

# Pacific Journal of Mathematics

## **ON THE ITERATES OF DERIVATIONS OF PRIME RINGS**

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## ON THE ITERATES OF DERIVATIONS OF PRIME RINGS

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In this paper we study properties of associative derivations whose iterates are related in rather special ways to the original derivation, or to the iterates of another derivation. An associative derivation  $d: R \rightarrow R$  is an additive (or linear when appropriate) mapping on a ring  $R$  satisfying  $d(xy) = xd(y) + d(x)y$  for all  $x, y \in R$ . A derivation  $d: R \rightarrow R$  is called *inner* if  $d(x) = (\text{ad } a)(x)$  for some  $a \in R$  where  $(\text{ad } a)(x) = [a, x] = ax - xa$ . In particular we ask when can the iterate of an inner derivation be an inner derivation? When can the iterates of two derivations commute? More precisely, we characterize elements  $a, b \in R$ ,  $R$  a prime ring, for which  $(\text{ad } a)^n(x) = (\text{ad } b)(x)$  for all  $x \in R$ , and we characterize derivations  $d: R \rightarrow R$ ,  $\delta: R \rightarrow R$  for which  $[d^n(x), \delta^n(y)] = 0$  for all  $x, y \in R$ ,  $R$  prime. Applications are made to  $C^*$ -algebras.

**1. Introduction.** In [15] it was shown that if  $d_1$  and  $d_2$  are derivations of a prime ring not of characteristic 2 with  $d_1 \circ d_2$  a derivation, then either  $d_1 = 0$  or  $d_2 = 0$ . Consequently if  $d^2(x) = 0$  for all  $x$ ,  $d$  a derivation on such a ring, then  $d = 0$ . In [6] it was shown that if  $(\text{ad } a)^n(x) = 0$  for all  $x$  in a simple ring there exists a scalar  $\lambda$  such that  $(a - \lambda e)^{[(n+1)/2]} = 0$ . And in [14] it was shown that if  $(\text{ad } a)^3(x) = (\text{ad } a)(x)$  for a self-adjoint  $a$  and all  $x$  in a von Neumann algebra then  $(a - z)^2 = a - z$  for some central element  $z$ . In [7] it was shown that if  $[d(x), d(y)] = 0$  for all  $x, y \in R$  where  $R$  is a prime ring and  $\text{char } R \neq 2$ , then  $d = 0$  or  $R$  is a commutative integral domain. Our results show that if  $d$  and  $\delta$  are as above, and  $R$  is prime of characteristic 0 then  $R$  is commutative, or  $d^{3n-1} = 0$ , or  $\delta^{3n-1} = 0$ . If  $d = \delta = \text{ad } b$  there exists  $\lambda$  in the extended centroid of  $R$  such that  $a = b - \lambda$  satisfies  $a^{[(2n+3)/3]} = 0$ .

In §§2 and 3 we prove results in the full generality of prime rings. Crucial use is made of the notions of extended centroid and central closure of a prime ring and of a key result on tensor products of closed prime algebras. We summarize these constructions by quoting from [11]. Let  $R$  be a prime ring and let  $T$  be the totality of all right  $R$ -homomorphisms  $f: U_R \rightarrow R_R$ , where  $U$  ranges over the non-zero ideals of  $R$ . An equivalence relation  $\sim$  is defined on  $T$  as follows:  $f$  (acting on  $U$ )  $\sim$   $g$  (acting on  $V$ ) if  $f = g$  on  $W$  where  $W$  is non-zero ideal contained in  $U \cap V$ . The set  $Q = \{\hat{f}\}$  of all equivalence classes forms a ring under the operations induced by addition and composition of representatives of the equivalence classes.  $R \subseteq Q$  via the map  $a \rightarrow \hat{a}_l$  where  $\hat{a}_l$  is left multiplication by  $a$  acting on  $R$ . The center  $C$  of  $Q$  is a field containing the centroid

of  $R$  and is called the *extended centroid* of  $R$ . The  $C$ -algebra  $A = RC + C$  is again a prime ring and is called the *central closure* of  $R$ . In general we define a prime algebra  $S$  over a field  $F$  to be a *closed prime algebra* over  $F$  if  $S$  is its own central closure, i.e., the extended center of  $S$  is just  $F$  itself. We list some examples of closed prime algebras which are important for the purposes of this paper:

(1) The central closure of a prime ring ([12], p. 503).

(2)  $A \otimes_C F$ , where  $A$  is a closed prime algebra over  $C$  and  $F$  is an extension field of  $C$  ([2], Theorem 3.5).

(3) Any 2-fold transitive algebra of linear transformations on a complex vector space ([4], Theorem 2.1.3 and [11], Theorem 12).

Finally, if  $A$  is an algebra over  $F$  we denote by  $A_l$  the algebra of left multiplications  $a_l$  of  $A$  determined by the elements of  $A$  and  $A_r$ , the algebra of right multiplications  $a_r$ , determined by the elements of  $A$ . A key result on tensor products is

**THEOREM 1.** *If  $S$  is a closed prime algebra over  $F$  and  $S^0$  is the opposite algebra of  $S$  then*

$$S \otimes_F S^0 \cong S_l S_r \text{ via the map } u \otimes v \rightarrow u_l v_r.$$

In §4 the results of §§2 and 3 are applied to  $C^*$ -algebras. Although  $C^*$ -algebras, in general, are not prime they have a complete set of (algebraically) irreducible representations and, in our case, a phenomenon which occurs in each of these representations can be translated to a corresponding result for the original algebra. For a full account of  $C^*$ -algebras we refer the reader to [1].

