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Two of the most important global properties of complex spaces $(X, \mathcal{O})$, holomorphic convexity and holomorphic separability, can each be characterized in terms of the standard natural map $\chi: X \to S_c(\mathcal{O}(X))$, $x \to \chi_x$, $\chi_x(f) := f(x)$, $f \in \mathcal{O}(X)$, of X into the continuous spectrum $S_c(\mathcal{O}(X))$ of the global function algebra $\mathcal{O}(X)$. The question as to whether there is any global function theoretical property of $(X, \mathcal{O})$ corresponding to the surjectivity of χ has remained unanswered. The purpose of this paper is to present an answer for finite dimensional spaces. For such spaces $(X, \mathcal{O})$ it will be shown that the surjectivity of χ is equivalent to requiring that for finitely many functions $f_1, \cdots, f_m \in \mathcal{O}(X)$ with no common zero on X there exist functions $g_1, \cdots, g_m \in \mathcal{O}(X)$ with $\sum_{i=1}^m f_i g_i = 1$. This property will be called the weak Nullstellensatz for the complex space $(X, \mathcal{O})$. An example due to H. Rossi shows that this result is not valid for infinite dimensional complex spaces. An application of the weak Nullstellensatz for Fréchet algebras A involving the Michael conjecture is that $S_c(A)$ is always dense in the spectrum $S(A)$ of A.

1. Introduction. Important global properties of complex spaces[1] $(X, \mathcal{O})$ can be characterized in terms of the standard natural map $\chi: X \to S_c(\mathcal{O}(X))$, $x \to \chi_x$, $\chi_x(f) := f(x)$, $f \in \mathcal{O}(X)$, of X into the continuous spectrum $S_c(A)$ of the global function algebra $A := \mathcal{O}(X)$ which takes points x of X to the corresponding point evaluations χ_x. For example: X is holomorphically separable if and only if χ is injective; X is holomorphically convex if and only if χ is proper (see § 3). A complex space which is both holomorphically separable and holomorphically convex is called Stein. The customary description of Stein spaces as being those complex spaces which have "sufficiently many" global holomorphic functions [14, VII] attains precision from a theorem of Igusa/Remmert/Iwahashi/Forster [15, 21, 17, 7] stating: X is Stein if and only if χ is a homeomorphism. In other words, a complex space X is Stein if and only if there are enough global holomorphic functions on X to enable X to be regained topologically from the continuous spectrum of these functions.[2] According to the above remarks, the main assertion of this theorem is that

[1] Throughout this paper, a complex space means a reduced complex space with countable topology.

[2] A Stein space X can also be regained as a complex space from $S_c(A)$ [**8**, **13**].

the surjectivity of χ is a consequence of χ being injective and closed.

Until now the question as to whether the surjectivity of χ corresponds to any global function theoretical property of the complex space $(X, \mathcal{O})$ has been unanswered. In order to treat this problem, the global property of satisfying the weak Nullstellensatz will be investigated here first for arbitrary function algebras and then for complex spaces. The weak Nullstellensatz[3] is defined in § 3 to be valid for a complex space $(X, \mathcal{O})$ if for finitely many functions $f_1, \cdots, f_m \in \mathcal{O}(X)$ with no common zero on X there exist functions $g_1, \cdots, g_m \in \mathcal{O}(X)$ with $1 = \sum_{i=1}^m f_i g_i$. The reason for choosing this terminology is as follows. In algebraic geometry there is a theorem referred to as the weak Nullstellensatz which states that every proper ideal in a polynomial ring $K[T_1, \cdots, T_n]$ in n variables with coefficients in an algebraically closed field K has at least one zero in K^n. Since $K[T_1, \cdots, T_n]$ is Noetherian, this is equivalent to saying that finitely many polynomials $f_1, \cdots, f_m \in K[T_1, \cdots, T_n]$ without a common zero never generate a proper ideal, i.e., there always exist polynomials $g_1, \cdots, g_m \in K[T_1, \cdots, T_n]$ with $1 = \sum_{i=1}^m f_i g_i$. The weak Nullstellensatz is equivalent to Hilbert's Nullstellensatz [26, VII § 3] which asserts that a power of a polynomial $f \in K[T_1, \cdots, T_n]$ lies in an ideal of $K[T_1, \cdots, T_n]$, if the polynomial vanishes for every zero of the ideal. In Rabinowitsch's proof of latter theorem, the weak version is proved first, and then it is shown that Hilbert's Nullstellensatz results from the weak Nullstellensatz [10, 26].

