Pacific Journal of Mathematics

THE FINE SPECTRA FOR WEIGHTED MEAN OPERATORS

BILLY E. RHOADES

Vol. 104, No. 1

May 1983

THE FINE SPECTRA FOR WEIGHTED MEAN OPERATORS

B. E. RHOADES

In a recent paper [5] the fine spectra of integer powers of the Cesàro matrix were computed. In this paper the fine spectra of weighted mean methods are determined. In most cases investigated, the interior points belong to III₁, the boundary points, except 1, belong to II₂, and 1 and any isolated points belong to III₃, where III₁, II₂, and III₃ are portions of the state space as described in [3].

From Goldberg [3], if $T \in B(X)$, X a Banach space, then there are three possibilities for R(T), the range of T:

- (I) $\underline{R(T)} = X$,
- (II) $\overline{R(T)} = X$, but $R(T) \neq X$, and
- (III) $R(T) \neq X$,

and three possibilities for T^{-1}

- (1) T^{-1} exists and is continuous,
- (2) T^{-1} exists but is discontinuous,
- (3) T^{-1} does not exist.

A weighted mean matrix A is a lower triangular matrix with entries $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \ge 0$ for n > 0, and $P_n = \sum_{k=0}^n p_k$. The necessary and sufficient condition for the regularity of A is that $\lim P_n = \infty$.

In [2] it was shown that, for any regular weighted mean matrix A, the spectrum, $\sigma(A)$, contains the set $\{\lambda || \lambda - (2 - \delta)^{-1} | \le (1 - \delta)/(2 - \delta)\}$ $\cup S$, and is contained in the set $\{\lambda || \lambda - (2 - \gamma)^{-1} | \le (1 - \gamma)/(2 - \gamma)\}$ $\cup S$, where δ = lim sup p_n/P_n , γ = lim inf p_n/P_n , and S = $\{p_n/P_n | n \ge 0\}$

We shall first consider those regular weighted mean methods for which $\delta = \gamma$, i.e., for which the main diagonal entries converge.

THEOREM 1. Let A be a regular weighted mean method such that $\delta = \lim p_n/p_n$ exists. If λ satisfies $|\lambda - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)$ and $\lambda \notin S$, then $\lambda \in \operatorname{III}_1 \sigma(A)$; i.e., λ is a point of $\sigma(A)$ for which $\overline{R(T)} \neq X$ and T^{-1} exists and is continuous.

Proof. First of all $\lambda I - A$ is a triangle, hence 1-1. Therefore $\lambda I - A \in 1 \cup 2$.

Consider the adjoint matrix $T^* = \lambda I - A^*$. Since A is regular, $A^* \in B(l)$ with entries $a_{00}^* = \chi(A) = \lim_n Ae - \sum_k \lim_n a_{nk} = 1$, $a_{n0}^* = a_{0n}^* = 0$ for n > 0 and $a_{nk}^* = a_{k-1,n-1}$ for n, k > 0, where $e = \{1, 1, 1, ...\}$.

Suppose $T^*x = 0$. Then

$$(\lambda-1)x_0=0,$$

and

(1)
$$\left(\lambda - \frac{p_{n-1}}{P_{n-1}}\right) x_n - \sum_{k=n+1}^{\infty} a_{nk}^* x_k = 0 \text{ for } n > 0.$$

Thus $x_0 = 0$ and, from (1), x_1 is arbitrary and, with $c_n = p_n/P_n$,

(2)
$$x_{n} = \frac{p_{n-1}x_{1}}{p_{0}\lambda^{n-1}} \prod_{j=0}^{n-2} (\lambda - c_{j}) = \frac{p_{n-1}}{p_{0}} x_{1} \prod_{j=0}^{n-2} \left(1 - \frac{c_{j}}{\lambda}\right)$$
$$= \left(1 - \frac{1}{\lambda}\right) \frac{p_{n-1}x_{1}}{P_{n-2}} \prod_{j=1}^{n-2} \left(1 + \left(1 - \frac{1}{\lambda}\right) \frac{p_{j}}{P_{j-1}}\right).$$

Now $|1 + (1 - \frac{1}{\lambda})p_j/P_{j-1}| < 1$ for all j sufficiently large if and only

if

$$\left(1+(1+\alpha)\frac{p_j}{P_{j-1}}\right)^2+\left(\beta\frac{p_j}{P_{j-1}}\right)^2<1, \text{ where } -\frac{1}{\lambda}=\alpha+i\beta.$$

