

# Pacific Journal of Mathematics

## **AN ERROR ESTIMATE UNIFORM IN TIME FOR SPECTRAL GALERKIN APPROXIMATIONS OF THE NAVIER-STOKES PROBLEM**

JOHN GROVES HEYWOOD

# AN ERROR ESTIMATE UNIFORM IN TIME FOR SPECTRAL GALERKIN APPROXIMATIONS OF THE NAVIER-STOKES PROBLEM

JOHN G. HEYWOOD

1. **Introduction.** The existence theory for the nonstationary Navier-Stokes equations can be developed, by the method of Galerkin approximation, using any of a wide variety of possible systems of basis functions. The basis functions used in the papers of Hopf [8] and of Kiselev and Ladyzhenskaya [10] are merely assumed to belong to and be complete in certain function spaces. However, to obtain refinements in the theory by the Galerkin method, particularly regarding the regularity and decay of solutions, it often appears essential to choose as basis functions the eigenfunctions of the Stokes operator; see Ito [9], Lions [12], Prodi [14], Foias [3], Ladyzhenskaya [11], Temam [17], and Heywood [4, 5, 6].

In these and other works, the convergence of the Galerkin approximations is generally proved by a compactness argument, based on *a-priori* bounds for the approximations. A notable exception is the paper [3] of Foias, where (on page 324) the approximations are shown to converge, uniformly over a time interval, in the Dirichlet norm. Recently, Rautmann [15, 16] has drawn attention to this type of result, and gone further, giving a systematic development of error estimates, for the Galerkin approximations and their time derivatives.

Rautmann's error estimates (and also Foias' convergence theorem) are presented locally, valid on a finite interval determined by certain norms of the data. At best, if one assumes the solution to be approximated is uniformly *regular*, for  $t \in [0, \infty)$ , the method yields an error estimate which grows exponentially with time. And, without further assumptions, this is the best that can be expected. However, if one assumes, additionally, the solution to be approximated is *stable*, then it is reasonable to expect an error estimate which is uniform in time. This is what is done in the present paper. It is hoped the result may prove suggestive for future developments in the Navier-Stokes theory. It should be mentioned, however, the original reason for undertaking this work arose in the author's joint study [7] with Rolf Rannacher, of finite element Galerkin approximations, a context in which error estimates uniform in time have important implications for computations. While the present work served to fix some ideas, in a simpler, purely theoretical context, it has turned out our arguments in the finite element context are substantially different.

We present here only the most basic result of its type. However, a number of elaborations seem possible, some of which are being investigated with Rannacher in the numerical context. We mention, the stability condition (A3) is formulated here in terms of the Dirichlet norm, could be formulated in terms of the  $L^2$ -norm. Also, our error estimate is given in the Dirichlet norm; it should be possible to obtain an improved rate of convergence in the  $L^2$ -norm. Finally, in some situations, there exist "stable" solutions which do not possess all the spatial symmetry of the domain and external forces. Of course, in such situations, there is only stability modulo shifts in the spatial symmetry. The definition of stability we work with, here, is too strict to allow for this possibility. We think it is possible to weaken our definition of stability, to be neutral relative to drifts with respect to the spatial symmetry, resulting in error estimates modulo shifts in the symmetry. Similarly, if the boundary values and forces are time independent, stability and error estimates may be considered modulo drifts and shifts in time.

After a short preliminary §2, our result is stated in §3, and proved in §§4 through 7.

**2. Preliminaries.** Let  $\Omega \subset R^n$ ,  $n = 2$  or  $3$ , be a bounded domain with smooth boundary  $\partial\Omega$ . The function spaces customarily denoted by  $L^p(\Omega)$ ,  $H^m(\Omega)$ ,  $C_0^\infty(\Omega)$  are defined as usual, though we will not distinguish in our notation between spaces of  $R$  and of  $R^n$ -valued functions, the distinction always being clear from the context. The  $L^2$  inner-product and norm are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$  respectively, and the  $L^p$  norm by  $\|\cdot\|_p$ . The following spaces of solenoidal functions are, of course,  $R^n$ -valued:

$$\begin{aligned} D(\Omega) &= \{\phi: \phi \in C_0^\infty(\Omega) \text{ and } \nabla \cdot \phi = 0\}, \\ J(\Omega) &\text{ is the completion of } D(\Omega) \text{ in } L^2(\Omega), \\ J_1(\Omega) &\text{ is the completion of } D(\Omega) \text{ in } H^1(\Omega). \end{aligned}$$

We let  $P$  denote the orthogonal projection  $L^2(\Omega) \rightarrow J(\Omega)$ , and let  $\tilde{A} = PA$ , where  $A$  is the Laplacean operator. We list, as lemmas, some results which will be needed later.

**LEMMA 1.** For  $u \in H^1(\Omega)$ ,  $\|Pu\|_{H^1} \leq c\|u\|_{H^1}$ .

**LEMMA 2.** For every  $u \in J(\Omega) \cap H^1(\Omega)$ , there exists a function  $\psi \in H^2(\Omega)$  such that  $\text{curl } \psi = u$ ,  $\psi|_{\partial\Omega} = 0$ , and  $\|\psi\|_{H^2} \leq c\|u\|_{H^1}$ .

**LEMMA 3.** For  $g \in L^2(\Omega)$ , the unique solution  $v \in J_1(\Omega)$ ,  $q \in L^2(\Omega)/R$  of the Stokes problem

$$-\Delta_q + \nabla_v = g \quad \text{and} \quad \nabla \cdot v = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0,$$

satisfies  $\|v\|_{H^2} + \|g\|_{H^1/\mathcal{R}} \leq c\|g\|$ . Hence, also,  $\|v\|_{H^2} \leq c\|\tilde{\Delta}v\|$  holds for all  $v \in J_1(\Omega) \cap H^2(\Omega)$ .