In contrast to the situation in algebraic geometry, the weak Nullstellensatz for complex spaces is not equivalent to the Hilbert Nullstellensatz for complex spaces [7] and the weak Nullstellensatz is not satisfied for every complex space. Cartan [4] was the first person to investigate the weak Nullstellensatz for complex spaces. As a result of Cartan's Theorem B, the weak Nullstellensatz holds for every Stein space. Using this result, it follows easily for domains X in C^n that the validity of the weak Nullstellensatz is equivalent to the holomorphic convexity of X [13, V § 5.3]. This equivalence can be readily generalized to holomorphically separable unramified domains X over Stein manifolds M—in particular to subdomains X of M (§ 3). Actually, every holomorphically convex complex space satisfies the weak Nullstellensatz; however, the converse is not true in general (see § 3).

The weak Nullstellensatz is a frequently used property in the theory of function algebras as well as in complex analysis. Two important examples of its application will be explicitly mentioned.

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[3] In the literature this property is usually not titelized; lately, however, it has been referred to by Grauert/Remmert [**13**, V § 5.3] as the representation of the unit.

Michael [20, 12.5] and Arens [1, 7.1] used the weak Nullstellensatz to prove that for certain Fréchet algebras A every algebra homomorphism $A \to \boldsymbol{C}$ is continuous, i.e., the continuous spectrum $S_c(A)$ is equal to the spectrum $S(A)$. Iss'sa [16, 1.2] employed the weak Nullstellensatz for Stein manifolds to show that normal Stein manifolds are completely determined up to a bimeromorphic map by the global meromorphic function field.

The purpose of this paper is to prove that for finite dimensional complex spaces $(X, \mathcal{O})$ the weak Nullstellensatz is equivalent to the surjectivity of the natural map $\chi\colon X \to S_c(\mathcal{O}(X))$, $x \to \chi_x$. In other words, such spaces $(X, \mathcal{O})$ satisfy the weak Nullstellensatz if and only if each closed maximal ideal in $\mathcal{O}(X)$ is of the form $\{f \in \mathcal{O}(X)\colon f(x) = 0\}$ for a fixed point $x \in X$. Another characterization of the weak Nullstellensatz for finite dimensional complex spaces $(X, \mathcal{O})$ which is proved here is that every closed proper ideal in $\mathcal{O}(X)$ has at least one zero in X.

The proofs use an analytical theorem of Grauert/Wiegmann [11, 25] and a topological-algebraic theorem of Arens [1]. Another consequence of Arens' theorem is that the continuous spectrum $S_c(A)$ of evey Fréchet algebra A is dense in $S(A)$.

In summary, a connection is established between a basic property in the theory of complex analysis and fundamental concepts in the theory of topological algebras. The interrelationship of these two discriplines is of special interest recently (see [3] and the references there).

2. The weak Nullstellensatz for function algebras. In order to demonstrate the strength of the weak Nullstellensatz, several implications of this property for function algebras will be mentioned in this section. A characterization of the weak Nullstellensatz for Fréchet function algebras will also be given, and it will be shown that $S_c(A)$ is dense in $S(A)$ for every Fréchet algebra A; both results are a straightforward application of a theorem of Arens [1, 6.3].

An *algebra* will mean a commutative complex algebra with identity. A *homomorphism between two algebras* is a $\boldsymbol{C}$-linear ring homomorphism which preserves the identity. The *spectrum* $S(A) :=$ Hom $(A, \boldsymbol{C})$ of an algebra A is the set of all homomorphisms of A to $\boldsymbol{C}$ endowed with the *Gelfand topology*, i.e., the topology which $S(A)$ inherits from the product topology on the Hausdorff space $\boldsymbol{C}^A$ of all maps $A \to \boldsymbol{C}$. In this topology, $S(A)$ is a closed subset of $\boldsymbol{C}^A$ (cf. [20, 6.2]).