Case I. Assume at most a finite number of the p_k are zero. Then the above inequality is equivalent to

$$2(1 + \alpha) + ((1 + \alpha)^{2} + \beta^{2})(p_{j}/P_{j-1}) < 0$$

for all *j* sufficiently large. The above inequality will be true for all *j* sufficiently large if $2(1 + \alpha) + ((1 + \alpha)^2 + \beta^2)\delta/(1 - \delta) < 0$, which is equivalent to $|\lambda - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)$. Let $z_n = \prod_{i=1}^{n-2} (1 + (1 - 1/\lambda)p_i/P_{i-1})$. Then

$$z_n = \prod_{j=1}^{n-2} (1 + (1 - 1/\lambda)p_j/P_{j-1}).$$
 Then
$$|z_{n+1}/z_n| = |1 + (1 - 1/\lambda)p_{n-1}/P_{n-2}|.$$

From the hypothesis on λ , and the discussion in the preceding paragraph,

(3)
$$\left|1 + \left(1 - \frac{1}{\lambda}\right)\frac{p_{n-1}}{P_{n-2}}\right| \le \xi < 1$$

for all *n* sufficiently large, and $\sum |z_n|$ is convergent.

Since $|(1 - 1/\lambda)p_{n-1}x_1/P_{n-2}|$ is bounded, it follows that $\Sigma |x_n|$ is convergent, so that $(\lambda I - A^*)x = 0$ has nonzero solutions.

By [3, Theorem II 3.7], $\lambda I - A$ does not have dense range. Therefore $\lambda I - A \in III$ and hence $\lambda I - A \in III_1 \cup III_2$.

To verify that $\lambda I - A \in III_1$ it is sufficient, from [3, Theorem II 3.11], to show that $\lambda I - A^*$ is onto.

Suppose $y = (\lambda I - A^*)x$, $x, y \in l$. Then $(\lambda - 1)x_0 = y_0$ and

(4)
$$(\lambda - c_{n-1})x_n - \sum_{k=n+1}^{\infty} p_{n-1}x_k/P_{k-1} = y_n, \quad n > 0.$$

Choose $x_1 = 0$ and solve for x in terms of y to get

(5)
$$-p_0 \sum_{k=2}^{\infty} x_k / P_{k-1} = y_1$$

(6)
$$(\lambda - c_{n-1})x_n = y_n + p_{n-1}\sum_{k=n+1}^{\infty} x_k/P_{k-1}.$$

For example, substituting (5) into (6), with n = 2, yields

$$(\lambda - c_1)x_2 = y_2 + p_1 \sum_{k=3}^{\infty} x_k / P_{k-1} = y_2 + p_1 \left(\sum_{k=2}^{\infty} x_k / P_{k-1} - x_2 / P_1 \right),$$

so that $x_2 = (y_2 - p_1 y_1 / p_0) / \lambda$.

Continuing this process, if B is a lower triangular matrix defined by By = x, then B has entries

$$b_{00} = \frac{1}{\lambda - 1}, \quad b_{nn} = \frac{1}{\lambda}, \qquad n > 1,$$

$$b_{21} = \frac{-p_1}{\lambda p_0}, \quad b_{n,n-1} = -\frac{p_{n-1}}{P_{n-2}\lambda^2}, \qquad n > 2,$$

$$b_{n1} = -\frac{p_{n-1}}{\lambda p_0} \prod_{j=1}^{n-2} \left(1 - \frac{c_j}{\lambda}\right), \qquad n > 2,$$

$$b_{nk} = -\frac{p_{n-1}}{\lambda^2 P_{k-1}} \prod_{j=k}^{n-2} \left(1 - \frac{c_j}{\lambda}\right), \qquad 1 < k < n - 1$$

and $b_{nk} = 0$ otherwise.

To show that $B \in B(l)$ it is sufficient to show that $\sum_{n} |b_{nk}|$ is finite, independent of k.

,

 $\sum_{n} |b_{n0}| = 1/|\lambda - 1|$. We may write $1 - c_j/\lambda = 1 - p_j/\lambda P_j = (P_{j-1}/P_j)(1 + (1 - 1/\lambda)p_j/P_{j-1})$. Also, $\sup_{n>1} |p_{n-1}/P_{n-2}| \le M < \infty$. Therefore

$$\sum_{n} |b_{n1}| \leq \frac{1}{|\lambda|} \left(M + M \sum_{n=3}^{\infty} \prod_{j=1}^{n-2} \left| 1 + \left(1 - \frac{1}{\lambda} \right) \frac{P_j}{P_{j-1}} \right| \right),$$

and, for k > 1,

$$\sum_{n} |b_{nk}| \leq \frac{1}{|\lambda|} + \frac{M}{|\lambda|2} + \frac{M}{|\lambda|2} \sum_{n=k+2}^{\infty} \prod_{j=k}^{n-2} \left|1 + \left(1 - \frac{1}{\lambda}\right) \frac{P_j}{P_{j-1}}\right|.$$

Since k > 1, the series in the second inequality is dominated by the series in the first inequality which, from (3), is absolutely convergent. Therefore $||B||_1 < \infty$.