## 2. Iterates of an inner derivation.

**LEMMA 1.** *Let  $R$  be a prime ring of characteristic  $> n > 1$ , and suppose  $(\text{ad } a)^n = \text{ad } b$  for some  $a, b \in R$ . Let  $C$  be the extended centroid of  $R$ ,  $A = RC + C$  the central closure of  $R$ ,  $F$  the algebraic closure of  $C$ , and  $S = A \otimes_C F$ . Then  $a$  is algebraic over  $F$  of degree  $\leq n$ , and if  $p(x) = (x - \lambda)^l g(x)$  is the minimum polynomial of  $a$  over  $F$  where  $\lambda \in F$  is a root of multiplicity  $l \geq 1$ , then  $b - \beta = c^n$ ,  $\beta \in F$ , and*

$$(1) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

*holds in  $S \otimes_F S^0$  with  $c = a - \lambda$ .*

*Proof.* The condition  $(\text{ad } a)^n = \text{ad } b$  clearly lifts to  $S$ , i.e.,  $(a_l - a_r)^n = b_l - b_r$  holds in the algebra  $S_l S_r$ . By Theorem 1,  $S \otimes_F S^0 \cong S_l S_r$  via  $u \otimes v \rightarrow u_l u_r$  since  $S$  is a closed prime algebra over  $F$ , and the condition further translates to

$$(2) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} a^{n-k} \otimes a^k = b \otimes 1 - 1 \otimes b \text{ in } S \otimes_F S^0.$$

It is clear that  $b \in \text{span}\{1, a, \dots, a^n\}$ . Writing  $b = \sum_{i=0}^n \beta_i a^i$ ,  $\beta_i \in F$ , and substituting this into (2) we see that

$$(3) \quad (a^n - b + \beta_0) \otimes 1 + (\beta_1 - na^{n-1}) \otimes a + \dots \\ + (\beta_{n-1} + (-1)^{n-1} na) \otimes a^{n-1} + (\beta_n + (-1)^n) \otimes a^n = 0.$$

If  $1, a, \dots, a^n$  are independent then, in particular,  $\beta_{n-1} + (-1)^{n-1} na = 0$ , whence  $a = \pm \beta_{n-1}/n \in F$  a contradiction. Therefore  $\{1, a, \dots, a^n\}$  is a dependent set and we have established that  $a$  is algebraic over  $F$  of degree  $\leq n$ . Hence we may set  $p(x) = (x - \lambda)'g(x)$  as indicated in the statement of the lemma. Setting  $c = a - \lambda$  we then see that  $m(x) = p(x + \lambda) = x'q(x)$  is the minimum polynomial of  $c$  over  $F$  with  $\deg m(x) = \deg p(x) \leq n$  and  $q(0) \neq 0$ . Since  $\text{ad } a = \text{ad } c$  we may rewrite (2) as

$$(4) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = b \otimes 1 - 1 \otimes b.$$

We set  $v = c^{l-1}q(c) \neq 0$ , note that  $cv = 0$ , and multiply (4) on the right by  $1 \otimes v$  to obtain

$$(5) \quad c^n \otimes v = b \otimes v - 1 \otimes \beta v,$$

whence  $(c^n - b + \beta) \otimes v = 0$ . Thus  $b = c^n + \beta$  and (4) becomes

$$(6) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

which completes the proof of the lemma.  $\square$

In the following Theorem,  $C, A = RC + C$  and  $S = A \otimes_C F$  have the same meaning as in Lemma 1.

**THEOREM 2.** *Let  $R$  be a prime ring of char.  $> n > 1$ , and suppose  $(\text{ad } a)^n = \text{ad } b$  for some  $a, b \in R$ . If the minimum polynomial  $p(x)$  of  $a$  over  $F$  (which necessarily exists in view of Lemma 1) contains a root  $\lambda \in F$  of multiplicity  $l > 1$ , then  $\lambda \in C$  and  $(a - \lambda)^{[(n+1)/2]} = 0$ .*

*Proof.* By Lemma 1 we have

$$(1) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

holding in  $S \otimes_F S^0$ , where  $c = a - \lambda$  and the minimum polynomial of  $c$  over  $F$  is  $m(x) = x^l q(x)$ ,  $l > 1$ ,  $q(0) \neq 0$ . We remark that  $q(c)$ ,  $cq(c)$ ,  $\dots$ ,  $c^{l-1}q(c)$  are  $F$ -independent. Indeed, one would have a dependency  $a_{i_1} c^{i_1} q(c) + \dots + a_{i_r} c^{i_r} q(c) = 0$ ,  $\alpha_{i_j} \neq 0$ ,  $i_1 < i_2 < \dots < i_r < l$ , with "length"  $r > 1$  minimal. Multiplication of the dependency by  $c^{l-i_r}$  would result in a dependency of shorter "length", a contradiction.