For proofs of Lemmas 1 and 3, see [17, p. 18] and [1], respectively. In Lemma 2, if  $\Omega$  is two-dimensional, it should be understood that  $\psi$  is scalar valued and that  $\text{curl } \psi = (\partial\psi/\partial x_2, -\partial\psi/\partial x_1)$ . The proof in the more difficult three-dimensional case is as follows. By a well-known construction [11, p. 25],  $u$  can be continued to all  $\mathbb{R}^3$  as a solenoidal function  $u \in H^1(\mathbb{R}^3)$  with compact support. Hence it can be expressed as  $u = \text{curl } \phi$ , in terms of the vector potential  $\phi(x) = (4\pi)^{-1} \int (\text{curl } u)/r \, dy$ . Since  $(\text{curl } \phi) \cdot n = 0$  on  $\partial\Omega$ , a scalar function  $p$  can be defined on component  $\partial\Omega_k$  of  $\partial\Omega$  by setting  $p(x) = \int_C \phi \cdot ds$ , where  $C$  is any curve lying in  $\partial\Omega$  joining  $x$  to a fixed point of  $\partial\Omega_k$ . Since  $\phi \in H^2(\Omega)$ , we have  $p \in W^{5/2}_2(\Omega\partial)$  and  $\phi \cdot n \in W^{3/2}_2(\partial\Omega)$ . It follows that  $p$  can be continued into  $\Omega$  as a function  $p \in H^3(\Omega)$  satisfying  $\partial p/\partial n = \phi \cdot n$  on  $\partial\Omega$ ; see [13, p. 104]. Clearly  $\nabla p = \phi$  on  $\partial\Omega$ , and thus  $\psi = \phi - \nabla p$  is a function with all the desired properties.

We denote by  $\{a^k(x)\}$  and  $\{\lambda_k\}$  the eigenfunctions and eigenvalues of the Stokes operator  $\tilde{\Delta}$  defined in  $J_1(\Omega) \cap H^2(\Omega)$ , i.e., of the problem  $-\Delta v + \nabla q = \lambda v$  and  $\nabla \cdot v = 0$  in  $\Omega$ ,  $v|_{\partial\Omega} = 0$ . Thus  $-\tilde{\Delta}a^k = \lambda_k a^k$ . Some well known results and elementary observations are collected in the following lemma.

**LEMMA 4.** *The eigenfunctions  $\{a^k\}$  are orthogonal in the inner products  $(u, v)$ ,  $(\nabla u, \nabla v)$ , and  $(\tilde{\Delta}u, \tilde{\Delta}v)$ , and complete in the spaces  $J(\Omega)$ ,  $J_1(\Omega)$  and  $J_1(\Omega) \cap H^2(\Omega)$ . If  $\sum_{k=1}^{\infty} c_k a^k \in J_1(\Omega)$ , then*

$$\left\| \sum_{k=n}^{\infty} c_k a^k \right\|^2 \leq \lambda_n^{-1} \left\| \nabla \sum_{k=n}^{\infty} c_k a^k \right\|^2.$$

If  $\sum_{k=1}^{\infty} c_k a^k \in J_1(\Omega) \cap H^2(\Omega)$ , then

$$\left\| \nabla \sum_{k=n}^{\infty} c_k a^k \right\|^2 \leq \lambda_n^{-1} \left\| \tilde{\Delta} \sum_{k=n}^{\infty} c_k a^k \right\|^2.$$

We list, in the following lemma, some Poincaré and Sobolev inequalities that will be needed, sometimes combined with the results of Lemmas 3 and 4. We will use only the three-dimensional versions of Sobolev's inequalities, these being, of course, also valid in bounded two-dimensional domains.

**LEMMA 5.** *If  $u \in J_1(\Omega) \cap H^2(\Omega)$ , then  $\|u\| \leq \lambda_1^{-1/2} \|\nabla u\| \leq \lambda_1^{-1} \|\tilde{\Delta}u\|$ ,*

$$\|u\|_6 \leq c \|\nabla u\|, \quad \sup |u| \leq c \|\tilde{\Delta} u\|, \quad \|u\|_4 \leq c \|u\|^{1/4} \|\nabla u\|^{3/4},$$

and  $\|\nabla u\|_6 \leq c \|\tilde{\Delta} u\|$ .

**3. Statement of the result.** Let  $u(x, t)$ ,  $p(x, t)$  be a solution of the Navier-Stokes problem:

$$(1) \quad \begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= f \quad \text{and} \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0, \infty), \\ u|_{t=0} &= a, \quad u|_{\partial\Omega} = 0. \end{aligned}$$

We assume the data for problem (1) satisfies

$$(A1) \quad a \in J_1(\Omega) \cap H^2(\Omega), \quad \sup_{t \geq 0} \|f\| < \infty, \quad \sup_{t \geq 0} \|f_t\| < \infty,$$

and that the solution satisfies

$$(A2) \quad \|\nabla u(t)\| \leq M, \quad \text{for all } t \geq 0.$$

We further assume  $u$  is *conditionally exponentially stable*, in the sense of condition (A3), below. The stability condition concerns the behaviour of perturbations of  $u$ . A function  $\zeta(x, t)$ , defined on some interval  $t \geq t_0$ , is called a perturbation of  $u$ , if  $\zeta + u$  is a solution of the Navier-Stokes equations, and if  $\zeta|_{\partial\Omega} = 0$ . Thus, setting  $\zeta_0 \equiv \zeta(\cdot, t_0)$ ,  $\zeta$  is a solution of the initial-boundary value problem:

$$(2) \quad \begin{aligned} \zeta_t - \Delta \zeta + u \cdot \nabla \zeta + \zeta \cdot \nabla u + \zeta \cdot \nabla \zeta + \nabla q &= 0 \quad \text{and} \\ \nabla \cdot \zeta &= 0 \quad \text{in } \Omega \times (t_0, \infty), \\ \zeta|_{t=t_0} &= \zeta_0, \quad \zeta|_{\partial\Omega} = 0. \end{aligned}$$

Our assumption, then, is:

(A3) *There exist positive numbers  $\alpha$ ,  $A$ ,  $\delta$  such that for every  $t_0 \geq 0$ , and every  $\zeta_0 \in J_1(\Omega) \cap H^2(\Omega)$  with  $\|\nabla \zeta_0\| < \delta$ , the perturbation problem (2) is uniquely solvable and its solution satisfies  $\|\nabla \zeta(t)\| \leq A \|\nabla \zeta_0\| e^{-\alpha(t-t_0)}$ , for all  $t \geq t_0$ .*

The  $n$ th spectral Galerkin approximation

$$u^n(x, t) = \sum_{k=1}^n c_{kn}(t) a^k(x)$$

to the solution of problem (1) is uniquely determined by the conditions

$$(3) \quad \begin{aligned} (u_t^n, \phi^n) + (\nabla u^n, \nabla \phi^n) + (u^n \cdot \nabla u^n, \phi^n) &= (f, \phi^n), \quad \text{for } t \geq 0, \quad \text{and} \\ (u^n(0) - a, \phi^n) &= 0, \end{aligned}$$

for all  $\phi^n$  of the form  $\phi^n(x) = \sum_{k=1}^n d_k a^k(x)$ .