Since $(\lambda I - A)^{-1}$ is bounded, it is continuous, and $\lambda \in III_1\sigma(A)$.

Case II. Suppose an infinite number of the p_k are zero. Since $\lim_n p_n = \infty$, there are an infinite number of nonzero p_k . Denote these by $\{p_{n_k}\}$.

From (2) for $n \neq 1 + n_k$, $x_n = 0$. Otherwise,

$$x_{1+n_k} = (1-1/\lambda) \frac{P_{n_k}}{P_{n_k-1}} \prod_{j=1}^r \left(1-(1-1/\lambda) \frac{P_{n_j}}{P_{n_j-1}} \right).$$

Now apply the same analysis as in Case I to verify that $\sum |z_n|$ converges, and hence $\lambda I - A^*$ has nonzero solutions.

To show that $\lambda I - A^*$ is onto, the presence of an infinite number of $p_k = 0$ merely introduces more zero entries in *B*. For the non-zero entries, the same argument as Case I applies.

THEOREM 2. Let A be a regular weighted mean method such that $\delta = \lim p_n/P_n$ exists, and $\delta < 1$. Suppose no diagonal entry of A occurs an infinite number of times. If $\lambda = \delta$, or $\lambda = a_{nn}$, n = 1, 2, 3, ... and $\delta/(2 - \delta) < \lambda < 1$, then $\lambda \in III_1\sigma(A)$.

Proof. First assume that A has distinct diagonal entries, and fix $j \ge 1$. Then the system $(a_{jj}I - A)x = 0$ implies $x_k = 0$ for k = 0, 1, ..., j - 1, and, for $n \ge j$,

$$(a_{jj} - a_{nn})x_n - \sum_{k=0}^{n-1} a_{nk}x_k = 0.$$

The above system yields the recursion relation

$$x_{n+1} = p_j P_n x_n / P_j P_{n+1} (c_j - c_{n+1}),$$

which can be solved for x_n to yield

(7)
$$x_{j+m} = \frac{P_j x_j c_j^m}{P_{j+m} \prod_{i=1}^m (c_j - c_{j+i})} = x_j \prod_{i=1}^m \left(\frac{1 - c_{j+i}}{1 - c_{j+i}/c_j} \right).$$

Since
$$(1 - c_{j+i})/(1 - c_{j+i}/c_j) = (P_{j+i} - p_{j+i})/(P_{j+i} - p_{j+i}/c_j) = P_{j+i-1}/(P_{j+i-1} + (1 - 1/c_j)p_{j+i}) = (1 + (1 - 1/c_j)p_{j+i}/P_{j+i-1})^{-1},$$

$$x_{j+m} = x_j / \prod_{i=1}^{m} (1 + (1 - 1/c_j)p_{j+i}/P_{j+i-1}).$$

Since $0 < c_i < 1$, the argument of Theorem 1 implies that

$$1 + (1 - 1/c_j)p_{j+i}/P_{j+i-1} \le \xi < 1$$

for all *i* sufficiently large. Therefore $x \in c$ implies x = 0 and $a_{ij}I - A$ is 1-1, so that $c_i I - A \in 1 \cup 2$.

Clearly $c_j I - A \in III$. It remains to show that $c_j I - A^*$ is onto. Suppose $(c_j I - A^*)x = y$, $x, y \in l$. By choosing $x_{j+1} = 0$ we can solve for x_0, \dots, x_j in terms of y_0, \dots, y_{j+1} . As in Theorem 1, the remaining equations can be written in the form x = By, where the nonzero entries of B are

