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A GRAPH AND ITS COMPLEMENT WITH SPECIFIED PROPERTIES VI: CHROMATIC AND ACHROMATIC NUMBERS

Dedicated to Ruth Bari

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We characterize the graphs G such that both G and its complement $\bar{G}$ are n-colorable, and we specify explicitly all 171 graphs for the case $n = 3$. We then determine the 41 graphs for which both G and $\bar{G}$ have achromatic number 3.

1. Introduction. We follow the terminology and notation of [1] but we include some basic definitions for completeness. A *coloring* of a graph G is an assignment of colors to its points so that whenever two points are adjacent they are colored differently. An *n-coloring* of G is a coloring of G which uses n colors. A *complete n-coloring* of G is an n-coloring of G such that, for every pair of distinct colors there exists a pair of adjacent points in G which receive the given pair of colors. The *chromatic number* $\chi = \chi(G)$ of a graph G is the least integer n such that G has an n-coloring. We say that G is *n-colorable* if $\chi(G) \leq n$. Alternatively, $\chi(G)$ can be characterized as the least integer n such that $V(G)$ has a partition into n subsets each of which induces a totally disconnected subgraph. Obviously if $n = \chi(G)$ then every n-coloring of G is complete. The *achromatic number* $\psi = \psi(G)$ of a graph G is the greatest integer m such that G has a complete m-coloring. Clearly every graph G of order p has a p-coloring, but this coloring is only complete if G is K_p.

A *homomorphism* of a graph G onto a graph G' is a function ϕ from $V(G)$ onto $V(G')$ such that, whenever u and v are adjacent points of G, their images $\phi(u)$ and $\phi(v)$ are adjacent in G'. Since no point of a graph is adjacent with itself, two adjacent points of G cannot have the same image under any homomorphism of G. If G' is the image of G under a homomorphism ϕ, we write $G' = \phi(G)$. The *order* of ϕ is $|V(\phi(G))|$. A homomorphism ϕ of G is *complete of order n* if $\phi(G) = K_n$. Thus every graph G has a complete homomorphism of order $\chi(G)$ and also a complete homomorphism of order $\psi(G)$, and $\chi(G)$ and $\psi(G)$ are the smallest and largest orders of the complete homomorphisms of G. It was shown by Harary, Hedetniemi and Prins [2] that G also has a complete homomorphism of order n for all intermediate n.

It is convenient to write $G > H$ when H is an induced subgraph of G. If X is a set of points in a graph G then we use $\langle X \rangle$ to denote the subgraph G induced by X. If necessary to avoid ambiguity we can write $\langle X \rangle_G$ and $\langle X \rangle_H$ if X is a set of points in two different graphs G and H. We write $\bar{\chi}(G)$ for $\chi(\bar{G})$ and $\bar{\psi}(G)$ for $\psi(\bar{G})$.

2. The chromatic number. We are concerned in this section with those graphs G for which both G and $\bar{G}$ are n-colorable.

THEOREM 1. *Let $G_1, G_2, \ldots, G_k$ be the components of a graph G. Then $\bar{\chi}(G) = \Sigma\bar{\chi}(G_i)$.*

Proof. We first prove the inequality $\chi(G) \leq \Sigma\, \chi(G_i)$ holds if $G_1, G_2, \ldots, G_k$ are induced subgraphs of G such that $V(G) = \bigcup V(G_i)$. For each $1 \leq i \leq k$ there exists a family $\mathbf{S}_i$ of subsets $V(G_i)$, whose union is $V(G_i)$, with $|\mathbf{S}_i| = \chi(G_i)$, and such that each $S \in \mathbf{S}_i$ induces in G_i a totally disconnected subgraph. Let $\mathbf{S} = \bigcup \mathbf{S}_i$. Then $\mathbf{S}$ is a family of subsets of $V(G)$, whose union is $V(G)$, such that each $S \in \mathbf{S}$ induces in G a totally disconnected subgraph. Thus $\chi(G) \leq |\mathbf{S}| \leq \Sigma\, |S_i| = \Sigma\, \chi(G_i)$.

Next we show that $\bar{\chi}(G) \geq \Sigma\, \bar{\chi}(G_i)$ if $G_1, G_2, \ldots, G_k$ are the components of G. There exists a family $\mathbf{S}$ of subsets of $V(G)$, whose union is $V(G)$, with $|\mathbf{S}| = \bar{\chi}(G)$, such that each $S \in \mathbf{S}$ induces in $\bar{G}$ a totally disconnected subgraph. For each $1 \leq i \leq k$, let $\mathbf{S}_i = \{S \in \mathbf{S} \mid S \cap V(G_i) \neq \varnothing\}$. Points from different components of G are adjacent in $\bar{G}$, so the subfamilies $\mathbf{S}_i$, $1 \leq i \leq k$, constitute a partition of $\mathbf{S}$. Each $\mathbf{S}_i$ is such that every $S \in \mathbf{S}_i$ induces in $\bar{G}_i$ a totally disconnected subgraph, so $|\mathbf{S}_i| \geq \bar{\chi}(G_i)$. Thus $\bar{\chi}(G) = |\mathbf{S}| = \Sigma\, |\mathbf{S}_i| \geq \Sigma\, \bar{\chi}(G_i)$.