**THEOREM.** *There exist constants  $N$  and  $K$  depending only on*

the domain  $\Omega$ , the norms of the data referred to in (A1), and the constants introduced in (A2) and (A3), such that

$$(4) \quad \|\nabla(u - u^n)(t)\| \leq K\lambda_{n+1}^{-1/2},$$

for all  $t \geq 0$ , provided  $n \geq N$ .

**4. Comparison of error with perturbations.** Let  $u = \sum_{k=1}^{\infty} c_k(t) a^k(x)$  be the eigenfunction expansion of the solution  $u$  of problem (1). Let  $v^n = \sum_{k=1}^n c_k(t) a^k(x)$  be the  $n$ th partial sum of the series for  $u$ . Let  $e^n = u - v^n$ , and let  $\eta^n = u^n - v^n$ , where  $u^n$  is the  $n$ th Galerkin approximation. Then  $u - u^n = e^n - \eta^n$ .

**LEMMA 6.** *The assumptions (A1), (A2) imply  $\|\tilde{A}u(t)\|$ ,  $\|u_t(t)\|$ , and  $e^{-t} \int_0^t e^{\tau} \|\nabla u_t\|^2 d\tau$  are uniformly bounded, for  $t \geq 0$ . Thus,  $\|\nabla e^n(t)\| \leq c\lambda_{n+1}^{-1/2}$  and  $\|\eta^n(t)\| \leq c\lambda_{n+1}^{-1}$ , for  $t \geq 0$ .*

These *a-priori* estimates for  $u$  are proved in [7]; most of the argument is repeated, in a slightly different context, in Lemma 8 below. The estimates for  $e^n$  follow, using Lemma 4.

It remains to estimate  $\eta^n$ . Observe that  $v^n$  satisfies the linearized equations

$$(5) \quad (v_t^n, \phi^n) + (\nabla v^n, \nabla \phi^n) + (u \cdot \nabla u, \phi^n) = (f, \phi^n), \quad \text{for } t \geq 0,$$

for all  $\phi^n$  of the form  $\phi^n(x) = \sum_{k=1}^n d_k a^k(x)$ . This is easily seen starting with the weak Navier-Stokes equations for  $u$  and using the orthogonality relations for the  $\{a^k\}$ . Subtracting (5) from (3) gives

$$\begin{aligned} (\eta_t^n, \phi^n) + (\nabla \eta^n, \nabla \phi^n) &= (u \cdot \nabla u, \phi^n) - (u^n \cdot \nabla u^n, \phi^n) \\ &= -(u \cdot \nabla \eta^n, \phi^n) - (\eta^n \cdot \nabla u^n, \phi^n) \\ &\quad + (u \cdot \nabla e^n, \phi^n) + (e^n \cdot \nabla u^n, \phi^n) \\ (6) \quad &= -(u \cdot \nabla \eta^n, \phi^n) - (\eta^n \cdot \nabla u, \phi^n) - (\eta^n \cdot \nabla \eta^n, \phi^n) \\ &\quad + (\eta^n \cdot \nabla e^n, \phi^n) + (e^n \cdot \nabla \eta^n, \phi^n) \\ &\quad + (u \cdot \nabla e^n, \phi^n) + (e^n \cdot \nabla v^n, \phi^n). \end{aligned}$$

This identity will be used, in §5, to get an *a-priori* estimate for  $\|\tilde{A}\eta^n(t)\|$ . In order to compare  $\eta^n$  with a perturbation  $\zeta$ , we must rewrite (6) in a form valid for all test functions  $\phi(x) \in J_1(\Omega) \cap H^2(\Omega)$ .

Let  $P_n$  and  $Q_n$  be the orthogonal projections of  $L^2(\Omega)$  onto  $\text{Span}\{a_1, \dots, a_n\}$  and  $\text{Span}\{a_{n+1}, a_{n+2}, \dots\}$ , respectively. Clearly  $P$ ,  $P_n$  and  $Q_n$  all commute,  $P = P_n + Q_n$ , etc. For  $\phi \in J(\Omega)$ , let us write  $\phi = P_n \phi + Q_n \phi = \phi^n + Q_n \phi$ . Then, for any  $w \in L^2(\Omega)$ ,

$$(w, \phi^n) = (P_n w, \phi) = (w, \phi) - (Q_n w, \phi).$$

Also, for  $\phi \in J_1(\Omega)$ ,

$$(\nabla \eta^n, \nabla \phi^n) = -(\tilde{\mathcal{A}} \eta^n, \phi^n) = -(\tilde{\mathcal{A}} \eta^n, \phi) = (\nabla \eta^n, \nabla \phi) .$$

Using such obvious identities, we rewrite (6) as

$$\begin{aligned} & (\eta_t^n, \phi) + (\nabla \eta^n, \nabla \phi) + (u \cdot \nabla \eta^n, \phi) + (\eta^n \cdot \nabla u, \phi) + (\eta^n \cdot \nabla \eta^n, \phi) \\ (7) \quad & = (Q_n[u \cdot \nabla \eta^n], \phi) + (Q_n[\eta^n \cdot \nabla u], \phi) + (Q_n[\eta^n \cdot \nabla \eta^n], \phi) \\ & + (P_n[\eta^n \cdot \nabla e^n], \phi) + (P_n[e^n \cdot \nabla \eta^n], \phi) \\ & + (P_n[u \cdot \nabla e^n], \phi) + (P_n[e^n \cdot \nabla v^n], \phi) , \end{aligned}$$

which is valid for all  $\phi \in J(\Omega)$ , and  $t \geq 0$ . This is to be compared with the weak form of (2), which is

$$(8) \quad (\zeta_t, \phi) + (\nabla \zeta, \nabla \phi) + (u \cdot \nabla \zeta, \phi) + (\zeta \cdot \nabla u, \phi) + (\zeta \cdot \nabla \zeta, \phi) = 0 .$$