(8)
$$b_{j+m,j+m} = 1/c_j, \quad m > 1;$$

$$b_{j+2,j+1} = -p_{j+1}/c_j p_j; b_{j+m,j+m-1} = -p_{j+m-1}/c_j^2 P_{j+m-2}, m > 2;$$

$$b_{j+m,j+k} = -\frac{p_{j+m-1}}{c_j^2 P_{j+k-1}} \prod_{i=j+k}^{j+m-2} \left(1 - \frac{c_i}{c_j}\right), \ 1 < k < m-1, \qquad m > 2;$$

$$b_{j+m,j+1} = -\frac{p_{j+m-1}}{c_j p_j} \prod_{i=j+1}^{j+m-2} \left(1 - \frac{c_i}{c_j}\right), \quad m > 2.$$

From (8),

(9)
$$\sum_{n=j+1}^{\infty} |b_{n,j+1}| = \frac{p_{j+1}}{c_j p_j} + \frac{1}{c_j p_j} \sum_{n=j+3}^{\infty} p_{n-1} \prod_{i=j+1}^{n-2} \left| 1 - \frac{c_i}{c_j} \right|.$$

For m > 1,

$$\sum_{n=m+j}^{\infty} |b_{n,m+j}| = \frac{1}{c_j} + \frac{p_{j+m}}{c_j^2 P_{j+m-1}} + \frac{1}{c_j^2} \sum_{n=m+j+2}^{\infty} \frac{p_{n-1}}{P_{j+m-1}} \prod_{i=j+m}^{n-2} \left|1 - \frac{c_i}{c_j}\right|.$$

Since p_{i+m}/P_{i+m-1} is bounded, and $p_i/P_{i+m-1} \le 1$ for each m > 1, to show that $\|B\|_1$ is finite, it is sufficient to show that the series in (9) converges. We may write

$$\frac{p_{n-1}}{p_j} = \frac{p_{n-1}P_{n-2}P_{n-3}\cdots P_j}{P_{n-2}P_{n-3}\cdots P_j P_j} = \frac{p_{n-1}}{P_{n-2}} \cdot \frac{1}{(1-c_{n-2})\cdots(1-c_{j+1})c_j}.$$

Substituting in (9) the series then becomes

$$\frac{1}{c_j^2} \sum_{n=j+3}^{\infty} \frac{p_{n-1}}{P_{n-2}} \prod_{i=j+1}^{n-2} \left| \frac{1-c_i/c_j}{1-c_i} \right|.$$

Note that

(10)
$$\frac{1-c_i/c_j}{1-c_i} = \frac{P_i - p_i/c_j}{P_i - p_i} = \frac{p_i}{P_{i-1}} + 1 - \frac{p_i}{c_j P_{i-1}}$$
$$= 1 + \left(1 - \frac{1}{c_j}\right) \frac{p_i}{P_{i-1}}.$$

From the hypothesis on λ , (3) is satisfied for all *i* sufficiently large, and the series in (9) is absolutely convergent.

Suppose A does not have distinct diagonal entries. The restriction on λ guarantees that no zero diagonal entries are being considered. Let $c_j \neq 0$ be any diagonal entry which occurs more than once, and let k, r denote, respectively, the smallest and largest integers for which $c_j = c_k = c_r$. From (7) it follows that $x_n = 0$ for $n \ge r$. Also, $x_n = 0$ for $0 \le n < k$. Therefore the system $(c_i I - A)x = 0$ becomes

(11)
$$(c_j - c_n)x_n - \sum_{i=j}^{n-1} a_{ni}x_i = 0, \quad k < n \le r.$$

Case I. r = k + 1. Then (11) reduces to the single equation

$$(c_j - c_{k+1})x_{j+1} - a_{k+1,k}x_k = 0,$$

which implies $x_k = 0$, since $c_j = c_r = c_{k+1}$, and $p_j \neq 0$. Therefore x = 0.

Case II. r > k + 1. From (11) one obtains the recursion formula $x_n = P_{n+1}(c_j - c_{n+1})x_{n+1}/c_jP_n$, k < n < r. Since $x_d r = 0$ it then follows that $x_n = 0$ for k < n < r. Using (11) with n = k + 1 yields $x_k = 0$ and so again x = 0.

To show that $c_j I - A^*$ is onto, suppose $(c_j I - A^*)x = y, x, y \in l$. By choosing $x_{j+1} = 0$ we can solve for x_0, x_1, \ldots, x_j in terms of $y_0, y_1, \ldots, y_{j+1}$. As in Theorem 1 the remaining equations can be written in the form x = By, where the entries of B are as in (8), with the other entries of B clearly zero.

Since $k \le j \le r$, there are two cases to consider.

Case I. j = r. Then the proof proceeds exactly as the argument following (8).