For a topological space X, let $\mathscr{C}(X)$ denote the algebra of continuous complex-valued functions on X with pointwise operations. In this paper, a *function algebra* will refer to a pair (X, A), where X is an arbitrary topological space and A is a subalgebra of $\mathscr{C}(X)$ containing the constants. The natural map

$$\chi: X \longrightarrow S(A)\,, \qquad x \longrightarrow \chi_x\,, \qquad \chi_x(f) := f(x)\,, \qquad f \in A\,,$$

is obviously continuous, since $A \subset \mathscr{C}(X)$.

DEFINITION. The *weak Nullstellensatz* is valid *for a function algebra* (X, A) if for every finite set of functions $f_1, \cdots, f_m \in A$ with no common zero in X there exist functions $g_1, \cdots, g_m \in A$ such that

$$\sum_{i=1}^{m} f_i g_i = 1\,.$$

The weak Nullstellensatz holds for a function algebra (X, A) if and only if finitely many functions lying in a *proper* ideal of A have at least one common zero in X. An immediate consequence of this property is that a function $f \in A$ vanishing nowhere on X has an inverse $1/f$ in A.

As already mentioned, the weak Nullstellensatz is valid for $(K^n, K[T_1, \cdots, T_n])$, where $K[T_1, \cdots, T_n]$ is the polynomial algebra in n variables over an algebraically closed field K. If $(X, \mathscr{O})$ is a complex space with the property that every global holomorphic function is constant (for example, X compact and connected), then the weak Nullstellensatz is trivially satisfied for $(X, \mathscr{O}(X)) = (X, \boldsymbol{C})$.

PROPOSITION 1. *If the weak Nullstellensatz is valid for a function algebra* (X, A), *then the natural map* $\chi: X \to S(A)$ *has a dense image.*

Proof. Suppose the theorem false. Then there exists a point $\varphi \in S(A)$ not in the closure of $\chi(X)$. This means there are finitely many functions $f_1, \cdots, f_m \in A$ and an $\varepsilon > 0$ such that $U \cap \chi(X) = \emptyset$ for

$$U := \{\psi \in S(A): |\psi(f_i) - \varphi(f_i)| < \varepsilon,\ i = 1, \cdots, m\}\,.$$

Let $\lambda_i := \varphi(f_i)$ and $g_i := f_i - \lambda_i$. Then $g_1, \cdots, g_m \in \ker \varphi$ have no common zero in X. This leads to a contradiction, because $\ker \varphi$ is a maximal ideal and therefore proper.

The converse of Proposition 1 doesn't hold, as the following classical example shows. Let X be a completely regular topological space and let A be the Banach algebra of bounded continuous functions on X. $(S(A), \chi)$ is the Stone-Čech compactification of X [22, 3.2.11], but the weak Nullstellensatz is not satisfied for (X, A) since $f \in A$, $0 \notin f(X)$ obviously doesn't always imply $1/f \in A$. Another example is given by $X := C^2 \backslash \{0\}$ and holomorphic functions $A := \mathscr{O}(X)$ on X. A is isomorphic to $\mathscr{O}(C^2)$. Since $C^2 \to S(A)$, $x \mapsto \chi_x$ is a homeomorphism (see § 1), $\chi(X)$ is dense in $S(A)$. However, the weak Nullstellensatz doesn't hold for (X, A) as seen by considering the projections $pr_i \in \mathscr{O}(C^2)$ onto the ith component of a point in C^2, $i = 1, 2$.