Now we multiply (1) on the right by  $1 \otimes q(c)$  to obtain

$$(7) \quad \sum_{k=0}^{l-1} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k q(c) = 0.$$

By the preceding remark  $\{c^k q(c) \mid k = 1, \dots, l-1\}$  is an independent set so in particular  $c^{n-(l-1)} = 0$ . Thus  $c$  is nilpotent, with  $m(x) = x^l$  where  $l \leq n - (l-1)$ , i.e.,  $l \leq (n+1)/2$ , and so  $c^{[(n+1)/2]} = 0$ . In terms of  $a$  this says that  $(a - \lambda)^{[(n+1)/2]} = 0$  and  $p(x) = (x - \lambda)^l$ .

It remains to prove that  $\lambda \in C$ . Since  $c = a \otimes 1 - 1 \otimes \lambda \in A \otimes_C F$  (when written more precisely we see from  $p(x) = (x - \lambda)^l$  that  $0 = (a \otimes 1 - 1 \otimes \lambda)^l = \sum_{k=0}^l (-1)^k \binom{l}{k} a^{l-k} \otimes \lambda^k$  holds in  $A \otimes_C F$ ). It follows that  $a$  is algebraic over  $C$  of degree  $\leq l$ . Let  $h(x)$  be the minimum polynomial of  $a$  over  $C$ . On the one hand we must have  $\deg h(x) \leq \deg p(x) = l$ , and so  $h(x) = p(x)$ . Therefore the coefficients of  $p(x)$  lie in  $C$  and so from  $p(x) = (x - \lambda)^l = x^l - l\lambda x^{l-1} + \dots$  we have  $l\lambda = \alpha \in C$ , whence  $\lambda = \alpha/l \in C$ .  $\square$

**COROLLARY 1.** *Let  $R$  be a prime ring of char.  $> n > 1$ , and suppose  $(\text{ad } a)^n = \text{ad } b$  for some  $a, b \in R$ . Then  $(a - \lambda)^{[(n+1)/2]} = 0$  for some  $\lambda$  in the extended centroid  $C$  if either of the following conditions hold:*

- (a)  $n$  is even;
- (b)  $b = 0$ .

*Proof.* By Lemma 1 we have

$$(8) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = c^n \otimes 1 - 1 \otimes c^n$$

where  $c = a - \lambda$  has minimum polynomial  $m(x) = x^l q(x)$ ,  $q(0) \neq 0$ . We show that if either (a) or (b) holds then  $c$  is nilpotent whence the conclusion follows from Theorem 2, since then  $m(x) = x^l$  and  $l > 1$ . If we are given (a) then (8) becomes

$$(9) \quad \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = -2 \otimes c^n.$$

Multiplication of (9) by  $c^{l-1}q(c) \otimes 1$  on the left yields  $0 = -2c^{l-1}q(c) \otimes c^n$  whence  $c^n = 0$ .

If we are given (b) then (8) reads

$$(10) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} \otimes c^k = 0.$$

Multiplication of (10) by  $1 \otimes c^{l-1}q(c)$  yields  $c^n \otimes c^{l-1}q(c) = 0$  so again we obtain  $c^n = 0$ .  $\square$

REMARK. Corollary 1(b) was first proved for simple rings by Herstein [6] and conjectured for prime rings by Kovacs [10] and Herstein and the proof announced in [13].

The conclusion of Corollary 1 is false for prime rings in general. As an example, let  $n = 5$ ,  $R = M_2(\mathbb{C})$ , the ring of  $2 \times 2$  matrices over  $\mathbb{C}$ , and  $a = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1 + \sqrt{-3}}{2} \end{bmatrix}$ . One may verify that  $(\text{ad } a)^5 = \text{ad}(a^5)$ , but there is no  $\lambda \in \mathbb{C}$  such that  $(a - \lambda)$  is nilpotent. A more complicated example of the same nature can be constructed as follows:

Let  $n = 5$ ,  $R_j = M_2(\mathbb{C})$  for  $j = 1, 2, \dots, R = \bigoplus_{j=1}^{\infty} R_j$ ,  $\{\theta_j\}_{j=1}^{\infty}$  a collection of distinct real numbers such that  $0 < \theta_j < \pi/6$ ,  $z_j = e^{i\theta_j}$  where  $i^2 = -1$ ,  $a_j = \begin{pmatrix} z_j & 0 \\ 0 & z_j \frac{1 + \sqrt{-3}}{2} \end{pmatrix}$ , and  $a = \bigoplus a_j$ . Then  $(\text{ad } a)^5 = \text{ad}(a^5)$  and  $a$  has infinitely many distinct eigenvalues.

COROLLARY 2. *Let  $R$  be a 2-fold transitive algebra of linear transformations on a complex vector space  $H$  and suppose  $(\text{ad } a)^n = \text{ad } b$ ,  $n > 1$ , for some  $a, b \in R$ . Then  $a$  is algebraic, and if the minimum polynomial  $p(x)$  of  $a$  contains a repeated root  $\lambda$  (in particular if either  $n$  is even or  $b = 0$ ) then  $(a - \lambda)^{[(n+1)/2]} = 0$ .*

The following corollary shows that the example following Corollary 1 is typical.