Case II. j < r. Then, from (8), $b_{j+m,j+k} = b_{j+m,j+1} = 0$ at least for $m \ge r - j + 2$. If there are other values of n, j < n < r for which $c_n = c_j$, then additional entries of B will be zero. These zero entries do not affect the validity of the argument showing that (9) converges.

If $\delta = 0$, then 0 does not lie inside the disc, and so it is not considered in this theorem.

Let $\lambda = \delta > 0$. If $a_{nn} \neq \delta$ for each $n \ge 1$, all *i* sufficiently large, then the argument of Theorem 1 applies and $\delta I - A \in III_1$. If $a_{nn} = \delta$ for some *n*, then the proof of Theorem 2 applies, with c_j replaced by δ , and, again, $\delta I - A \in III_1$.

Therefore, in all cases, $c_i I - A \in 1 \cup 2$.

THEOREM 3. Let A be a regular weighted mean method such that $\delta = \lim p_n/P_n$ exists and $p_n/P_n \ge \delta$ for all n sufficiently large. If λ satisfies $|\lambda - (2 - \delta)^{-1}| = (1 - \delta)/(2 - \delta), \lambda \ne 1, \delta/(2 - \delta)$, then $\lambda \in II_2\sigma(A)$.

Proof. Fix $\lambda \neq 1$, $\delta/(2-\delta)$, and satisfying $|\lambda - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)$. Since $\lambda I - A$ is a triangle, it is 1-1 and $\lambda I - A \in 1 \cup 2$.

Now consider $(\lambda I - A^*)x = 0$. As in Theorem 1, $x_0 = 0$, x_1 is arbitrary and $\{x_n\}$ satisfies (2) for all n > 0. From the hypothesis there exists a positive integer N such that $n \ge N$ implies $c_n \ge \delta$. This fact, together with the condition on λ , implies that $|1 + (1 - 1/\lambda)p_n/P_{n-1}| \ge 1$ for $n \ge N$. Thus $|x_n| \ge cp_{n-1}/P_{n-2}$ for $n \ge N$, where c is a constant independent of n. We may write

$$p_{n-1}/P_{n-2} = (p_{n-1}/P_{n-1})(p_{n-1}/P_{n-2})$$

= $(p_{n-1}/P_{n-1})(1 + p_{n-1}/P_{n-2}) \ge p_{n-1}/P_{n-1}.$

From [4, p. 290], $\sum p_n/P_n$ diverges, so $\{x_n\} \in l$ implies $x_1 = 0$, hence x = 0 and $\lambda I - A^* \in 1 \cup 2$. Since $\lambda \in \sigma(A)$, and $\lambda \neq 1$, $\delta/(2 - \delta)$, $\lambda \in \operatorname{II}_2\sigma(A)$.

THEOREM 4. Let A be a regular weighted mean method. Then $1 \in III_3\sigma(A)$.

Proof. Since (I - A)e = 0, I - A is not 1-1 and hence $I - A \in 3$. It remains to show that $\overline{R(I - A)} \neq c$. Let $z \in c$ such that $z_0 \neq 0$. Then $||(I - A)x - z|| \ge |z_0| > |z_0|/2$ for all $x \in c$. Therefore $z \notin \overline{R(I - A)}$.

THEOREM 5. Let A be a regular weighted mean method with $\gamma = \lim \inf p_n/P_n$. If there exists values of n such that $0 \le c_n \le \gamma/(2 - \gamma)$, then $\lambda = c_n$ implies $\lambda \in III_3\sigma(A)$.

Proof. Let c_k be any diagonal entry satisfying $0 < c_k \le \gamma/(2 - \gamma)$. Let j be the smallest integer such that $c_j = c_k$. Since $c_0 = 1, j > 0$. By setting $x_n = 0$ for n > j + 1, $x_0 = 0$, the system $(c_j I - A^*)x = 0$ reduces to a homogeneous linear system of j equations in j + 1 unknowns, so that nontrivial solutions exist. Therefore $c_j I - A \in III$.

If $c_j = \gamma/(2 - \gamma)$ then clearly $c_j I - A \in 3$. Assume $0 < c_j < \gamma/(2 - \gamma)$ and let r denote the largest integer such that $c_r = c_k$. Solving $(c_r I - A)x = 0$ leads to (7) with j = r.