Since each $\bar{G}_i$ is an induced subgraph of $\bar{G}$, the theorem is an immediate consequence of the discussion above.

The corollaries which follow include a characterization of graphs G such that G and $\bar{G}$ are both n-colorable.

COROLLARY 1a. *Let $G_1, G_2, \ldots, G_k$ be the components of G. Then G and $\bar{G}$ are both n-colorable if and only if*

(i) *$\chi(G_i) \leq n$ for every $1 \leq i \leq k$, and*

(ii) *$\Sigma\, \bar{\chi}(G_i) \leq n$.*

Proof. This follows directly from Theorem 1 and the fact that $\chi(G) = \max \chi(G_i)$.

COROLLARY 1b. *If G has k components, then $\bar{\chi}(G) \geq k$. If $k = \bar{\chi}(G)$, then each component of G is complete.*

Proof. As G has k components G_i, $\bar{G}$ must contain K_k. If $k = \bar{\chi}(G)$, then $\Sigma\, \bar{\chi}(G_i) = k$, so for each i, $\bar{\chi}(G_i) = 1$, whence $\bar{G}_i$ is totally disconnected and therefore G_i is complete.

For the special case of disconnected graphs G such that G and $\overline{G}$ are both 3-colorable, Theorem 1 leads to a particularly simple characterization.

COROLLARY 1c. *If a graph G is disconnected then G and $\overline{G}$ are both 3-colorable if and only if one of the following conditions is satisfied.*

(i) *G has exactly 3 components each of which is a complete graph of order no greater than 3.*

(ii) *G has exactly 2 components, G_1 and G_2, and G_1 is a complete graph of order no greater than 3, and G_2 is 3-colorable and $\overline{G}_2$ is 2-colorable.*

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of a disconnected graph G.

Suppose first that G and $\overline{G}$ are both 3-colorable. By Corollary 1b we need consider only two possible values of k.

Case 1. $k = 3$.

In this case $k = \overline{\chi}(G)$ so Corollary 1b applies and each G_i is complete. Then $\chi(G) \leq 3$ implies that each G_i is of order no greater than 3. In this case G satisfies condition (i).

Case 2. $k = 2$.

From Theorem 1 we get $\overline{\chi}(G_1) + \overline{\chi}(G_2) = \overline{\chi}(G) \leq 3$. Without loss of generality we may conclude that $\overline{\chi}(G_1) = 1$ and $\overline{\chi}(G_2) \leq 2$. As in Case 1 it follows that G_1 is complete of order no greater than 3. Thus G_2, being a subgraph of G, is 3-colorable, and $\overline{G}_2$ is 2-colorable because $\overline{\chi}(G_2) \leq 2$. In this case G satisfies condition (ii).

Suppose conversely that G satisfies either (i) or (ii).

Case 1′. G satisfies (i).

Let G_1, G_2 and G_3 be the components of G. Then each G_i is complete so $V(G_i)$ induces in $\overline{G}$ a totally disconnected subgraph, thus $\overline{\chi}(G) \leq 3$. Because each G_i is of order no greater than 3 we can partition $V(G)$ into three subsets V_1', V_2' and V_3' such that $| V_i' \cap V(G_j) | \leq 1$ for $1 \leq j, j \leq 3$. Then each V_i' induces in G a totally disconnected subgraph, so $\chi(G) \leq 3$. In this case G and $\overline{G}$ are both 3-colorable.

Case 2′. G satisfies (ii).

In this case Corollary 1a clearly implies that G and $\overline{G}$ are both 3-colorable.

THEOREM 2. *If a graph G is n-colorable, then $\overline{\chi}(G)$ is the least integer t such that $V(G)$ can be partitioned into t subsets $V_1, V_2, \ldots, V_t$ and for each $1 \leq i \leq t$, $| V_i | \leq n$ and V_t induces a complete subgraph.*

Proof. By definition $\bar{\chi}(G)$ is the least integer t such that $V(G)$ can be partitioned into t subsets $V_1, V_2, \ldots, V_t$ each of which induces in $\bar{G}$ a totally disconnected subgraph. Also for any subset S of $V(G)$, S induces in $\bar{G}$ a totally disconnected subgraph if and only if S induces in G a complete subgraph, in which case $|S| \leq \chi(G) \leq n$.