COROLLARY 3. *Let  $R$  be a 2-fold transitive algebra of linear transformations on a complex vector space  $H$ , and suppose  $(\text{ad } a)^n = \text{ad } b$  for some  $a, b \in R$  with  $n$  odd,  $n > 1$ . If the minimum polynomial  $p(x)$  of  $a$  has distinct roots  $\lambda_1, \dots, \lambda_k$ ,  $k \leq n$ , there exist idempotents  $p_1, \dots, p_k$  with  $p_i p_j = 0$  for  $i \neq j$ ,  $\sum_{i=1}^k p_i = 1$  and  $a = \sum_{i=1}^k \lambda_i p_i$ . If  $k > 2$  and  $0, \lambda, \mu$  are distinct eigenvalues of  $a$ , then  $\lambda^n - \mu^n = (\lambda - \mu)^n$ . Conversely, if  $R$  is an algebra over a field  $F$  of characteristic 0, with idempotents  $p_1, \dots, p_k$  such*

that  $1 = \sum_{i=1}^k p_i$ ,  $p_i p_j = 0$  and  $\lambda_1, \dots, \lambda_k$  distinct elements of  $F$  such that  $\lambda_i^n - \lambda_j^n = (\lambda_i - \lambda_j)^n$ ,  $1 \leq i, j \leq n$ , and  $n$  odd, then  $(\text{ad } a)^n = \text{ad}(a^n)$  and  $a = \sum_{i=1}^k \lambda_i p_i$ .

*Proof.* Standard linear algebra gives the existence of the  $p_i$  with the desired properties. Equation (1) of Lemma 1 becomes  $\sum_{k=0}^n (-1)^k \binom{n}{k} c^{n-k} x c^k = [c^n, x]$  for all  $x \in R$  where  $c = a - \lambda$ . Since  $n$  is odd we have

$$0 = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} x c^k \quad \text{for all } x \in R.$$

Let  $v \neq 0$  be such that  $av = \mu v$ . Then  $cv = (a - \lambda)v = (\mu - \lambda)v \neq 0$ . Hence

$$\begin{aligned} 0 &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} x c^k v \\ &= \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} (\mu - \lambda)^k x v \quad \text{for all } x \in R. \end{aligned}$$

By transitivity,

$$0 = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} c^{n-k} (\mu - \lambda)^k = (c - (\mu - \lambda))^n - c^n + (\mu - \lambda)^n.$$

In terms of  $a$  this says  $0 = (a - \mu)^n - (a - \lambda)^n + (\mu - \lambda)^n$ . Therefore, the minimum polynomial  $p(x)$  must divide  $(x - \mu)^n - (x - \lambda)^n + (\mu - \lambda)^n$ . Since 0 is an eigenvalue of  $a$  we have  $(-\mu)^n - (-\lambda)^n + (\mu - \lambda)^n = 0$ . The other part of the corollary is a straightforward calculation.  $\square$

### 3. Commuting iterates of derivations.

**THEOREM 3.** *Let  $R$  be a prime ring and let  $d$  and  $\delta$  be derivations on  $R$  such that  $[d^n(x), \delta^n(y)] = 0$  for all  $x, y \in R$ . Then either  $R$  is commutative, or  $d^{3n-1} = 0$ , or  $\delta^{3n-1} = 0$ . Furthermore, if  $n = 1$ , and the characteristic of  $R \neq 2$ , then either  $R$  is commutative, or  $d = 0$ , or  $\delta = 0$ .*

*Proof.* Let  $W$  be the subring generated by  $\{d^n(x) \mid x \in R\}$  and note that  $d(W) \subseteq W$ . By the Leibniz formula we have

$$(1) \quad d^n(xy) = \sum_{k=0}^n \binom{n}{k} d^k(x) d^{n-k}(y) \in W \quad \text{for all } x, y \in R.$$

For  $l = 1, 2, \dots, n$  we substitute  $d^{l-1}(x)$  for  $x$  and  $d^{2n-l}(y)$  for  $y$  in (1) to obtain

$$(2) \quad \sum_{k=0}^{n-l} \binom{n}{k} d^{k+l-1}(x) d^{3n-k-l}(y) \in W,$$

$$l = 1, 2, \dots, n \text{ for all } x, y \in R.$$

For  $l = n$ , (2) reads

$$(3) \quad d^{n-1}(x) d^{2n}(y) \in W,$$

and for  $l = n - 1$ , (2) reads

$$(4) \quad d^{n-2}(x) d^{2n+1}(y) + \binom{n}{1} d^{n-1}(x) d^{2n}(y) \in W.$$

Together (3) and (4) imply that  $d^{n-2}(x) d^{2n+1}(y) \in W$ . Continuing in this fashion by comparing successive decreasing values of  $l$  from  $l = n$  to  $l = 1$  we have that  $xd^{3n-1}(y) \in W$  for  $x, y \in R$ . Therefore,  $Rd^{3n-1}(y) \subseteq W$  for all  $y \in R$ . Similarly for all  $x$ ,  $R\delta^{3n-1}(x) \subseteq V$ , the subring generated by  $\{\delta^n(t) \mid t \in R\}$ . Since  $[W, V] = 0$  by assumption, the left ideals  $Rd^{3n-1}(y)$  and  $R\delta^{3n-1}(x)$  commute for all  $x, y \in R$ . Without loss of generality we may assume that  $d^{3n-1}(y) \neq 0$  for some  $y$  and that  $\delta^{3n-1}(x) \neq 0$  for some  $x$ . Since we have two commuting non-zero left ideals  $Rd^{3n-1}(y)$  and  $R\delta^{3n-1}(x)$  in the prime ring  $R$ ,  $R$  must be commutative.