Pick $\varepsilon = \min\{\gamma(1-\gamma)/(2-\gamma)2, \gamma/2-1/(1+1/c_j)\}$. (Since $c_j < \gamma/(2-\gamma), \gamma/2-1/(1+1/c_j) > 0$.) Choose N large enough so that, for $m \ge N, c_{m+j+1} > \gamma - \varepsilon$. From $c_j < \gamma/(2-\gamma)$ it follows that

$$c_{j+m+1}/c_j - 1 > ((2-\gamma)/\gamma)c_{j+m+1} - 1 > ((2-\gamma)/\gamma)(\gamma-\varepsilon) - 1 > 0$$

since $\varepsilon < \gamma(1 - \gamma)/(2 - \gamma)$. For m > N from (7)

For $m \ge N$, from (7),

$$\frac{|x_{j+m+1}|}{|x_{j+m}|} = \left|\frac{1-c_{j+m+1}}{1-\frac{c_{j+m+1}}{c_j}}\right| = \frac{1-c_{j+m+1}}{\frac{c_{j+m+1}}{c_j}-1} < \frac{1-\gamma+\varepsilon}{\frac{\gamma-\varepsilon}{c_j}-1} < 1,$$

I

since $\varepsilon < \gamma - 2/(1 + 1/c_j)$. Consequently $\{x_n\} \in l$, hence $\{x_n\} \in c$, and $c_j I - A$ is not 1-1.

Suppose A has a zero on the main diagonal and $\gamma > 0$. Let j denote the smallest positive integer for which $c_j = 0$. Let e^j denote the coordinate sequence with a 1 in the jth positive and all other entries zero. Then $Ae^j = 0$, and $c_jI - A = -A$ is not 1-1. By setting $x_0 = 0$, $x_n = 0$ for n > j + 1, the system $(c_jI - A^*)x = 0$ reduces to a homogeneous linear system of j equations in j + 1 unknowns.

When the diagonal entries of A do not converge, it was shown in [2] that the spectrum need no longer be a disc. This fact was illustrated by considering weighted mean methods with diagonal entries $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, n > 0, where $1 . Under these conditions, <math>\sigma(A) = \{\lambda | (p-1)(q-1) | \lambda |^2 \ge |1 - p\lambda| | |1 - q\lambda| \}$. The boundary of the spectrum is either an oval, two ovals tangent at a point on the x-axis between 0 and 1, or two disjoint ovals, depending on the relative sizes of p and q. It will now be shown that the fine spectra of these methods behave exactly as the fine spectra for the weighted mean methods considered in Theorems 1-5.

THEOREM 6. Let A be a regular weighted mean method defined by $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, n > 0, where $1 . If <math>\lambda \neq 1/p, 1/q, 1$ and satisfies $(p-1)(q-1)|\lambda|^2 > |1-p\lambda||1-q\lambda|$, then $\lambda \in III_1\sigma(A)$.

Proof. Since $\lambda \neq 1/p$, 1/q, or $1, \lambda I - A$ is a triangle, so it is 1-1, and $\lambda I - A \in 1 \cup 2$.

Suppose $(\lambda I - A^*)x = 0$. Then, as in the proof of Theorem 1, $x_0 = 0$, x_1 is arbitrary and, from (2),

$$x_{2n} = \frac{p_{2n-1}}{p_0} x_1 \left(1 - \frac{1}{\lambda}\right) \left(1 - \frac{1}{p\lambda}\right)^{n-1} \left(1 - \frac{1}{q\lambda}\right)^{n-1},$$
$$x_{2n+1} = \frac{p_{2n}}{p_0} x_1 \left(1 - \frac{1}{\lambda}\right) \left(1 - \frac{1}{p\lambda}\right)^{n-1} \left(1 - \frac{1}{q\lambda}\right)^n.$$

From the hypotheses on A it follows that

$$p_{2n} = \frac{p_0 p^{n-1} q^n}{(p-1)^n (q-1)^n}, \quad p_{2n-1} = \frac{p_0 (pq)^{n-1}}{(p-1)^{n-1} (q-1)^n},$$

so that

$$\frac{|x_{2n+2}|}{|x_{2n}|} = \frac{|x_{2n+1}|}{|x_{2n-1}|} = \frac{pq}{(p-1)(q-1)} \left| 1 - \frac{1}{p\lambda} \right| \left| 1 - \frac{1}{q\lambda} \right|$$
$$= \frac{pq}{(p-1)(q-1)} \frac{|1-p\lambda||1-q\lambda|}{pq|\lambda|^2}$$
$$< \frac{pq}{(p-1)(q-1)} \frac{|\lambda|^2 (p-1)(q-1)}{pq|\lambda|^2} = 1.$$

Consequently there exist nonzero sequences $\{x_n\} \in l$ such that $(\lambda I - A^*)x = 0$ and $\lambda I - A^*$ is not 1-1. Therefore $\lambda I - A \in III$.