The corollaries which follow include another characterization of graphs G such that G and $\bar{G}$ are both n-colorable which can usefully be applied to connected graphs.

COROLLARY 2a. *A graph G and its complement are both n-colorable if and only if there exist positive integers $s, t \leq n$ such that*

For each $1 \leq i \leq s$ there is a positive integer $a_i \leq t$ such that $\cup K_{a_i}$ is a spanning subgraph of $\bar{G}$.

(ii) *For each $1 \leq i \leq t$ there is a positive integer $b_i \leq s$ such that $\cup K_{b_i}$ is a spanning subgraph of G.*

Moreover the minimum values of s and t which satisfy these conditions are $\chi(G)$ and $\bar{\chi}(G)$ respectively.

Proof. Suppose first that G and $\bar{G}$ are both n-colorable. Let $s = \chi(G)$ and $t = \bar{\chi}(G)$, so $s, t \leq n$. As G is s-colorable, by Theorem 2 there is a partition of $V(G)$ into $t = \bar{\chi}(G)$ subsets $V_1, \ldots, V_t$ such that for each $1 \leq i \leq t$, $|V_i| \leq s$ and V_i induces a complete subgraph in G. Writing $b_i = |V_i|$, we have $\cup K_{b_i} = \cup \langle V_i \rangle$ as a spanning subgraph of G.

Similarly, since $\bar{G}$ is t-colorable and $\bar{\chi}(G) = s$, the same argument applied to $\bar{G}$ yields $\cup K_{a_i}$ as a spanning subgraph of $\bar{G}$ for some sequence of positive integers $a_i \leq t$.

Now suppose conversely that G is a graph which satisfies conditions (i) and (ii). By condition (i), there is a partition of $V(G)$ into s subsets $V_1, \ldots, V_s$ such that for each $1 \leq i \leq s$, V_i induces a complete subgraph in $\bar{G}$. Then each V_i induces in G a totally disconnected subgraph. Thus $\chi(G) \leq s \leq n$, so G is n-colorable. Also note that the least value of s which can satisfy (i) is $\chi(G)$ since $\chi(G) \leq s$. Similarly by (ii) we deduce $\bar{\chi}(G) \leq t \leq n$, so $\bar{G}$ is n-colorable and $\bar{\chi}(G)$ is the minimum possible value for t.

COROLLARY 2b. *If a graph G and its complement are both n-colorable then the order of G is at most n^2.*

Although this corollary is clearly a consequence of the partition described in Theorem 2, we should also point out that it is also a special case of the well known result of Nordhaus and Gaddum [3] that the order p of a graph satisfies the inequality, $p \leq \chi\bar{\chi}$. It is convenient to include here another useful consequence of the Nordhaus-Gaddum theorem.

COROLLARY 2c. *If a graph G and its complement are both n-colorable and the order of G exceeds $n(n-1)$, then $\chi(G) = \bar{\chi}(G) = n$.*

Proof. Since $\chi(G) \leq n$ and $\bar{\chi}(G) \leq n$, if either were actually less than n then $\chi(G) \cdot \bar{\chi}(G)$ would be no greater than $n(n-1)$.

Our final corollary of this theorem deals again with the special case $n = 3$.

COROLLARY 2d. *If a graph G of order p and its complement $\bar{G}$ are both 3-colorable, then $p \leq 9$ and*

(i) *if $p = 9$, then G and $\bar{G}$ each contain $3K_3$ as a subgraph,*

(ii) *if $p = 8$, then G and $\bar{G}$ each contain $2K_3 \cup K_2$ as a subgraph,*

(iii) *if $p = 7$, then G and $\bar{G}$ each contain either $K_3 \cup 2K_2$ or $2K_3 \cup K_1$ as a subgraph.*

Proof. Suppose that G and $\bar{G}$ are both 3-colorable. Then by Corollary 2b the order p of G is at most 9. If $p \geq 7$ then by Lemma 2c, $\chi(G) = \bar{\chi}(G) = 3$. Thus by Corollary 2a, depending on the value of p, G and $\bar{G}$ must contain the subgraphs described above.