For any algebra A, the *Gelfand transform* $\hat{f}$ of $f \in A$ is the continuous function on $S(A)$ defined by $\hat{f}(\varphi) := \varphi(f)$, $\varphi \in S(A)$. Thus, every algebra A induces a function algebra $(S(A), \hat{A})$ for $\hat{A} := \{\hat{f} \mid f \in A\}$. The natural map $\chi\colon S(A) \to S(\hat{A})$ is always bijective and the *Gelfand transformation* $A \to \mathscr{C}(S(A))$, $f \mapsto \hat{f}$, is always an algebra homomorphism. If (X, A) is a function algebra, the Gelfand transformation is injective, since the diagram

$$\begin{tikzcd} X \arrow[rr, "\chi"] \arrow[dr, "f"'] & & S(A) \arrow[dl, "\hat{f}"] \\ & C & \end{tikzcd}$$

is commutative. $\hat{f}$ is a continuous extension of f to $S(A)$ and $f(X) \subset \hat{f}(S(A))$. Finitely many functions $f_1, \cdots, f_m \in A$ define a map $f\colon X \to C^m$, $x \to (f_1(x), \cdots, f_m(x))$. The induced map $S(A) \to C^m$, $\varphi \to (\hat{f}_1(\varphi), \cdots, \hat{f}_m(\varphi))$, will be denoted by $\hat{f}$.

PROPOSITION 2. *If the weak Nullstellensatz is valid for a function algebra (X, A), then*

$$f(X) = \hat{f}(S(A))$$

for all $f \in A^m$ and $m \in N$.

Proof. For $f \in A^m$, $m \in N$, and $\varphi \in S(A)$, let $\lambda := (\lambda_1, \cdots, \lambda_m) := \hat{f}(\varphi) \in C^m$. It suffices to show that λ is assumed as a value by f. Denote the components of f by $f_1, \cdots, f_m$. The ideal in A generated by $f_1 - \lambda_1, \cdots, f_m - \lambda_m$ is proper, since it is contained in the maximal ideal $\ker \varphi$. Consequently, $f - \lambda$ has a zero in X.

The major point of interest in this paper are *topological function algebras*, i.e., function algebras (X, A) where A is a topological algebra under the topology induced on A by the topology of compact convergence on $\mathscr{C}(X)$, and, in particular, *Fréchet function algebras*, i.e., A is furthermore a Fréchet algebra. Every algebra A induces a topological function algebra $(S(A), \hat{A})$. For a complex space $(X, \mathcal{O})$, $A := \mathcal{O}(X)$ is canonically a Fréchet algebra [12], and therefore (X, A) is a Fréchet function algebra.

The *continuous spectrum* $S_c(A) := \mathrm{Hom}_c(A, \boldsymbol{C})$ *of a topological algebra* A is the set of all continuous homomorphisms of A to $\boldsymbol{C}$ endowed with the Gelfand topology. $S_c(A)$ is a closed subset of the set A' of all continuous linear maps $A \to \boldsymbol{C}$ where A' has the product topology [20, 6.2]. In this topology A' is normally referred to as the *weak dual* of A. The evaluations $\chi_x: A \to \boldsymbol{C}$, $f \mapsto f(x)$, of points $x \in X$ lie in $S_c(A)$ whenever (X, A) is a topological function algebra.

For Banach algebras A it is, of course, always true that $S(A) = S_c(A)$. If $(X, \mathcal{O})$ is a finite dimensional Stein space, then $S(\mathcal{O}(X)) = S_c(\mathcal{O}(X))$ [9].[4)] Michael [20, p. 53] conjectured that $S(A) = S_c(A)$ for every Fréchet algebra A. This conjecture has not been proved or disproved even today.

Two immediate results of the last propositions are:

COROLLARY 1. *If the weak Nullstellensatz is valid for a topological function algebra* (X, A), *then the continuous spectrum* $S_c(A)$ *of* A *is a dense subset of the spectrum* $S(A)$ *of* A.

COROLLARY 2. *If the weak Nullstellensatz is valid for a topological function algebra* (X, A), *then*

$$f(X) = \hat{f}(S(A)) = \hat{f}(S_c(A))$$

for all $f \in A^m$ *and* $m \in \boldsymbol{N}$.

A Fréchet function algebra (X, A) satisfies the weak Nullstellensatz if and only if the *joint spectral image* $\hat{f}(S_c(A))$ of every $f \in A^m$, $m \in \boldsymbol{N}$, is exactly the image of f:

THEOREM 1. *The weak Nullstellensatz is valid for a Fréchet function algebra* (X, A) *if and only if*

$$f(X) = \hat{f}(S_c(A))$$

for all $f \in A^m$ *and* $m \in \boldsymbol{N}$.