It will now be shown that $\lambda I - A^*$ is onto. Suppose $y = (\lambda I - A^*)x$, $x, y \in l$. Then $(\lambda - 1)x_0 = y_0$ and (4) holds. Solving (4) for x in terms of y yields the matrix B, from x = By, with entries as described in the paragraph following (6). $\sum_n |b_{n0}| = 1/|\lambda - 1| < \infty$.

(12)
$$\sum_{n=1}^{\infty} |b_{n1}| = \frac{p_1}{p_0 |\lambda|} + \sum_{n=3}^{\infty} \frac{p_{n-1}}{p_0 |\lambda|} \prod_{j=1}^{n-2} \left|1 - \frac{c_j}{\lambda}\right|.$$

The series on the right can be written in the form $\Sigma_1 + \Sigma_2$, where

$$\begin{split} \Sigma_{1} &= \sum_{n=1}^{\infty} \frac{p_{2n}}{p_{0} |\lambda|} \prod_{j=1}^{2n-1} \left| 1 - \frac{c_{j}}{\lambda} \right|, \\ \Sigma_{2} &= \sum_{n=2}^{\infty} \frac{p_{2n-1}}{p_{0} |\lambda|} \prod_{j=1}^{2n-2} \left| 1 - \frac{c_{j}}{\lambda} \right|. \\ \Sigma_{1} &= \frac{1}{p_{0} |\lambda|} \sum_{n=1}^{\infty} \frac{p_{0} p^{n-1} q^{n}}{(p-1)^{n} (q-1)^{n}} \left| 1 - \frac{1}{p\lambda} \right|^{n-1} \left| 1 - \frac{1}{q\lambda} \right|^{n} \\ &= \frac{|q\lambda - 1|}{(p-1)(q-1) |\lambda|^{2}} \sum_{n=1}^{\infty} R^{n-1}, \end{split}$$

where $R = |p\lambda - 1| |q\lambda - 1|/(p-1)(q-1)|\lambda|^2$. Similarly,

$$\Sigma_2 = \frac{1}{(q-1)|\lambda|} \sum_{n=2}^{\infty} R^{n-1}$$

From the hypothesis on λ , both series are convergent geometric series. For k > 1,

$$\sum_{n=k}^{\infty} |b_{nk}| = \frac{1}{|\lambda|} \left(1 + \frac{p_k}{|\lambda| P_{k-1}} \right) + \sum_{n=k+2}^{\infty} \frac{p_{n-1}}{|\lambda| P_{k-1}} \prod_{j=k}^{n-2} \left| 1 - \frac{c_j}{\lambda} \right|.$$

Since the above series is dominated by the series in (12), $||B||_1 < \infty$.

THEOREM 7. Let A be as in Theorem 6. If $\lambda = 1/p, 1/q$ then $\lambda \in III_1\sigma(A)$.

Proof. Suppose $\lambda = 1/p$. Then $\lambda I - A$ maps x into $\{(1/p - 1)x_0, -a_{10}x_0 + (1/p - 1/q)x_1, -a_{20}x_0 - a_{21}x_1, -a_{30}x_0 - a_{31}x_1 - a_{32}x_2 + (1/p - 1/q)x_3, \cdots\}$, so that $(\lambda I - A)x = 0$ implies $x_0 = x_1 = 0$. By induction, one solves successively a pair of equations of the form

$$-a_{2n+1,2n}x_{2n} + (1/p - 1/q)x_{2n+1} = 0,$$

$$-a_{2n+2,2n}x_{2n} - a_{2n+2,2n+1}x_{2n+1} = 0,$$

whose only solution is $x_{2n} = x_{2n+1} = 0$, since the determinant of the coefficients is equal to $p_{2n}/pP_{2n+2} \neq 0$.

If $\lambda = 1/q$, then $\lambda I - A$ maps x into $\{(1/q - 1)x_0, -a_{10}x_0, -a_{20}x_0 - a_{21}x_1 + (1/q - 1/p)x_2, \cdots\}$, and $(\lambda I - A)x = 0$ forces $x_0 = 0$. Again

one solves successively pairs of equations, this time of the form

$$-a_{2n+2,2n+1}x_{2n+1} + (1/q - 1/p)x_{2n+2} = 0,$$

$$-a_{2n+3,2n+1}x_{2n+1} - a_{2n+3,2n+2}x_{2n+2} = 0.$$

Since the determinant of the coefficients is equal to $p_{2n+1}/qP_{2n+3} \neq 0$, x = 0, and $\lambda I - A$ is 1-1. Clearly $\lambda I - A \in III$, and it remains to show that $\lambda I - A^*$ is onto.