We complete this section by cataloguing all graphs G of order 6 or less and all disconnected graphs G of order 7, 8 or 9 for which G and $\bar{G}$ are both 3-colorable. Because there are 171 graphs in this category we will not illustrate them. Rather we describe each such graph by specifying an ordered triple (p, q, n) where p denotes the order and q the size of the graph and n denotes its numerical designation in the Graph Diagrams in Appendix I of [**1**]. Every graph of order 6 or less appears in these diagrams and the triple (p, q, n) completely describes such graphs. The disconnected graphs of order 7, 8, and 9 for which $\chi \leq 3$ and $\bar{\chi} \leq 3$ do not appear in the diagrams, but their components do, and we indicate such graphs by specifying their components. There are pairs (p, q) for which only one graph of order p and size q exists. Such graphs do not have a numerical designation in the Graph Diagrams. We hereby confer the designation 1 on all such graphs. Thus in the lists which follow the triple (2, 1, 1) represents the unique graph of order 2 and size 1, namely K_2. Our list of disconnected graphs of order 7 through 9 with $\chi = \bar{\chi} = 3$ are really complete, by the following argument. By Corollary 1c, all such graphs have 3 components each of order 3 or less or 2 components, G_1 and G_2, with G_1 complete of order 3 or less and $\chi(G_2) \leq 3$, $\bar{\chi}(G_2) \leq 2$. By the Nordhaus-Gaddum theorem we conclude that the order of G_2 is no greater than 6, so G_2 is in List C, our list of all graphs of order 6 or less with $\chi = 3$, $\bar{\chi} = 2$.

List A. $\chi + \bar{\chi} \leq 4$.
$\chi = \bar{\chi} = 1$: $(1, 0, 1)$ which is K_1.
$\chi = 1$ and $\bar{\chi} = 2$: $(2, 0, 1)$ which is $\bar{K}_2$.
$\chi = 2$ and $\bar{\chi} = 1$: $(2, 1, 1)$ which is K_2.
$\chi = 1$ and $\bar{\chi} = 3$: $(3, 0, 1)$ which is $\bar{K}_3$.
$\chi = 3$ and $\bar{\chi} = 1$: $(3, 3, 1)$ which is K_3.
$\chi = \bar{\chi} = 2$, connected: $(3, 2, 1)$, $(4, 3, 2)$, and $(4, 4, 2)$ which are P_3, P_4 and C_4.
$\chi = \bar{\chi} = 2$, disconnected: $(3, 1, 1)$ and $(4, 2, 2)$ which are $K_1 \cup K_2$ and $2K_2$.

List B. $\chi = 2$ and $\bar{\chi} = 3$.
Connected: $(4, 3, 3)$, $(5, 4, 4)$, $(5, 4, 6)$, $(5, 5, 3)$, $(5, 6, 5)$ and $p = 6$ with $(q, n) = (5, 7)$, $(5, 10)$, $(5, 14)$, $(6, 7)$, $(6, 9)$, $(6, 11)$, $(7, 5)$, $(7, 14)$, $(8, 23)$, $(9, 17)$.
Disconnected: $(4, 1, 1)$, $(4, 2, 1)$, $(5, 2, 2)$, $(5, 3, 1)$, $(5, 3, 4)$, $(5, 4, 1)$, $(6, 3, 5)$, and $(6, 4, 8)$.

List C. $\chi = 3$ and $\bar{\chi} = 2$.
Connected: $(4, 4, 1)$, $(4, 5, 1)$, $(5, 5, 4)$, $(5, 6, 1)$, $(5, 6, 4)$, $(5, 6, 6)$, $(5, 7, 1)$, $(5, 8, 2)$, and $p = 6$ with $(q, n) = (7, 23)$, $(8, 5)$, $(8, 14)$, $(9, 7)$, $(9, 9)$, $(9, 11)$, $(10, 7)$, $(10, 10)$, $(10, 14)$, $(11, 8)$, $(12, 5)$.
Disconnected: $(4, 3, 1)$, $(5, 4, 5)$ and $(6, 6, 17)$.

List D. $\chi = \bar{\chi} = 3$, order 6 or less.
Connected: $p = 5$ with $(q, n) = (5, 2)$, $(5, 5)$, $(5, 6)$, $(6, 2)$, $(7, 2)$; $(6, 5, 3)$;
$(p, q) = (6, 6)$ with $n = 8$, 10, 13, 14, 18, 20;
$(p, q) = (6, 7)$ with $n = 6$, 7, 8, 9, 10, 11, 12, 13, 16, 19, 20, 21, 24;
$(p, q) = (6, 8)$ with $n = 1$, 2, 6, 7, 8, 9, 10, 11, 12, 13, 16, 19, 20, 21, 24;
$(p, q) = (6, 9)$ with $n = 2$, 3, 5, 8, 10, 13, 14, 18, 19, 20; $(6, 10, 3)$, $(6, 10, 12)$, $(6, 10, 15)$.
Disconnected: $(5, 3, 2)$, $(5, 4, 2)$, $(5, 5, 1)$;
$p = 6$ with $(q, n) = (4, 6)$, $(5, 12)$, $(5, 15)$, $(6, 2)$, $(6, 3)$, $(6, 5)$, $(6, 19)$, $(7, 1)$, $(7, 2)$.