For  $n = 1$ , if  $R$  is not commutative, we may assume  $d^2 = 0$ . From  $d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y)$  we have  $d(x)d(y) = 0$ . In particular,  $0 = d(xy)d(x) = [d(x)y + xd(y)]d(x)$  whence  $d(x)yd(x) = 0$  for all  $x, y \in R$ . Since  $R$  is prime it follows that  $d(x) = 0$  for all  $x \in R$ .  $\square$

**THEOREM 4.** *Let  $R$  be a prime ring of characteristic  $\geq 3n$  and let  $d = \text{ad } b$  be an inner derivation of  $R$  satisfying  $[d^n(x), d^n(y)] = 0$  for all  $x, y \in R$ . Then there exists an element  $\lambda$  in the extended centroid of  $R$  such that  $a = b - \lambda$  satisfies  $a^{[(2n+3)/3]} = 0$ .*

*Proof.* We can assume, by [7], that  $n > 1$ . The condition on  $d$  clearly extends to the central closure  $A = RC + C$  of  $R$ . By Theorem 3,  $d^{3n-1} = 0$  and hence by Corollary 1(b) there exists  $\lambda \in C$  such that  $a = b - \lambda \in A$  satisfies  $a^{[3n/2]} = 0$ . If  $l$  is the degree of nilpotency of  $a$  we have

$$(5) \quad l \leq \frac{3n}{2}$$

and, assuming the theorem to be false, we may also suppose that

$$(6) \quad l > \frac{2n+3}{3},$$



in other words,  $3l - 2n - 4 \geq 0$ . We then set

$$(7) \quad p = q = \frac{3l - 2n - 4}{2} \quad \text{if } l \text{ is even}$$

and

$$(8) \quad p = \frac{3l - 2n - 5}{2}, q = \frac{3l - 2n - 3}{2} \quad \text{if } l \text{ is odd.}$$

In either case  $p + q = 3l - 2n - 4$ , which we wish to view in the form  $2n + p + q = 3(l - 1) - 1$ .

Expansion of  $[d^n(x), d^n(y)] = 0$  by the Leibniz formulas, followed by replacement of  $x$  by  $ax$ , yields

$$0 = a^p[d^n(ax), d^n(y)]a^q = g - h$$

where

$$g = \sum_{j, k=0}^n (-1)^{n-j+k} \binom{n}{j} \binom{n}{k} a^{p+j+1} x a^{2n-j-k} y a^{q+k}$$

and

$$h = \sum_{j, k=0}^n (-1)^{n-j+k} \binom{n}{j} \binom{n}{k} a^{p+j} y a^{2n+1-j-k} x a^{q+k}.$$

Since  $2n + 1 + p + q = 3(l - 1)$  the only possible surviving summand of  $g$  occurs when  $j + p + 1 = q + k = l - 1$ ; for  $h$  it occurs when  $j + p = q + k = l - 1$ . To see that these terms actually occur it is necessary to show that the  $j$  and  $k$  thus determined are indeed within the range  $0 \leq j, k \leq n$ . This means verifying

$$(9) \quad 0 \leq l - p - 2 \leq n, \quad 0 \leq l - p - 1 \leq n, \quad 0 \leq l - q - 1 \leq n.$$

We leave it to the reader to check that the various substitutions of (7) and (8) in (9) lead to the following inequalities

$$(10) \quad 0 \leq 2n - l + i \leq 2n, \quad i = 0, 1, 2, 3$$

where  $i = 0, 2$  when  $l$  is even and  $i = 1, 3$  when  $l$  is odd. But the inequalities (10) follow readily from (5) and (6), so that we have established

$$(11) \quad g = (-1)^{n+p-q+1} \binom{n}{l-p-2} \binom{n}{l-q-1} a^{l-1} x a^{l-1} y a^{l-1}$$

and

$$(12) \quad h = (-1)^{n+p-q} \binom{n}{l-p-1} \binom{n}{l-q-1} a^{l-1} y a^{l-1} x a^{l-1}.$$

Setting  $x = y$  in (11) and (12), noting that the coefficients of  $a^{l'-1}xa^{l'-1}xa^{l'-1}$  in (11) and (12) are of opposite parity, and knowing  $g = h$ , we may conclude that  $a^{l'-1}xa^{l'-1}xa^{l'-1} = 0$  for all  $x \in A$ . This means that the non-zero right ideal  $a^{l'-1}A$  is nil of bounded degree which in view of [5], Lemma 1.1 provides a contradiction since  $A$  is a prime ring.