Let us denote the right side of (7) by  $(g^n, \phi)$ , and let  $w = \eta^n - \zeta$ . Then, subtracting (8) from (7) gives

$$(9) \quad \begin{aligned} & (w_t, \phi) + (\nabla w, \nabla \phi) + (u \cdot \nabla w, \phi) + (w \cdot \nabla u, \phi) \\ & + (\eta^n \cdot \nabla w, \phi) + (w \cdot \nabla \zeta, \phi) = (g^n, \phi) , \end{aligned}$$

for all  $\phi \in J(\Omega)$ , and  $t \geq t_0$ . The regularity of  $\eta^n$  and  $\zeta$ , which is implicitly assumed here and in Lemma 7 below, will be verified later. Setting  $\phi = -\tilde{\mathcal{A}}w$ , we obtain

$$\begin{aligned} (10) \quad & \frac{1}{2} \frac{d}{dt} \|\nabla w\|^2 + \|\tilde{\mathcal{A}}w\|^2 = (u \cdot \nabla w, \tilde{\mathcal{A}}w) + (w \cdot \nabla u, \tilde{\mathcal{A}}w) \\ & + (\eta^n \cdot \nabla w, \tilde{\mathcal{A}}w) + (w \cdot \nabla \zeta, \tilde{\mathcal{A}}w) - (g^n, \tilde{\mathcal{A}}w) \\ & \leq c \|\tilde{\mathcal{A}}u\| \cdot \|\nabla w\| \cdot \|\tilde{\mathcal{A}}w\| + c \|\tilde{\mathcal{A}}\eta^n\| \cdot \|\nabla w\| \cdot \|\tilde{\mathcal{A}}w\| \\ & + c \|\tilde{\mathcal{A}}\zeta\| \cdot \|\nabla w\| \cdot \|\tilde{\mathcal{A}}w\| + \|g^n\| \cdot \|\tilde{\mathcal{A}}w\| , \end{aligned}$$

or

$$\frac{d}{dt} \|\nabla w\|^2 + \|\tilde{\mathcal{A}}w\|^2 \leq c(\|\tilde{\mathcal{A}}u\|^2 + \|\tilde{\mathcal{A}}\eta^n\|^2 + \|\tilde{\mathcal{A}}\zeta\|^2) \|\nabla w\|^2 + \|g\|^2 ,$$

and hence:

**LEMMA 7.** *Let  $t_0 \geq 0$  and  $\zeta$  be as in problem (2). Then, for  $w$  and  $g^n$  as above, and  $\beta(t) = c(\|\tilde{\mathcal{A}}u\|^2 + \|\tilde{\mathcal{A}}\eta^n\|^2 + \|\tilde{\mathcal{A}}\zeta\|^2)$ , there holds*

$$(11) \quad \|\nabla w(t)\|^2 \leq e^{\int_{t_0}^t \beta(\tau) d\tau} \left\{ \|\nabla w(t_0)\|^2 + \int_{t_0}^t \|g^n\|^2 d\tau \right\} ,$$

for all  $t \geq t_0$ .

**REMARK.** It is well to point out, here, why we don't assume  $\|\zeta(t)\| \leq A \|\zeta(t_0)\| e^{-\alpha(t-t_0)}$  in (A3), in place of the condition for the

Dirichlet norm, and then try to use a continuous dependence theorem in the  $L^2$  norm, in place of (11). The difficulty is that we would still need to estimate  $\|g^n\|$  and hence  $\|\tilde{\mathcal{A}}\eta^n\|$ , and this does not seem possible, in three-dimensions, starting with an estimate for  $\|w(t)\|$ . But we can, starting with an estimate for  $\|\mathcal{F}w(t)\|$ , because this implies an estimate for  $\|\mathcal{F}\eta^n\|$ , which leads to an estimate for  $\|\tilde{\mathcal{A}}\eta^n\|$ .

### 5. A-priori estimate of $\|\tilde{\mathcal{A}}\eta^n(t)\|$ .

**LEMMA 8.** *Suppose  $\|\mathcal{F}\eta^n(t)\| < \gamma$  holds on some interval  $0 \leq t \leq t^*$ , for some number  $\gamma \leq \delta$ . Then there also holds  $\|\tilde{\mathcal{A}}\eta^n(t)\|^2 < R(\gamma^2 + \lambda_{n+1}^{-1})$ , for  $0 \leq t \leq t^*$ , with a constant  $R$  dependent only on  $\Omega$ ,  $\delta$ ,  $\sup_{t \geq 0} \|\tilde{\mathcal{A}}u(t)\|$ ,  $\sup_{t \geq 0} \|u_t(t)\|$ , and  $\sup_{t \geq 0} e^{-t} \int_0^t e\tau \|\mathcal{F}u_t\|^2 d\tau$ , i.e., quantities all bounded via Lemma 6 in terms of the assumptions (A1), (A2), (A3).*

*Proof.* What follows is based on the identity (6). Since the Galerkin approximation  $u^n$  does not appear, there will be no ambiguity in setting  $\eta = \eta^n$ ,  $e = e^n$ ,  $v = v^n$ ,  $\phi = \phi^n$ . Setting  $\phi = -\tilde{\mathcal{A}}\eta$  in (6), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathcal{F}\eta\|^2 + \|\tilde{\mathcal{A}}\eta\|^2 &\leq c \|\tilde{\mathcal{A}}u\| \cdot \|\mathcal{F}\eta\| \cdot \|\tilde{\mathcal{A}}\eta\| + c \|\mathcal{F}\eta\|^{3/2} \|\tilde{\mathcal{A}}\eta\|^{3/2} \\ &\quad + c \|\tilde{\mathcal{A}}u\| \cdot \lambda_{n+1}^{-1/2} \|\tilde{\mathcal{A}}u\| \cdot \|\tilde{\mathcal{A}}\eta\|, \end{aligned}$$

and hence

$$(12) \quad \frac{d}{dt} \|\mathcal{A}\eta\|^2 + \|\tilde{\mathcal{A}}\eta\|^2 \leq c \|\tilde{\mathcal{A}}u\|^2 \cdot \|\mathcal{F}\eta\|^2 + c \|\mathcal{F}\eta\|^6 + c \lambda_{n+1}^{-1} \|\tilde{\mathcal{A}}u\|^4.$$

Multiplying (12) by  $e^t$  and integrating gives

$$\begin{aligned} (13) \quad e^{-t} \int_0^t e^\tau \|\tilde{\mathcal{A}}\eta\|^2 d\tau &\leq e^{-t} \int_0^t e^\tau \{\gamma^2 + c \|\tilde{\mathcal{A}}u\|^2 \gamma^2 + c \gamma^6 + c \lambda_{n+1}^{-1} \|\tilde{\mathcal{A}}u\|^4\} d\tau \\ &\leq e^{-t} \int_0^t e^\tau c(\gamma^2 + \lambda_{n+1}^{-1}) d\tau. \end{aligned}$$