List E. $\chi = \bar{\chi} = 3$, of order 7, 8, or 9, disconnected $3K_3, 2K_3 \cup K_2, K_3 \cup 2K_2, 2K_3 \cup K_1$, and $K_3 \cup G$ where G is any connected graph in List C, and $K_2 \cup G$ where G is any connected graph of order 5 or 6 in List C, and $K_1 \cup G$ where G is any connected graph of order 6 in List C.

Of the 171 graphs which appear in these lists, 116 have $\chi = \bar{\chi} = 3$. In addition to these the complements of the 51 graphs in List E are connected graphs of order 7 through 9 with $\chi = \bar{\chi} = 3$. And Corollary 2d implies that there are many other graphs of order 7 through 9 with

$\chi = \bar{\chi} = 3$ which are not in our lists, of which one example is $G = C_7 + e$ where the edge e joins two points whose distance in C_7 is 2. In this case clearly both G and $\bar{G}$ contain $K_3 \cup 2K_2$ as a subgraph so $\chi(G) = \bar{\chi}(G) = 3$.

3. The achromatic number. We first characterize graphs G with $\psi(G) = 2$.

THEOREM 3. *A graph G has achromatic number* 2 *if and only if each component of G is complete bipartite.*

Proof. Obviously the union of complete bipartite graphs has $\psi = 2$. For the converse, assume that $\psi = 2$, then $\chi \leq 2$ since $\chi \leq \psi$ for any graph. Thus G must be bipartite. Moreover each component of G cannot contain P_4 as an induced subgraph since $\psi(P_4) = 3$. Thus each component of G must be complete bipartite.

COROLLARY 3a. *The only graphs with $\psi = \bar{\psi} = 2$ are $C_4, 2K_2$, $K_{1,2}$ and $K_2 \cup K_1$.*

We now develop some results in the form of five lemmas for finding all graphs with $\psi = \bar{\psi} = 3$. We write uAv to indicate adjacency and $u\bar{A}v$ for nonadjacency. The first lemma was proved by exhaustion and we omit the detailed verification.

LEMMA 4a. *Among all graphs of order* 6, *only the six graphs $2K_3$, $2K_2 + \bar{K}_2$, $C_4 + \bar{K}_2$ and their complements $K_{3,3}$, $C_4 \cup K_2$ and $3K_2$ satisfy the property that either G or $\bar{G}$ contains two point-disjoint triangles and $\psi = \bar{\psi} \leq 3$.*

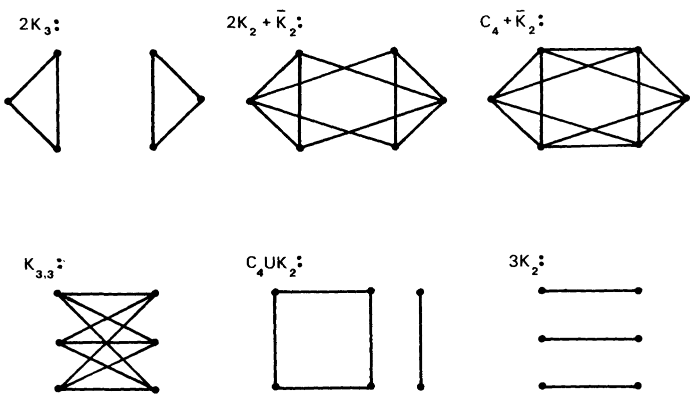

FIGURE 1. The six graphs of order 6 with ψ, $\bar{\psi} \leq 3$

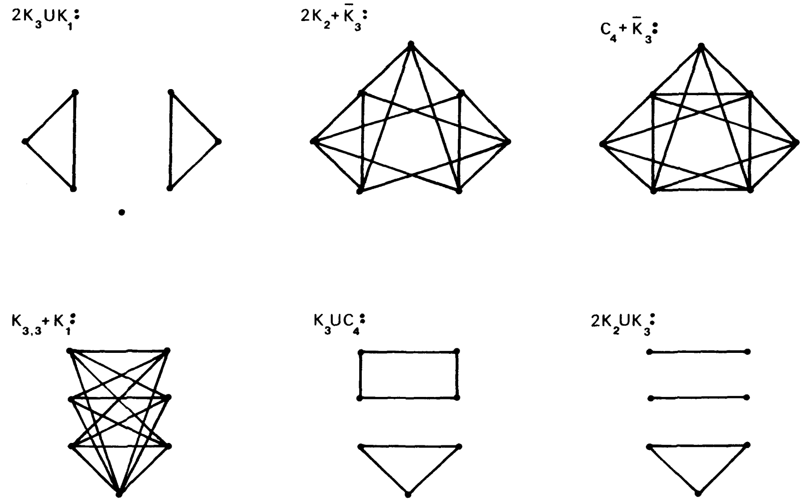

FIGURE 2. The six graphs of Lemma 4b

LEMMA 4b. *Among all graphs of order* 7, *only the six graphs* $2K_3 \cup K_1$, $2K_2 + \bar{K}_3$, $C_4 + \bar{K}_3$ *and their complements satisfy the property that either* G *or* $\bar{G}$ *contains two point-disjoint triangles and* ψ, $\bar{\psi} \leq 3$.