Proof. Let the joint spectral image of f be equal to the image

4) See [**6**] for a generalization. However, whether the requirement on the dimension of X can be dropped entirely is still unclear, since the proof in [**13**, V, §7.3] is erroneous (see Bemerkung **2**, p. 183).

of f for all $f \in A^m$, $m \in N$. In order to prove the validity of the weak Nullstellensatz, consider functions $f_1, \cdots, f_m \in A$ which generate a proper ideal in A. According to a theorem of Arens [1, 6.3] for Fréchet algebras

$$\hat{f}(S_c(A)) = \{(\lambda_1, \cdots, \lambda_m) \in C^m \mid \langle f_1 - \lambda_1, \cdots, f_m - \lambda_m \rangle \neq A\},$$

where $f \in A^m$ has the components $f_1, \cdots, f_m$ and $\langle f_1 - \lambda_1, \cdots, f_m - \lambda_m \rangle$ is the ideal in A generated by $f_1 - \lambda_1, \cdots, f_m - \lambda_m$. Then, the joint spectral image of f contains zero, and f has a zero in X.

If A is a Banach algebra, the weak Nullstellensatz holds for $(S(A), \hat{A})$ (see [14, I, H.10]). The weak Nullstellensatz is also valid for $(S_c(A), \hat{A})$, when A is a Fréchet algebra, as the theorem of Arens which was just mentioned shows. Another result of this theorem is

THEOREM 2. *If A is a Fréchet algebra, then the continuous spectrum $S_c(A)$ of A is a dense subset of the spectrum $S(A)$ of A.*

Proof. Let U be the neighborhood of a point $\varphi \in S(A)$ given by finitely many elements $f_1, \cdots, f_m \in A$ and an $\varepsilon > 0$, i.e.,

$$U := \{\psi \in S(A) \colon |\hat{f}_i(\psi) - \hat{f}_i(\varphi)| < \varepsilon, i = 1, \cdots, m\}.$$

Let $g \in A^m$ have the components $g_i := f_i - \varphi(f_i)$, $i = 1, \cdots, m$. It suffices to show that $\hat{g}_1, \cdots, \hat{g}_m$ have a common zero on $S_c(A)$. If $0 \notin \hat{g}(S_c(A))$, then elements $h_1, \cdots, h_m \in A$ with $\sum_{i=1}^m g_i h_i = 1$ would exist according to Arens' theorem [1, 6.3]. Thus $\sum_{i=1}^m \varphi(g_i)\varphi(h_i) = 1$ would follow leading to a contradiction, since $\varphi(g_i) = 0$, $i = 1, \cdots, m$.

PROPOSITION 3. *The weak Nullstellensatz is valid for a Fréchet function algebra (X, A) if either one of the following conditions holds*:
1. *The natural map $\chi\colon X \to S_c(A)$ is surjective.*
2. *Every closed proper ideal in A has at least one zero in X.*

Proof. An application of Theorem 1 shows that condition 1 implies the weak Nullstellensatz. Condition 1 follows from condition 2: For $\varphi \in S_c(A)$, $\ker \varphi$ is a closed maximal ideal in A and therefore has at least one zero $x \in X$. That means $\ker \varphi \subset \ker \chi_x$ and $\varphi = \chi_x$ results, since $\ker \chi_x$ is also a maximal ideal.

The converse of Proposition 3 is true for (X, A), where A is the algebra of all global holomorphic functions on a finite dimensional complex space X. This will be shown in the next section.

3. The weak Nullstellensatz for complex spaces. First it will be noted that every holomorphically convex complex space satisfies the weak Nullstellensatz; counterexamples show that the converse isn't true. Then two characterizations of the weak Nullstellensatz for finite dimensional complex spaces are proved.

A complex space $(X, \mathcal{O})$ is *holomorphically convex* if for every compact subset K of X, the following set

$$\hat{K} := \bigcap_{f \in \mathcal{O}(X)} \{x \in X : |f(x)| \leqq \|f\|_K\}$$

is compact, where $\|f\|_K := \sup_{y \in K} |f(y)|$.