Setting  $\phi = \eta_t$  in (6), we obtain

$$\begin{aligned} \|\eta_t\|^2 &\leq \|\tilde{\mathcal{A}}\eta\| \cdot \|\eta_t\| + c \|\tilde{\mathcal{A}}u\| \cdot \|\mathcal{F}\eta\| \cdot \|\eta_t\| + c \|\tilde{\mathcal{A}}\eta\| \cdot \|\mathcal{F}\eta\| \cdot \|\eta_t\| \\ &\quad + c \|\tilde{\mathcal{A}}e\| \cdot \|\mathcal{F}\eta\| \cdot \|\eta_t\| + c \|\tilde{\mathcal{A}}u\| \cdot \|\mathcal{F}e\| \cdot \|\eta_t\|, \end{aligned}$$

and hence

$$(14) \quad \|\eta_t\|^2 \leq c \|\tilde{\mathcal{A}}\eta\|^2 + c \|\tilde{\mathcal{A}}u\|^2 \cdot \|\mathcal{F}\eta\|^2 + c \|\tilde{\mathcal{A}}\eta\|^2 \cdot \|\mathcal{F}\eta\|^2 + c \lambda_{n+1}^{-1} \|\tilde{\mathcal{A}}u\|^4.$$

Multiplying by  $e^t$  and integrating, and using (13), we get



$$(15) \quad e^{-t} \int_0^t e^\tau \|\eta_t\|^2 d\tau \leq e^{-t} \int_0^t e^\tau c(\gamma^2 + \lambda_{n+1}^{-1}) d\tau .$$

Now differentiating (6) with respect to  $t$  and setting  $\phi = \eta_t$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta_t\|^2 + \|\nabla \eta_t\|^2 &= -(u_t \cdot \nabla \eta, \eta_t) - (\eta_t \cdot \nabla u, \eta_t) - (\eta \cdot \nabla u_t, \eta_t) \\ &\quad - (\eta_t \cdot \nabla \eta, \eta_t) + (\eta_t \cdot \nabla e, \eta_t) + (\eta \cdot \nabla e_t, \eta_t) \\ &\quad + (e_t \cdot \nabla \eta, \eta_t) + (u_t \cdot \nabla e, \eta_t) + (u \cdot \nabla e_t, \eta_t) \\ &\quad + (e_t \cdot \nabla v, \eta_t) + (e \cdot \nabla v_t, \eta_t) \\ &\leq c \|u_t\| \cdot \|\tilde{\Delta} \eta\| \cdot \|\nabla \eta_t\| + c \|\tilde{\Delta} u\| \cdot \|\eta_t\| \cdot \|\nabla \eta_t\| \\ &\quad + c \|\nabla \eta\| \cdot \|\eta_t\|^{1/2} \|\nabla \eta_t\|^{3/2} + \|u_t\|_3 \cdot \|e\|_6 \cdot \|\nabla \eta_t\| \\ &\quad + c \|\tilde{\Delta} u\| \cdot \|e_t\| \cdot \|\nabla \eta_t\| , \end{aligned}$$

and hence

$$(17) \quad \begin{aligned} \frac{d}{dt} \|\eta_t\|^2 &\leq c \|u_t\|^2 \cdot \|\tilde{\Delta} \eta\|^2 + c \|\tilde{\Delta} u\|^2 \cdot \|\eta_t\|^2 + c \|\nabla \eta\|^4 \cdot \|\eta_t\|^2 \\ &\quad + c \lambda_{n+1}^{-1} (\|u_t\|^2 + \|\nabla u_t\|^2) \|\tilde{\Delta} u\|^2 + c \lambda_{n+1}^{-1} \|\tilde{\Delta} u\|^2 \cdot \|\nabla u_t\|^2 . \end{aligned}$$

Multiplying by  $e^t$  and integrating, using (13), and (15), we get

$$\|\eta_t(t)\|^2 \leq e^{-t} \|\eta_t(0)\|^2 + e^{-t} \int_0^t e^\tau c(\gamma^2 + \lambda_{n+1}^{-1}) d\tau + c \lambda_{n+1}^{-1} e^{-t} \int_0^t e^\tau \|\nabla u_t\|^2 d\tau .$$

From (14), it is clear  $\|\eta_t(0)\|^2 \leq c \lambda_{n+1}^{-1} \|\tilde{\Delta} a\|^4$ . Using this and the *a-priori* estimates of Lemma 6, we get

$$(18) \quad \|\eta_t(t)\|^2 < c(\gamma^2 + \lambda_{n+1}^{-1}) .$$

Since

$$-\frac{d}{dt} \|\nabla \eta\|^2 = -2(\nabla \eta, \nabla \eta_t) = 2(\tilde{\Delta} \eta, \eta_t) \leq 1/2 \|\tilde{\Delta} \eta\|^2 + 2 \|\eta_t\|^2 ,$$

(12) implies

$$(19) \quad \|\tilde{\Delta} \eta\|^2 \leq \|\eta_t\|^2 + c \|\tilde{\Delta} u\|^2 \cdot \|\nabla \eta\|^2 + c \|\nabla \eta\|^6 + c \lambda_{n+1}^{-1} \|\tilde{\Delta} u\|^4 .$$

Clearly (18), (19) and the assumption  $\|\nabla \eta\| < \gamma \leq \delta$  imply

$$(20) \quad \|\tilde{\Delta} \eta(t)\|^2 < c(\gamma^2 + \lambda_{n+1}^{-1}) .$$

**6. A-priori estimate of  $\|g^n\|$ .** The forcing term on the right side of (7) is

$$\begin{aligned} g^n &= Q_n[u \cdot \nabla \eta^n] + Q_n[\eta^n \cdot \nabla u] + Q_n[\eta^n \cdot \nabla \eta^n] \\ &\quad + P_n[\eta^n \cdot \nabla e^n] + P_n[e^n \cdot \nabla \eta^n] + P_n[u \cdot \nabla e^n] + P_n[e^n \cdot \nabla v^n] . \end{aligned}$$

LEMMA 9. Whenever  $\|\tilde{A}\eta(t)\|^2 < R(\delta^2 + \lambda_1)$ , there holds

$$(21) \quad \|g^n(t)\|^2 \leq \theta(n)\|\tilde{A}\eta^n\|^2 + \nu\lambda_{n+1}^{-1},$$

with  $\theta(n)$  a function satisfying  $\theta(n) \rightarrow 0$  as  $n \rightarrow \infty$ . The function  $\theta$  and constant  $\nu$  depend only on  $\delta$ ,  $R$ ,  $\Omega$  and  $\sup_{t \geq 0} \|\tilde{A}u\|$ .

To prove Lemma 9, we need:

LEMMA 10. For any vector field  $w \in H^1(\Omega)$ , there holds

$$(22) \quad \|Q_n w\|^2 \leq \hat{\theta}(n)\|w\|_{H^1}^2,$$

with  $\hat{\theta}(n)$  a function satisfying  $\hat{\theta}(n) \rightarrow 0$  as  $n \rightarrow \infty$ . The function  $\hat{\theta}(n)$  depends only on  $\Omega$ .