**COROLLARY 4.** *Let  $R$  be a 2-fold transitive ring of linear transformations on a complex vector space  $H$  and let  $d = \text{ad } b$  be an inner derivation of  $R$  satisfying  $[d^n(x), d^n(y)] = 0$  for all  $x, y \in R$ . Then there exists a complex scalar  $\lambda$  such that  $a = b - \lambda$  satisfies  $a^{(2n+3)/3} = 0$ .  $\square$*

**4. Applications.** If  $A$  is an algebraically irreducible algebra of operators on a complex Banach space  $H$  then  $A$  is 1-fold transitive (since it is irreducible) and hence by [3] it is  $m$ -fold transitive if  $H$  is infinite dimensional or is of dimension at least  $m$ . In particular,  $\mathcal{L}(H)$ , the algebra of all bounded linear operators on  $H$  is 2-fold transitive so that the previous results apply.

Let  $A$  be a  $C^*$ -algebra of operators, containing the identity operator 1, acting on a complex Hilbert space  $H$ . Let  $R = A''$  be the ultraweak closure of  $A$  and let  $M$  be the universal enveloping von Neumann algebra of  $R$ . If  $\phi$  is any  $*$ -representation of  $R$  and  $\pi$  the natural injection of  $R$  into  $M$ , there exists a normal  $*$ -representation  $\tilde{\phi}$  of  $M$  such that  $\phi(x) = \tilde{\phi}(\pi(x))$ . We have that  $\tilde{\phi}(M) = \phi(R)''$ . If  $\phi$  is irreducible,  $\tilde{\phi}(M) = \phi(R)'' = \mathcal{L}(H_\phi)$ , the ring of all bounded linear operators on  $H_\phi$ , where  $H_\phi$  is the representation Hilbert space. If  $\tilde{\phi}$  is a normal homomorphism of  $M$  onto a von Neumann algebra  $N$ , there exists a central projection  $c \in M$  and a  $*$ -isomorphism  $\tilde{\psi}$  of  $M_c$  onto  $N$  such that  $\tilde{\phi}(x) = \tilde{\psi}(xc)$  for all  $x$  in  $M$ .

**THEOREM 5.** *Let  $A$  be a  $C^*$ -algebra of operators acting on a complex Hilbert space  $H$ , and assume  $A$  contains the identity operator 1. Let  $R = A''$ , the ultraweak closure of  $A$ , and suppose  $(\text{ad } a)^n(x) = (\text{ad } b)(x)$  for some  $a, b \in A$  and all  $x \in A$ . If  $n$  is odd there exists a central projection  $c \in R$  and a central element  $z$  in  $R$  such that  $((a - z)c)^{(n+1)/2} = 0$ ,  $1 - c = \sum_{\beta \in B'} d_\beta$ , and  $\text{ad}_\beta = \sum_{i=1}^{j(\beta)} \lambda_i^\beta r_j^\beta$  where the  $\lambda_i^\beta$  are distinct complex numbers, the  $r_j^\beta$  are (not necessarily self-adjoint) orthogonal idempotents, and the  $d_\beta$  are orthogonal central projections. If  $n$  is even there exists a central element  $z \in R$  such that  $(a - z)^{n/2} = 0$ .*

*Proof.* Let  $\{\phi_\beta\}_{\beta \in B}$  be a complete set of irreducible representations of  $R$ . Then  $\phi_\beta(x) = \tilde{\phi}_\beta(\pi(x))$  where  $\tilde{\phi}_\beta$  is a normal  $*$ -homomorphism of  $M$  on  $\mathcal{L}(H_{\phi_\beta})$ . As above, for each  $\beta$ , there exists a central projection  $c_\beta$  in  $M$  and a  $*$ -isomorphism  $\tilde{\psi}_\beta$  of  $M_{c_\beta}$  on  $\mathcal{L}(H_{\phi_\beta})$  such that  $\tilde{\phi}_\beta(x) = \tilde{\psi}_\beta(xc_\beta)$  for

