in [11] invokes [11, Proposition 1.1 (d)] to conclude that f is index-σ-locally finite. That is not quite justified, however, since the hypothesis of [11, Proposition 1.1(d)] requires that $\{f(E): E \in \mathscr{E}\}$ be *index-σ*-locally finite (see [11, Footnote 3]). To establish this sharper fact, one needs the following two additional properties of $\mathscr{B}$ and f which were not recorded in [11] but which are clear from Reference 5 of [11] from which $\mathscr{B}$ and f were obtained.

(i) $\mathscr{B}$ is σ-locally finite.

(ii) The fibers of f are Lindelöf.

It now follows from (i) that $\mathscr{E}$ must be σ-locally finite, so (ii) and our Corollary 4.2 imply that $\{f(E): E \in \mathscr{E}\}$ is index-point-countable. Since we know that $\{f(E): E \in \mathscr{E}\}$ is σ-locally finite, it follows from our Lemma 3.1 that $\{f(E): E \in \mathscr{E}\}$ is *index-σ*-locally finite, and that completes the proof.

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