To prove Lemma 10, we need:

LEMMA 11. For every  $0 < \varepsilon < 1$ , there exists a continuously differentiable, piecewise twice differentiable “cut-off” function  $\lambda_\varepsilon(s)$ , defined for  $s \geq 0$ , such that  $\lambda_\varepsilon(0) = 1$ ,  $\lambda'_\varepsilon(0) = 0$ ,  $\lambda_\varepsilon(s) = 0$  for  $s \geq \varepsilon$ , and such that everywhere  $|\lambda_\varepsilon(s)| \leq \min\{\varepsilon/s, 1\}$ ,  $|\lambda'_\varepsilon(s)| \leq \min\{\varepsilon/s, C_\varepsilon\}$ , and  $|\lambda''_\varepsilon(s)| \leq C_\varepsilon$ , with constants  $C_\varepsilon$  dependent only on  $\varepsilon$ .

The construction of  $\lambda_\varepsilon(s)$  is well-known; see [2]. We only remark, it is probably easiest to start with the function

$$\tilde{\lambda}_\varepsilon(s) \equiv \int_s^\varepsilon \frac{\varepsilon}{\sigma} \left(1 - \frac{\sigma}{\varepsilon}\right) d\sigma = \varepsilon(\log \varepsilon - \log s) + (s - \varepsilon),$$

defined for  $0 < s \leq \varepsilon$ . Observing that  $|\tilde{\lambda}_\varepsilon(s)| \leq \varepsilon/s$ ,  $|\tilde{\lambda}'_\varepsilon(s)| \leq \varepsilon/s$ , and  $\tilde{\lambda}''_\varepsilon(s) = \varepsilon/s^2$ , it is clear how to proceed to construct  $\lambda_\varepsilon(s)$ .

*Proof of Lemma 10.* Let  $\lambda_\varepsilon(x) \equiv \lambda_\varepsilon(s)$ , where  $\lambda_\varepsilon(s)$  is as in Lemma 11 and  $s = \text{distance}(x, \partial\Omega)$ . For sufficiently small  $\varepsilon$ , this is well-defined and the estimates for the derivatives of  $\lambda_\varepsilon(s)$  are valid for  $\lambda_\varepsilon(x)$ , i.e.,  $|\nabla \lambda_\varepsilon(x)| = |\lambda'_\varepsilon(s)|$  and  $|D_x^2 \lambda_\varepsilon(x)| \leq c(|\lambda''_\varepsilon(s)| + |\lambda'_\varepsilon(s)|)$ .

Choose a vector field  $\psi$ , in accordance with Lemmas 1 and 2, such that  $\text{curl } \psi = Pw$ ,  $\psi|_{\partial\Omega} = 0$ , and  $\|\psi\|_{H^2} \leq c\|Pw\|_{H^1} \leq c\|w\|_{H^1}$ . Clearly,

$$(23) \quad Pw - \text{curl}(\lambda_\varepsilon \psi) \in J_1(\Omega), \quad \text{curl}(\lambda_\varepsilon \psi) \in L^2(\Omega).$$

We have the following estimates:

$$(24) \quad \begin{aligned} &\|Pw - \text{curl}(\lambda_\varepsilon \psi)\|_{H^1} = \|Pw - \lambda_\varepsilon Pw - (\nabla \lambda_\varepsilon) \times \psi\|_{H^1} \\ &\leq C_\varepsilon \|Pw\|_{H^1} + \|(\nabla \lambda_\varepsilon) \times \psi\| + \|(D_x^2 \lambda_\varepsilon) \psi\| + \|(\nabla \lambda_\varepsilon) \nabla \psi\| \\ &\leq C_\varepsilon \|Pw\|_{H^1} \leq C_\varepsilon \|w\|_{H^1}, \end{aligned}$$

$$\begin{aligned}
(25) \quad & \| \operatorname{curl} (\lambda_\varepsilon \psi) \| \leq \| \nabla \lambda_\varepsilon \times \psi \| + \| \lambda_\varepsilon \operatorname{curl} \psi \| \\
& \leq \left( \int_{\Omega_\varepsilon} |\varepsilon/s|^2 \cdot |\psi|^2 dx \right)^{1/2} + \left( \int_{\Omega_\varepsilon} |\nabla \psi|^2 dx \right)^{1/2} \\
& \leq c\varepsilon \| \psi \|_{H^1} + c\varepsilon \| \psi \|_{H^2} \leq c\varepsilon \| w \|_{H^1} .
\end{aligned}$$

In (25) we have set  $\Omega_\varepsilon \equiv \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$  and used the inequality

$$\int_0^1 \phi^2(s)/s^2 ds \leq 4 \int_0^1 (\phi'(s))^2 ds ,$$

valid if  $\phi(0) = 0$ , and also the inequality  $\| \phi \|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon \| \phi \|_{H^1(\Omega)}$ , valid for  $\phi \in H^1(\Omega)$ . Finally, using Lemma 4 and (23), (24), (25), we have

$$\begin{aligned}
\| Q_n w \| &= \| Q_n P_w \| \\
&\leq \| Q_n [Pw - \operatorname{curl} (\lambda_\varepsilon \psi)] \| + \| Q_n \operatorname{curl} (\lambda_\varepsilon \psi) \| \\
&\leq \lambda_{n+1}^{-1/2} \| Pw - \operatorname{curl} (\lambda_\varepsilon \psi) \| + \| \operatorname{curl} (\lambda_\varepsilon \psi) \| \\
&\leq \lambda_{n+1}^{-1/2} C_\varepsilon \| w \|_{H^1} + c\varepsilon \| w \|_{H^1} ,
\end{aligned}$$

which clearly implies the result.