Proof. Assume that $\psi = \bar{\psi} = 3$ and that G contains two point-disjoint triangles $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_4, v_5, v_6\}$. Then the subgraph H of G induced by these six points in one of the three graphs, $2K_3$, $K_2 + \bar{K}_2$ or $C_4 + \bar{K}_2$, of Lemma 4a; otherwise either G or $\bar{G}$ contains an induced subgraph of order 6 which has achromatic number at least 4 and so ψ or $\bar{\psi}$ would be at least 4, a contradiction to the hypothesis. By w we denote the seventh point in $V(G) - V(H)$, and divide the proof into three cases according to whether H is $2K_3$, $2K_2 + \bar{K}_2$, or $C_4 + \bar{K}_2$.

Case 1. $H = 2K_3$.

If $G = H \cup K_1$, it is easily verified that $\psi = \bar{\psi} = 3$. Now we may assume that $G \supset H \cup K_1$ properly. Then there is a point v_i in G which is adjacent to w. Without loss of generality we may assume that wAv_i. On the other hand, there is at least one point v_i, $i = 4, 5$ or 6, which is not adjacent to w, say v_4 as shown in Figure 3, otherwise all three points v_i, $i = 4, 5$, and 6 are adjacent to w and so $\{v_4, v_5, v_6, w\}$ induces K_4, a contradiction.

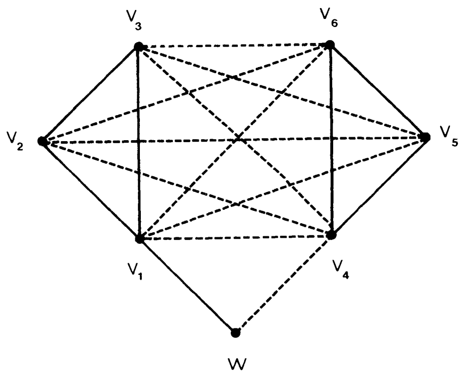

FIGURE 3. A step in the proof of Case 1

Then it is easy to see that $\psi(G) = 4$ regardless of whether or not wAv_i for $i = 2, 3, 5, 6$, a contradiction.

Case 2. $H = 2K_2 + \overline{K}_2$.

As $\psi = \overline{\psi} = 3$, we know that $\chi, \overline{\chi} \leq 3$ so by Lemma 2c, $\chi = \overline{\chi} = 3$. Thus by Corollary 2d, $\overline{G}$ contains a triangle. As $H = 2K_2 + \overline{K}_2 = G - w$, it follows that G contains $C_4 \cup K_2$ as an induced subgraph. Hence there are two possibilities: either $\overline{G} \supset F_1$ or $\overline{G} \supset F_2$, where F_1, F_2 are the graphs illustrated in Figure 4, which we now consider as two subcases.

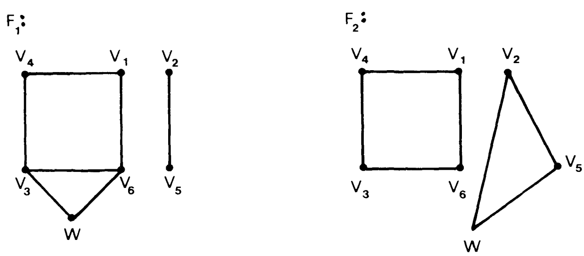

FIGURE 4. A step in the proof of Case 2

Case 2a. $\overline{G} \supset F_1$.

If $\overline{G} \neq F_1$, then w is adjacent to at least one more point of G, i.e., to v_1, v_2, v_4, or v_5. We may assume that w is adjacent to v_1 or v_2 from the symmetry of F_1. In either case, $\overline{\psi} = 4$, a contradiction. On the other hand, if $\overline{G} = F_1$ then $\overline{\psi} = 4$, a contradiction.

Case 2b. $\overline{G} \supset F_2$.

If $\overline{G} = F_2$, then $\psi = \overline{\psi} = 3$. If $\overline{G} \neq F_2$, then w is adjacent to one of the points v_i, $i = 1, 3, 4$ or 6. From the symmetry of F_2, we may assume that wAv_1. Then it is easy to see that $\psi = 4$, a contradiction.