all  $x \in M$ . Now  $(\text{ad } a)^n(x) = (\text{ad } b)(x)$  for all  $x \in A$  implies  $(\text{ad } \pi(a))^n(x) = (\text{ad } \pi(b))(x)$  for all  $x \in M$  so that  $(\text{ad } \tilde{\phi}_\beta(\pi(a)))^n(x) = (\text{ad } \tilde{\phi}_\beta(\pi(b)))(x)$ . Since  $\tilde{\phi}_\beta$  is irreducible, the results of Theorem 2 and Corollaries 2 and 3 apply. If  $n$  is odd then either (i) there exists  $\lambda_\beta \in \mathbb{C}$  such that  $(\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = 0$  (in the case that the minimum polynomial of  $\tilde{\phi}_\beta(\pi(a))$  has a repeated root or  $\tilde{\phi}_\beta(\pi(a))$  is central) or (ii)  $\tilde{\phi}_\beta(\pi(a)) = \sum_{i=1}^{j(\beta)} \lambda_i^\beta p_i^\beta$  where the  $p_i^\beta$  are mutually orthogonal idempotents and the  $\lambda_i^\beta$  are distinct. Since  $\{\phi_\beta\}_{\beta \in B}$  is complete we have  $\text{LUB } c_\beta = 1$ . Choose mutually orthogonal central projections  $d'_\beta \in M$  such that  $d'_\beta \leq c_\beta$  and  $\sum_{\beta \in B} d'_\beta = 1$ . Let  $c' = \sum_{\beta \in B_1} d'_\beta$  where  $B_1 = \{\beta \mid \text{(i) holds}\}$ . Then  $1 - c' = \sum_{\beta \in B \setminus B_1} d'_\beta$ . If  $\beta \in B_1$  then  $0 = (\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = (\phi_\beta(a) - \lambda_\beta)^{(n+1)/2}$  so that  $|\lambda_\beta| \leq \|\phi_\beta(a)\| \leq \|a\|$ . Hence  $z' = \sum_{\beta \in B_1} \lambda_\beta d'_\beta \in Z_M$ . Moreover,  $\beta \in B_1$  implies  $0 = (\tilde{\phi}_\beta(\pi(a)) - \lambda_\beta)^{(n+1)/2} = (\tilde{\psi}_\beta((\pi(a) - \lambda_\beta)c_\beta))^{(n+1)/2}$  so that  $0 = (\pi(a) - \lambda_\beta)^{(n+1)/2} c_\beta$  since  $\tilde{\phi}_\beta$  is an isomorphism.  $(\pi(a) - z')^{(n+1)/2} c' = ((\pi(a) - z')c')^{(n+1)/2} c_\beta d'_\beta = 0$  for each  $\beta \in B_1$ . Similarly if  $\beta \in B \setminus B_1$ ,  $\tilde{\phi}_\beta(\pi(a)) = \sum_{i=1}^{j(\beta)} \lambda_i^\beta p_i^\beta = \tilde{\psi}_\beta(\pi(a)c_\beta)$  so that  $\pi(a)c_\beta = \sum_{i=1}^{j(\beta)} \lambda_i^\beta q_i^\beta$  and  $\pi(a)d'_\beta = \sum_{i=1}^{j(\beta)} \lambda_i^\beta q_i^\beta d'_\beta$ .

Let  $i: R \rightarrow R$  be the identity map,  $\tilde{i}$  the normal homomorphism of  $M$  on  $R$  for which  $\tilde{i}(\pi(x)) = i(x)$  for all  $x \in R$ . Let  $c''$  be a central projection in  $M$  and  $\tilde{j}$  an isomorphism of  $M_{c''}$  on  $R$  such that  $\tilde{i}(\pi(x)) = \tilde{j}(\pi(x)c'')$ .  $\tilde{j}$  induces an isomorphism between  $Z_{M_{c''}}$  the center of  $M_{c''}$  and  $Z_R$ . Hence  $c'c'' + (1 - c')c''$  which is the identity of  $M_{c''}$  is sent by  $\tilde{j}$  to 1, the identity of  $R$ . Let  $c = \tilde{j}(c', c'')$  and  $z = \tilde{j}(z', c'')$ ,  $d_\beta = \tilde{j}(d'_\beta c'')$ , and  $r_j^\beta = \tilde{j}(q_i^\beta d'_\beta c'')$ .  $\square$

**THEOREM 6.** *Let  $A$  be a  $C^*$ -algebra,  $d$  a derivation on  $A$  such that  $[d^n(x), d^n(y)] = 0$  for all  $x, y \in A$ . There exists  $s \in R$ ,  $z \in Z_R$  the centre of  $R$ , such that  $d(x) = [s, x]$  for all  $x \in A$  and  $(s - z)^{[(2n+3)/3]} = 0$ .*

*Proof.* By [16: 4.1.7] there exists such an  $s$ . Moreover  $d$  extends in this way to a derivation on  $R$ . The result follows from Theorem 4 and an argument as in Theorem 5.  $\square$

We finish with a result which does not fit the title of the paper but which contains the same methods in its proof.

**LEMMA** [See 8: Theorem]. *If  $R$  is a prime ring not of characteristic 2 and  $d: R \rightarrow R$  a derivation, then either  $d = 0$  or  $\{x \in R \mid [x, d(r)] = 0 \text{ for all } r \in R\} \subseteq Z_R$ , the centre of  $R$ .*  $\square$

*Proof.* Let  $b \in \{x \in R \mid [x, d(r)] = 0 \text{ for all } r \in R\}$ . Then  $0 = [b, d(r)] = ((\text{ad } b) \circ d)(r)$  for all  $r \in R$ . By [15: Theorem 1] either  $d = 0$  or  $\text{ad } b = 0$ . If  $\text{ad } b = 0$  then  $[b, r] = 0$  for all  $r$  so  $b \in Z_R$ .  $\square$

**THEOREM 7.** *Let  $A$  be a  $C^*$ -algebra with identity  $e$ ,  $R = A''$ , and  $d: A \rightarrow A$  a derivation. There exists a central projection  $c \in Z_R$  such that  $dc(a) = 0$  for all  $a \in A$ , and  $\{a(e - c) \mid [a, d(b)](e - c) = 0 \text{ for all } b \in A\} \subseteq Z_A(e - c)$ .*

*Proof.*  $d$  extends to a derivation (denoted by  $d$ ) from  $R$  to  $R$ . Let  $\phi$  be an irreducible representation of  $R$  and consider  $d_\phi: \phi(R) \rightarrow \phi(R)$  given by  $d_\phi(\phi(r)) = \phi(d(r))$ . Then  $d_\phi$  is a derivation on the irreducible algebra  $\phi(R)$ . By the lemma either  $d_\phi = 0$  or  $\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0 \text{ for all } s \in R\} \subseteq Z_{\phi(R)}$ .