PROPOSITION 4. *A complex space* $(X, \mathcal{O})$ *is holomorphically convex if and only if the natural map*

$$\chi : X \longrightarrow S_c(\mathcal{O}(X))$$

is proper, i.e., *the inverse image of every compact subset of* $S_c(\mathcal{O}(X))$ *is a compact subset of* X.

Proof. Let χ be proper and $A := \mathcal{O}(X)$. Suppose $K \subset X$ is compact. Then

$$U := \{f \in A \mid \|f\|_K \leqq 1\}$$

is a neighborhood of the origin in A. Its polar

$$U^0 := \bigcap_{f \in U} \{\varphi \in A' \mid |\varphi(f)| \leqq 1\}$$

is a compact subset of the weak dual A' of A, and therefore the closed subset $S_c(A) \cap U^0$ of U^0 is compact in $S_c(A)$ [20, 7.5]. Since

$$\hat{K} = \chi^{-1}(S_c(A) \cap U^0) ,$$

the holomorphic convexity of X follows.

If $(X, \mathcal{O})$ is holomorphically convex and $A := \mathcal{O}(X)$, then $S_c(A)$ can be endowed with a Stein complex structure such that $X \rightarrow S_c(A)$, $x \rightarrow \chi_x$ is proper and the algebra $\mathcal{O}(S_c(A))$ of holomorphic functions on $S_c(A)$ is topologically isomorphic to A [23, 21, 5]; $S_c(A)$ is homeomorphic to the *Remmert quotient* of X which is the quotient space X/R, where R is the following equivalence relation on X: $x \sim y$ iff $f(x) = f(y)$ for every $f \in A$.[5)]

Since $S_c(\mathcal{O}(X))$ is locally compact if χ is proper, Proposition 4

[5)] Mazet [19] proved that a complex space $(X, \mathcal{O})$ is holomorphically convex if and only if $\chi : X \rightarrow S_c(\mathcal{O}(X))_\beta$ is proper, where $S_c(\mathcal{O}(X))_\beta$ denotes the continuous spectrum of $\mathcal{O}(X)$ with the strong topology β.

can also be stated as follows: $(X, \mathcal{O})$ is holomorphically convex if and only if the natural map $\chi: X \to S_c(\mathcal{O}(X))$ is closed and $\chi^{-1}(\varphi)$ is a compact subset of X for every $\varphi \in S_c(\mathcal{O}(X))$ [2, I, 10.3, Proposition 7].

DEFINITION. The *weak Nullstellensatz* is valid *for a complex space* $(X, \mathcal{O})$ if it is valid for the function algebra $(X, \mathcal{O}(X))$.

Since the natural map $\chi: X \to S_c(\mathcal{O}(X))$ is surjective for a holomorphically convex complex space $(X, \mathcal{O})$, Proposition 3 (or a direct calculation) implies

COROLLARY 3. *The weak Nullstellensatz is valid for every holomorphically convex complex space.*

The converse of Corollary 3 is not true. Every non-compact complex space $(X, \mathcal{O})$ which just has constant global holomorphic functions satisfies the weak Nullstellensatz but is not holomorphically convex, since $\chi: X \to S_c(\mathcal{O}(X)) = \{\mathrm{id}_C\}$ is obviously not proper. For example: every non-compact, pseudoconcave manifold X (in particular, the punctured two-dimensional complex projective space $\boldsymbol{P}^2\backslash\{p\}$, $p \in \boldsymbol{P}^2$) or the union X of countably many copies of the Riemann sphere $\boldsymbol{P}$ with the north pole of the nth sphere identified to the south pole of the $(n+1)$st sphere; in both cases X carries the natural complex structure.