Case 3. $H = C_4 + \overline{K}_2$.

Since $\overline{G} \supset K_3$ from Corollary 2d, and $\overline{H} = 3K_2$, it follows that $\overline{G} \supset 2K_2 \cup K_3$. We may assume without loss of generality that $\{v_2, v_5, w\}$ induces K_3 in $\overline{G}$; see Figure 5. If $\overline{G} = 2K_2 \cup K_3$, then $\psi = \overline{\psi} = 3$. If $\overline{G} \neq 2K_2 \cup K_3$, then w must be adjacent to at least one of v_i, $i = 1, 3, 4$ or 6. Assuming now that wAv_1, we see that $\overline{\psi} = 4$, a contradiction.

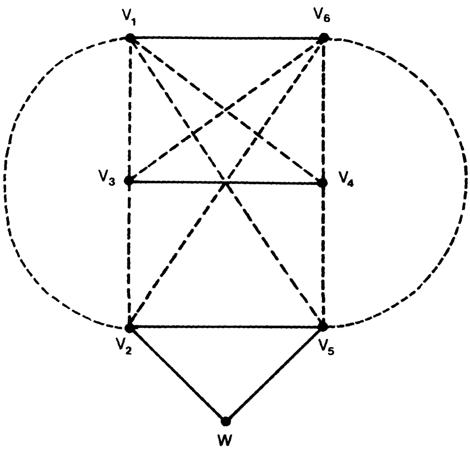

FIGURE 5. A step in the proof of Case 3

LEMMA 4c. *If G is a graph of order 7 such that neither G nor $\overline{G}$ contains two point-disjoint triangles, then ψ or $\overline{\psi}$ is at least* 4.

Proof. Assume that $\psi = \overline{\psi} = 3$, then $\chi, \overline{\chi} \leq 3$ since $\chi \leq \psi$. By applying Lemma 2c, $\chi = \overline{\chi} = 3$. Thus $G \supset K_3 \cup 2K_2$ or $G \supset 2K_3 \cup K_1$ by Corollary 2d. But by the hypothesis, G cannot contain two point-disjoint triangles and so, $G, \overline{G} \supset K_3 \cup 2K_2$. Now we label the points of $K_3 \cup 2K_2$ as in Figure 6.

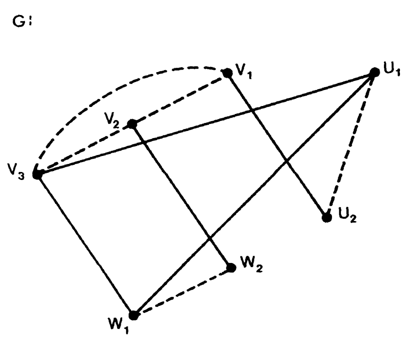

FIGURE 6. A labelling of $K_3 \cup 2K_2$

By the symmetry of G and $\bar{G}$, it is sufficient to handle only the case u_2Aw_2. By the hypothesis that G cannot contain two point-disjoint triangles, v_1Aw_2 and v_2Au_2. Then regardless of the presence or absence of other lines, we can easily verify that $\bar{\psi} = 4$, a contradiction.

LEMMA 4d. *There are no graphs of order at least* 8 *such that* $\psi = \bar{\psi} = 3$.

Proof. Assume that G has order 8 and $\psi = \bar{\psi} = 3$. Then $\chi = \bar{\chi} = 3$ by Lemma 2c. Thus both G and $\bar{G}$ contain $2K_3 \cup K_2$ as a spanning subgraph by Corollary 2d. The subgraph of G induced by the set of points of $2K_3$ must be one of the three graphs, $2K_3$, $2K_2 + \bar{K}_2$ or $C_4 + \bar{K}_2$ of Lemma 4a. We now divide the proof into three cases:

Case 1. G contains $2K_3$ as an induced subgraph.

By Corollary 2d, both G and $\bar{G}$ contain $2K_3 \cup K_2$ hence of course $\bar{G} \supset 2K_3$. It is convenient to label $\bar{G}$ as in Figure 7.

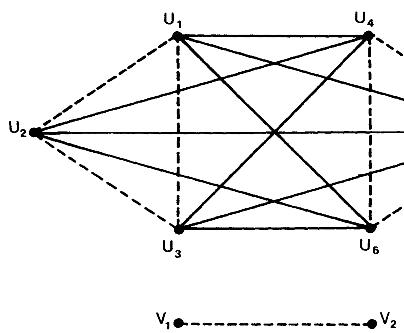

FIGURE 7. A subgraph of $\bar{G}$

By symmetry, we may assume that both point sets $\{u_3, u_6, v_1\}$ and $\{u_2, u_5, v_2\}$ induce K_3 in $\bar{G}$. Then it is easily verified that $\bar{\psi} = 4$.