*Proof of Lemma 9.* Using Lemmas 4, 5 and 10 we obtain:

$$\begin{aligned}
\| Q_n [u \cdot \nabla \eta^n] \|^2 &\leq \hat{\theta}(n) \| u \cdot \nabla \eta^n \|_{H^1}^2 \\
&\leq \hat{\theta}(n) (\| u \eta_x^n \|^2 + \| u_x \eta_x^n \|^2 + \| u \eta_{xx}^n \|^2) \\
&\leq \hat{\theta}(n) (\| u \eta_x^n \|^2 + \| \nabla u \|^2_4 \| \nabla \eta^n \|^2_4 + \| u \eta_{xx}^n \|^2) \\
&\leq c\hat{\theta}(n) \| \tilde{\mathcal{A}}u \|^2 \cdot \| \tilde{\mathcal{A}}\eta^n \|^2 , \\
\| Q_n [\eta^n \cdot \nabla u] \|^2 &\leq \hat{\theta}(n) \| \eta^n \cdot \nabla u \|_{H^1}^2 \\
&\leq \hat{\theta}(n) (\| \eta^n u_x \|^2 + \| \eta_x^n u_x \|^2 + \| \eta^n u_{xx} \|^2) \\
&\leq c\hat{\theta}(n) \| \tilde{\mathcal{A}}u \|^2 \cdot \| \tilde{\mathcal{A}}\eta^n \|^2 , \\
\| Q_n [\eta^n \cdot \nabla \eta^n] \|^2 &\leq \hat{\theta}(n) \| \eta^n \cdot \nabla \eta^n \|_{H^1}^2 \\
&\leq \hat{\theta}(n) (\| \eta^n \eta_x^n \|^2 + \| \eta_x^n \eta_x^n \|^2 + \| \eta^n \eta_{xx}^n \|^2) \\
&\leq c\hat{\theta}(n) \| \tilde{\mathcal{A}}\eta^n \|^4 , \\
\| P_n [\eta^n \cdot \nabla e^n] \|^2 &\leq \| \eta^n \cdot \nabla e^n \|^2 \leq c \| \tilde{\mathcal{A}}\eta^n \|^2 \cdot \| \nabla e^n \|^2 \leq c\lambda_{n+1}^{-1} \| \tilde{\mathcal{A}}u \|^2 \cdot \| \tilde{\mathcal{A}}\eta^n \|^2 , \\
\| P_n [e^n \cdot \nabla \eta^n] \|^2 &\leq \| e^n \cdot \nabla \eta^n \|^2 \leq c \| e^n \|_6^2 \cdot \| \nabla \eta^n \|^2_3 \leq c\lambda_{n+1}^{-1} \| \tilde{\mathcal{A}}u \|^2 \cdot \| \tilde{\mathcal{A}}\eta^n \|^2 , \\
\| P_n [u \cdot \nabla e^n] \|^2 &\leq \| u \cdot \nabla e^n \|^2 \leq c \| \mathcal{A}u \|^2 \cdot \| \nabla e^n \|^2 \leq c\lambda_{n+1}^{-1} \| \tilde{\mathcal{A}}u \|^4 , \\
\| P_n [e^n \cdot \nabla v^n] \|^2 &\leq \| e^n \cdot \nabla v^n \|^2 \leq c \| e^n \|_6 \cdot \| \nabla v^n \|^2 \leq c\lambda_{n+1}^{-1} \| \tilde{\mathcal{A}}u \|^4 .
\end{aligned}$$

**7. Proof of Main Result.** An *a-priori* estimate, similar to Lemma 6, will be needed for solutions  $\zeta$  of (2). If  $\| \nabla \zeta(t_0) \| < \delta$ , (A3) ensures that  $\| \nabla \zeta(t) \| < A\delta$ , for all  $t \geq t_0$ . So  $v = u + \zeta$  is a solution of the Navier-Stokes equations with  $\| \nabla v(t) \| < M + A\delta$ , for  $t \geq t_0$ .

Also, in view of Lemma 6,  $\|\tilde{A}v(t_0)\| \leq \|\tilde{A}u(t_0)\| + \|\tilde{A}\zeta(t_0)\|$  is bounded if  $\|\tilde{A}\zeta(t_0)\|$  is bounded. Hence the *a-priori* estimates for solutions of the Navier-Stokes equations, already appealed to in Lemma 6, imply  $\|\tilde{A}\zeta(t)\|$  is bounded, for all  $t \geq t_0$ , in terms of  $\|\tilde{A}\zeta(t_0)\|$  and the constants and norms of assumptions (A1), (A2), (A3). In summary:

**LEMMA 12.** *For perturbations  $\zeta$  (i.e., solutions of (2)) which satisfy initially  $\|\nabla\zeta(t_0)\| < \delta$  and  $\|\tilde{A}\zeta(t_0)\|^2 \leq R(\delta^2 + \lambda_1^{-1})$ , there holds  $\|\tilde{A}\zeta(t)\| < c$ , for all  $t \geq t_0$ , where  $c$  is a constant dependent only on  $R$ ,  $\delta$ ,  $\Omega$  and the bounds assumed for various norms in conditions (A1), (A2), (A3).*

Now consider the expression  $\beta(t) = c(\|\tilde{A}u(t)\|^2 + \|\tilde{A}\eta^n\|^2 + \|\tilde{A}\zeta(t)\|^2)$  appearing in Lemma 7. In view of Lemmas 6, 8 and 12, there exists a constant  $B$  bounding all possible values of  $\beta(t)$  in any interval  $t_0 \leq t \leq t^*$  for which it is known

$$(26) \quad \|\nabla\eta^n(t)\| < \delta ,$$

provided we restrict our considerations to perturbations  $\zeta$  satisfying

$$(27) \quad \|\nabla\zeta(t_0)\| < \delta \quad \text{and} \quad \|\tilde{A}\zeta(t_0)\|^2 < R(\delta^2 + \lambda_1^{-1}) .$$

Consider, also, the forcing term  $g^n$  appearing in Lemma 7. Together, Lemmas 8 and 9 imply

$$(28) \quad \|g^n(t)\|^2 \leq \theta(n)R(\gamma^2 + \lambda_{n+1}^{-1}) + \nu\lambda_{n+1}^{-1}$$

on any interval  $0 \leq t \leq t^*$  for which it is known

$$(29) \quad \|\nabla\eta^n(t)\| < \gamma , \quad \text{with} \quad \gamma \leq \delta .$$

We now choose  $T$  sufficiently large that

$$(30) \quad Ae^{-\alpha T} \leq \frac{1}{2} ,$$

with  $\alpha$  and  $A$  as in (A3), and then choose  $N$  sufficiently large that

$$(31) \quad e^{BT}T\theta(n)R < \frac{1}{8}$$

and

$$(32) \quad \gamma_n \equiv \sqrt[3]{8}(e^{BT}T\theta(n)R + \nu)^{1/2}\lambda_{n+1}^{-1/2} < \delta$$

hold for  $n \geq N$ . Here, we have defined  $\gamma_n$  in such a way that, in the presence of (31), there holds

$$(33) \quad e^{BT}T[\theta(n)R(\gamma_n^2 + \lambda_{n+1}^{-1}) + \nu\lambda_{n+1}^{-1}] < \frac{1}{4}\gamma_n^2 ,$$

as may be easily checked.