Two characterizations of those finite dimensional complex spaces for which the weak Nullstellensatz is valid will be proved now:

THEOREM 3. *For a finite dimensional complex space $(X, \mathcal{O})$ the following assertions are equivalent:*

1. *The weak Nullstellensatz is valid for $(X, \mathcal{O})$.*
2. *Every closed proper ideal in $\mathcal{O}(X)$ has at least one zero in X.*
3. *The natural map $\chi: X \to S_c(\mathcal{O}(X))$ is surjective.*

Proof. $1 \Rightarrow 2$: Let the weak Nullstellensatz be valid for a finite dimensional complex space $(X, \mathcal{O})$ and suppose that I is a closed proper ideal in $\mathcal{O}(X)$. According to a theorem of Grauert for Stein manifolds [11, Satz 2] which was generalized by Wiegmann to finite dimensional complex spaces [25, Darstellungssatz], there exist $f_1, \cdots, f_{n+1} \in I$, for $n := \dim X$, such that

$$\{x \in X: f_i(x) = 0, 1 \leqq i \leqq n+1\} = \{x \in X: g(x) = 0, g \in I\},$$

since I is a closed vector subspace of $\mathcal{O}(X)$. Because I is proper, $f_1, \cdots, f_{n+1}$ have at least one common zero $x \in X$ due to the weak

Nullstellensatz. This point x is then a common zero for all the functions in I.

$2 \Rightarrow 3$: See proof of Proposition 3.
$3 \Rightarrow 1$: Proposition 3.

A complex space $(X, \mathscr{O})$ is *holomorphically separable* if every two points $x, y \in X$, $x \neq y$, can be separated by a global holomorphic function, i.e., there exists an $f \in \mathscr{O}(X)$ with $f(x) \neq f(y)$. This is equivalent to the injectivity of the natural map $\chi: X \to S_c(\mathscr{O}(X))$. A theorem of Igusa [15] says that the holomorphic convexity of a domain X in C^n is equivalent to the surjectivity of the canonical map $\chi: X \to S_c(\mathscr{O}(X))$. Thus, for the special case of domains X in C^n, it is known that the surjectivity of χ is equivalent to the weak Nullstellensatz (see § 1). However, since the proof relies on the fact that domains in C^n are holomorphically separable, it obviously can't be generalized to arbitrary finite dimensional complex spaces.

As already mentioned, the weak Nullstellensatz for domains X in C^n is equivalent to the holomorphic convexity of X. This result can be easily generalized to holomorphically separable unramified domains over a Stein manifold:

COROLLARY 4. *Let $(X, \mathscr{O})$ be a holomorphically separable unramified domain over a Stein manifold M. The weak Nullstellensatz is valid for $(X, \mathscr{O})$ if and only if X is holomorphically convex.*

Proof. Let the weak Nullstellensatz be valid for $(X, \mathscr{O})$. Rossi [24] proved that the spectrum $S_c(A)$, $A := \mathscr{O}(X)$, can be endowed with a Stein complex structure such that it is also an unramified domain over M, $\chi: X \to S_c(A)$ is holomorphic, and $S_c(A)$ is the envelope of holomorphy of X. χ is injective, because X is holomorphically separable. Since X is finite dimensional, χ is surjective by Theorem 3. χ is open, as a result of $\dim X = \dim_{\chi_x} S_c(A) = \dim M$ for every $x \in X$. Therefore χ is a homeomorphism, and X is also Stein.

For an infinite dimensional complex space $(X, \mathscr{O})$ the weak Nullstellensatz doesn't imply the surjectivity of $\chi: X \to S_c(\mathscr{O}(X))$ as illustrated by the following example due to the referee, H. Rossi.

EXAMPLE. Let $Y := \coprod_{n=2}^{\infty} (C^n \backslash \{0\})$ denote the disjoint union of the punctured spaces $C^n \backslash \{0\}$ with the natural complex structure $\mathscr{O}_Y$. For each $n \geqq 2$, let $(x_{nm})_{m \geqq 2}$ be a sequence of distinct points in $C^n \backslash \{0\}$ converging to the origin $0 = 0_n$ in C^n. Then $S := \{x_{nm} \mid n, m \in N + 2\}$

is a closed complex subspace of Y. Consider the disjoint union $Z := \coprod_{m=2}^{\infty} \{m\} \times (\boldsymbol{P}^2 \backslash \{p\})$, $p \in \boldsymbol{P}^2$, of countably many copies of the punctured two-dimensional complex projective space. Suppose $(p_{nm})_{n \geqq 2}$ is a sequence in $\{m\} \times (\boldsymbol{P}^2 \backslash \{p\})$ converging to p such that $p_{kl} = p_{nm}$ iff $k = n$, $l = m$. The pushout belonging to the inclusion $S \hookrightarrow Y$ and the map $S \to Z$, $x_{nm} \mapsto p_{nm}$ exists in the category of complex spaces [18, 1.8] and will be denoted by $(X, \mathscr{O}_X)$. X is obtained by attaching Y to Z via the identification of x_{nm} with p_{nm} for $n, m \geqq 2$. Because every holomorphic function on $\boldsymbol{P}^2 \backslash \{p\}$ is constant, it follows that