Case 2. G contains $2K_2 + \bar{K}_2$ as an induced subgraph.

Let F_1, F_2 be the graphs illustrated in Figure 8.

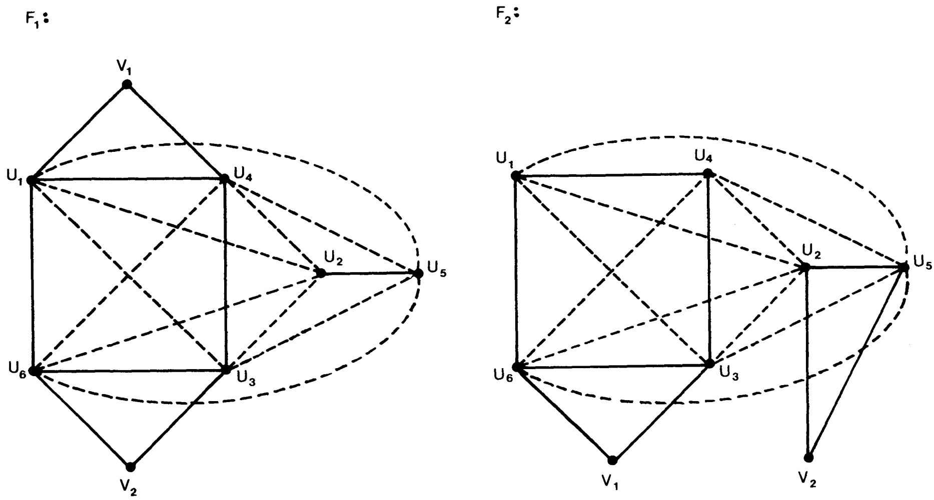

FIGURE 8. Subgraphs F_1 and F_2 of $\overline{G}$

Since $\overline{G} \supset 2K_3$ by Corollary 2d, there are two possibilities: either $\overline{G} \supset F_1$ or $\overline{G} \supset F_2$. However in either case, $\overline{\psi} = 4$.

Case 3. G contains $C_4 + \overline{K}_2$ as an induced subgraph.

Since $\overline{G} \supset 2K_3$ by Corollary 2d, we may assume that both $\{v_1, u_2, u_5\}$ and $\{v_2, u_3, u_4\}$ induce K_3 in $\overline{G}$, see Figure 9, and thus $\overline{\psi} = 4$, a contradiction.

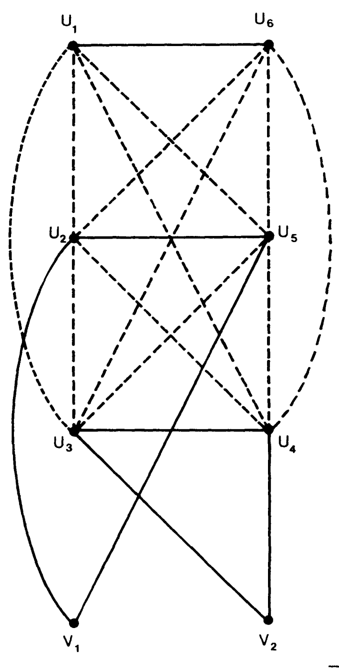

FIGURE 9. A subgraph of $\overline{G}$

Combining the preceding four lemmas, we obtain the following result.

LEMMA 4e. *Let G be a graph of order at least* 7, *then G has* $\psi = \bar{\psi} = 3$ *if and only if G is one of the six graphs*, $2K_3 \cup K_1$, $K(3,3,1)$, $C_4 \cup C_3$, $2K_2 + \bar{K}_3$, $2K_2 \cup K_3$ *and* $K(3,2,2)$.

We are now ready to specify all the graphs with $\psi = \bar{\psi} = 3$.

THEOREM 4. *There are exactly* 41 *graphs G such that both G and* $\bar{G}$ *have achromatic number* 3: *six have order* 7, *twenty are of order* 6, *fourteen of order* 5 *and just one of order* 4.

Proof. By Lemma 4d, we know that there are no such graphs of order $p \geq 8$. Lemma 4e lists all six graphs with $p = 7$ and Figure 2 shows them. To complete the list of all the graphs with $\psi = \bar{\psi} = 3$, we had to resort to the method of brute force by an exhaustive inspection of Appendix I of [**1**] for $p = 4, 5$, and 6.

As the determination of all graphs with $\psi = \bar{\psi} = n \geq 4$ appears to be hopelessly complicated, we can realistically ask only for the construction of additional families of graphs with $\psi = \bar{\psi}$.

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