For  $n \geq N$ , we claim

$$(34) \quad \|\nabla \gamma^n(t)\| < \gamma_n, \quad \text{for all } t \geq 0.$$

If not, that is, if (34) fails for some  $n \geq N$ , let  $t^*$  be the first value of  $t$  for which  $\|\nabla \gamma^n(t^*)\| = \gamma_n$ . To show it is impossible that  $t^* \leq T$ , consider Lemma 7 with  $t_0 = 0$  and  $\zeta \equiv 0$ . The implication, in view of (28) and (33), and remembering  $\gamma^n(0) = 0$ , is that

$$\begin{aligned} \|\nabla \gamma^n(t^*)\|^2 &= \|\nabla w(t^*)\|^2 \\ &\leq e^{BT} T [\theta(n) R(\gamma_n^2 + \lambda_{n+1}^{-1}) + \nu \lambda_{n+1}^{-1}] < \frac{1}{4} \gamma_n^2, \end{aligned}$$

which contradicts our supposition about  $t^*$ . On the other hand, if  $t^* > T$ , then  $\|\nabla \gamma^n(t^* - T)\| < \gamma_n$ , and of course  $\|\tilde{A} \gamma^n(t^* - T)\| < R(\delta^2 + \lambda_1^{-1})$ , by Lemma 8. So, considering Lemma 7 again, but with  $t_0 = t^* - T$  and  $\zeta(t^* - T) = \gamma^n(t^* - T)$ , we find

$$(35) \quad \begin{aligned} \|\nabla \gamma^n(t^*) - \zeta(t^*)\|^2 &\leq e^{BT} T [\theta(n) R(\gamma_n^2 + \lambda_{n+1}^{-1}) + \nu \lambda_{n+1}^{-1}] \\ &\leq \frac{1}{4} \gamma_n^2. \end{aligned}$$

In addition, in view of (30), the assumption (A3) implies

$$(36) \quad \|\nabla \zeta(t^*)\| \leq \frac{1}{2} \|\nabla \zeta(t^* - T)\| < \frac{1}{2} \gamma_n.$$

Together, (35) and (36) imply  $\|\nabla \gamma^n(t^*)\| < \gamma_n$ , again contradicting our supposition about  $t^*$ . So (34) must hold, and combined with Lemma 6, it implies the theorem of §3.

## REFERENCES

1. L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova, **31** (1961), 308-340.
2. R. Finn, *On the steady-state solutions of the Navier-Stokes equations*, Acta Math., **3** (1961), 197-244.
3. C. Foias, *Statistical study of Navier-Stokes equations. Part I*, Rend. Sem. Math. Univ. Padova, **48** (1973), 219-348.
4. J. Heywood, *The Navier-Stokes equations: On the existence, regularity and decay of solutions*, Indiana U. Math. J., **29** (1980), 639-681.
5. ———, *Classical solutions of the Navier-Stokes equations*, Proc. IUTAM Symp, Paderborn (W. Germany) 1979, Springer Lecture Notes in Math., **771** (1980).
6. ———, *On classical solutions of the nonstationary Navier-Stokes equations in two and three dimensions*, Fluid Dynamics Transactions, **10**, (1980), 177-203.
7. J. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes Problem I*, SIAM J. Num. Anal.
8. E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr., **4** (1951), 213-231.

9. S. Ito, *The existence and the uniqueness of regular solution of nonstationary Navier-Stokes equation*, J. Fac. Sci. Univ. Tokyo Sect. 1A, **9** (1961), 103-140.
10. A. A. Kiselev and O. A. Ladyzhenskaya, *On the existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluid*, Izv. Akad. Nauk SSSR Ser. Mat., **21** (1957), 665-680.
11. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Second Edition, Gordon and Breach, New York, 1969.
12. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites nonlineaires*, Dunod (1969), Paris.
13. J. Necas, *Les méthodes directes en théorie des équations elliptiques*, Masson et C<sup>ie</sup>, Editeurs, (1967), Paris.
14. G. Prodi, *Teoremi di tipo locale per il sistema de Navier-Stokes e stabilità delle soluzione stazionarie*, Rend. Sem. Mat. Univ. Padova, **32** (1962), 374-397.
15. R. Rautmann, *On the convergence-rate of nonstationary Navier-Stokes approximations*, Proc. IUTAM Symp, Paderborn (W. Germany) 1979, Springer Lecture Notes in Math., **771** (1980).
16. ———, *Eine Fehlerschranke für Galerkinapproximationen lokaler Navier-Stokes-Lösungen*, *Constructive methods for Nonlinear Boundary Value Problems and Nonlinear Oscillations*, edited by J. Albrecht, L. Collatz, K. Kirchgässner, Basel IRN 48, Birkhauser, (1979), 110-125.
17. R. Temam, *Navier-Stokes Equations*, North-Holland Publ. Comp., Amsterdam, 1977.

Received August 20, 1980. This research was supported by Natural Sciences and Engineering Research Council Canada, grant no. A4150

UNIVERSITY OF BRITISH COLUMBIA  
VANCOUVER, BC, CANADA V6T1W5



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DONALD BABBITT (Managing Editor)

University of California  
Los Angeles, CA 90024

HUGO ROSSI

University of Utah  
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, CA 94305

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF HAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1982 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan



Thomas E. Armstrong, Barycentric simplicial subdivision of infinite-dimensional simplexes and octahedra .....	251
Hom Nath Bhattarai and James William Fernandez, Joins of double coset spaces .....	271
Alexandru Buium, Ritt schemes and torsion theory .....	281
Jacob Burbea, Operator-valued Pick's conditions and holomorphicity .....	295
Su-Shing Chen, Duality condition and property (S) .....	313
Ky Fan, Evenly distributed subsets of $S^n$ and a combinatorial application ...	323
Leslie Foged, On $g$ -metrizability .....	327
John Groves Heywood, An error estimate uniform in time for spectral Galerkin approximations of the Navier-Stokes problem .....	333
Aggie Ho, The Kreĭn-Milman property and complemented bushes in Banach spaces .....	347
David R. Jackett, Rings on certain mixed abelian groups .....	365
Shoji Kyuno, Prime ideals in gamma rings .....	375
George Lucius O'Brien, Zero-inducing functions on finite abelian groups .....	381
P. Robba, Sur les équations différentielles linéaires $p$ -adiques. II .....	393
Wolfgang Ruess, [Weakly] compact operators and DF spaces .....	419
Claude Schochet, Topological methods for $C^*$ -algebras. II. Geometry resolutions and the Künneth formula .....	443
Harry F. Smith, Jr., Equivalent nilpotencies in certain generalized right alternative rings .....	459
Max Warshauer, Diagonalization up to Witt .....	469
Hugh C. Williams, A class of primality tests for trinomials which includes the Lucas-Lehmer test .....	477