$$A := \mathscr{O}_X(X) = \{f \in \mathscr{O}_Y(Y) \mid f(M_m) \in \boldsymbol{C} \text{ for } m \geqq 2\}$$

where $M_m := \{x_{nm} \mid n \geqq 2\}$. $\mathscr{O}_Y(Y)$ is topologically isomorphic to $\mathscr{O}_{\tilde{Y}}(\tilde{Y})$ when $\tilde{Y} := \coprod_{n=2}^{\infty} \boldsymbol{C}^n$ carries the natural complex structure. Therefore, A is topologically isomorphic to

$$\{f \in \mathscr{O}_{\tilde{Y}}(\tilde{Y}) \mid f(M_n) \in \boldsymbol{C} \text{ for } m \geqq 2\} .$$

Note that $f(0_n) = f(0_m)$, $n, m \in \boldsymbol{N} + 2$, for $f \in A$.

In order to calculate the set $S_c(A)$, consider the closed subalgebra

$$A_m := \{f \in \mathscr{O}_{\tilde{Y}}(\tilde{Y}) \mid f(M_n) \in \boldsymbol{C} \text{ for } 2 \leqq n \leqq m\}$$

of $\mathscr{O}_{\tilde{Y}}(\tilde{Y})$ which is a Fréchet algebra under the relative topology. $S_c(A_m)$ is obtained from $\tilde{Y}$ by collapsing each set M_n, $2 \leqq n \leqq m$, to a point. A is the inverse limit of $\{A_m, \iota_m\}$, where $\iota_m \colon A_{m+1} \to A_m$ denotes the inclusion. Because $\bigcup_{n=2}^{m} M_n$ is a discrete sequence in the Stein space $\tilde{Y}$, ι_m has a dense image [13, V § 4.1, Folgerung 1]. Consequently, the natural map $A \to A_m$ also has a dense image [1, 2.4]. This implies that $S_c(A)$ is the direct limit of $\{S_c(A_m), \iota'_m\}$ in the category of sets for $\iota'_m \colon S_c(A_m) \to S_c(A_{m+1})$, $\varphi \mapsto \varphi \circ \iota_m$ [1, 5.21]. Therefore, $S_c(A) = \tilde{Y}/R$, where R is the following equivalence relation on $\tilde{Y}$: $x \sim y$ iff $f(x) = f(y)$ for every $f \in A$. The map $\chi \colon X \to S_c(A)$ is not surjective, since the equivalence class represented by $0_2 \in \boldsymbol{C}^2$ has no inverse image.

To prove that the weak Nullstellensatz nevertheless holds for $(X, \mathscr{O}_X)$, let $f_1, \cdots, f_m \in \mathscr{O}_X(X)$ have no common zero on X. Considered as holomorphic functions on Y, they have no common zero on Y. The holomorphic continuation of these functions to $\tilde{Y}$ have no common zero on $\tilde{Y}$ either. Otherwise $0_{m+1} \in \boldsymbol{C}^{m+1}$ would be a common zero. Then the zero set of $f_i \mid \boldsymbol{C}^{m+1}$, $1 \leqq i \leqq m$, in $\boldsymbol{C}^{m+1}$ would be positive dimensional, but this would mean that a common zero on Y must exist—a contradiction. $(\tilde{Y}, A)$ is a Fréchet function algebra and the evaluation map $\tilde{Y} \to S_c(A)$ is surjective. Therefore, the $\hat{f}_i$, $1 \leqq i \leqq m$, have no common zero on $S_c(A)$, and due to the theorem of Arens mentioned in § 2, functions $g_1, \cdots, g_m \in A$ exist with $\sum_{i=1}^{m} f_i g_i = 1$.

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