If  $\tilde{\psi}$ ,  $\pi$ , and  $c$  are as in the beginning of this section,

$$\begin{aligned} & \{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0 \text{ for all } s \in R\} \\ &= \{\tilde{\psi}(\pi(r)c) \mid [\tilde{\psi}(\pi(r)c), \tilde{\psi}(\pi(d(s))c)] = 0 \text{ for all } s \in R\} \\ &= \{\tilde{\psi}(\pi(r)c) \mid [\pi(r)c, \pi(d(s))c] = 0 \text{ for all } s \in R\}. \end{aligned}$$

Since  $\tilde{\psi}$  is a  $*$ -isomorphism from  $M_c$  onto  $\mathcal{L}(H_\phi)$  it carries centers to centers so that if  $\{\phi(r) \mid [\phi(r), d_\phi(\phi(s))] = 0 \text{ for all } s \in R\} \subseteq Z_{\phi(R)}$  we must have  $\{\pi(r)c \mid [\pi(r)c, \pi(d(s))c] = 0 \text{ for all } s \in R\} \subseteq Z_{M_c}$ .

Let  $\{\phi_\beta\}$  be a complete set of irreducible  $*$ -representations of  $R$  and  $d\phi_\beta$  as above. If  $d\phi_\beta = 0$  there exists a central projection  $c_\beta$  in  $M$  such that  $0 = d\phi_\beta(\phi_\beta(x)) = \phi_\beta(d(x)) = \tilde{\phi}_\beta(\pi(d(x))) = \tilde{\psi}_\beta(\pi(d(x))c_\beta)$  so that  $\pi(d(x))c_\beta = 0$  for all  $x \in R$ . If  $d\phi_\beta \neq 0$  there exists  $c_\beta$  in  $M$  such that  $\{\pi(r)c_\beta \mid [\pi(r)c_\beta, \pi(d(s))c_\beta] = 0 \text{ for all } s \in R\} \subseteq Z_{M_{c_\beta}}$ .

Since  $\{\phi_\beta\}$  is complete,  $\text{LUB } c_\beta = e$ , choose mutually orthogonal central projections  $c'_\beta$  in  $M$  such that  $c'_\beta \leq c_\beta$  and  $\sum c'_\beta = e$ . Let  $c_0 = \sum c'_\beta$  where the sum is over all  $\beta$  such that  $\pi(d(x))c_\beta = 0$  for all  $x \in R$ .

Let  $i$ ,  $\tilde{i}$  and  $\tilde{j}$  be as above with  $c_1$  a central projection in  $M$  such that  $\tilde{j}$  is an isomorphism of  $M_{c_1}$  on  $R$ . There exists  $c \in Z_R$  such that  $\tilde{j}(c_0c_1) = c$ . We have  $\tilde{i}(c_0) = c$ . Now  $0 = \pi(d(r))c_0$  for all  $r \in R$  so that  $0 = \tilde{i}(\pi(d(r))c_0) = d(r)c$  for all  $r \in R$ . Moreover,

$$\begin{aligned} & \{\pi(r)(e - c_0) \mid [\pi(r), \pi(d(s))](e - c_0) = 0 \text{ for all } s \in R\} \\ &= \{\pi(r)(e - c_0) \mid \tilde{i}([\pi(r), \pi(d(s))](e - c_0)) = 0 \text{ for all } s \in R\} \\ &= \{\pi(r)(e - c_0) \mid [r, d(s)](e - c) = 0 \text{ for all } s \in R\}. \end{aligned}$$

Hence  $\{\pi(r)(e - c_0) \mid [\pi(r), \pi(d(s))](e - c_0) = 0 \text{ for all } s \in R\} \subseteq Z_{M_{e-c_0}}$  implies  $\{r(e - e) \mid [r, d(s)](e - c) = 0 \text{ for all } s \in R\} \subseteq \tilde{i}(Z_{M_{e-c_0}}) = Z_{R_{e-c}}$ .

Finally,

$$\begin{aligned} &\{a(e - c) \mid [a, d(b)](e - c) = 0 \text{ for all } b \in A\} \\ &\quad \subseteq \{r(e - c) \mid [r, d(s)](e - c) = 0 \text{ for all } s \in R\} \end{aligned}$$

by the ultra weak continuity of  $d$ . Therefore

$$\begin{aligned} &\{a(e - c) \mid [a, d(b)](e - c) = 0 \text{ for all } b \in A\} \\ &\quad \subseteq A(e - c) \cap Z_R(e - c) = Z(e - c). \quad \square \end{aligned}$$

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Received October 31, 1980 and in revised form February 10, 1982. Partially supported by NSERC Grant 7682.

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