Suppose $(\lambda I - A^*)x = y$, $x, y \in l$. As in the proof of Theorem 2, choosing $x_{j+1} = 0$ we can solve for x_0, \dots, x_j in terms of y_0, \dots, y_{j+1} , and the remaining values of x are determined from x = By, where B is as defined in (8). Since $c_{j+2} = c_j$ for each j > 0, it is clear from (8) that $b_{j+m,j+m-k} = b_{j+m,j+1} = 0$ for $k \ge 3$, $m \ge 4$. Also, $b_{j+m,j+m-2} = 0$ for m even. Consequently B has at most three nonzero diagonals, with bounded elements, and $B \in B(l)$.

THEOREM 8. If A is defined as in Theorem 6, and λ satisfies

$$(p-1)(q-1)|\lambda|^2 = |1-p\lambda||1-q\lambda|, \quad \lambda \neq 1,$$

$$\in \prod_{q \in A} \sigma(A)$$

then $\lambda \in II_2\sigma(A)$.

Proof. Since $\lambda I - A$ is a triangle, it is 1-1, so that $\lambda I - A \in 1 \cup 2$. Now consider $(\lambda I - A^*)x = 0$. Then, as in the proof of Theorem 1, $x_0 = 0$ and x_n satisfies (2) for n > 0. It then follows that

$$|x_{2n}| = \frac{|x_1|}{q-1} \left| 1 - \frac{1}{\lambda} \right|$$

and

$$|x_{2n+1}| = \frac{|x_1||\lambda - 1|}{|p\lambda - 1|}.$$

Therefore $\{x_n\} \in l$ implies x = 0 and $\lambda I - A^* \in 1 \cup 2$. It then follows that $\lambda \in II_2\sigma(A)$.

From Theorem 4 it follows that $1 \in III_3\sigma(A)$.

Cartlidge [1] demonstrated that certain weighted mean methods belong to $B(l^p)$, $p \ge 1$, and computed their spectra. For example, he showed that, if $\delta = \lim p_n/P_n > 0$, then $A \in B(l^p)$ and

$$\sigma(A) = \left\{\lambda \mid\mid \lambda - (2-\delta)^{-1} \mid \leq (1-\delta)/(2-\delta)\right\} \cup S.$$

It can be shown that the results of Theorems 1-5 are true for each such A.

Based on the results established in this paper, the following is a reasonable conjecture.

Let A be a weighted mean method, $A \in B(l^p)$, for some p satisfying $1 \le p \le \infty$. Then all interior points of $\sigma(A)$ belong to III₁, all boundary points, except 1, and possibly $\gamma/(2 - \gamma)$, belong to II₂, and 1 and all isolated points belong to III₃. If $\gamma/(2 - \gamma)$ is a diagonal element of A, then $\gamma/(2 - \gamma) \in III_3$. Otherwise $\gamma/(2 - \gamma) \in II_2$.

References

1. J. P. Cartlidge, Weighted Mean Matrices as Operators on l^p , Ph.D. Dissertation, Indiana University, 1978.

2. Frank P. Cass and B. E. Rhoades, *Mercerian theorems via spectral theory*, Pacific J. Math., **73** (1977), 63-71.

3. S. Goldberg, Unbounded Linear Operators, McGraw Hill, New York, 1966.

4. K. Knopp, *Theory and Application of Infinite Series*, Hafner Publishing Co., New York, 1971.

5. Robert B. Wenger, *The fine spectra of the Hölder summability operators*, Indian J. Pure and Appl. Math., **6** (1975), 695–712.

Received May 1, 1979 and in revised form January 29, 1982.

Indiana University Bloomington, In 47405

PACIFIC JOURNAL OF MATHEMATICS EDITORS

DONALD BABBITT (Managing Editor) University of California Los Angeles, CA 90024

HUGO ROSSI University of Utah Salt Lake City, UT 84112

C. C. MOORE and ARTHUR OGUS University of California Berkeley, CA 94720 J. DUGUNDJI Department of Mathematics University of Southern California Los Angeles, CA 90089-1113

R. FINN and H. SAMELSON Stanford University Stanford, CA 94305

ASSOCIATE EDITORS

B. H. NEUMANN

R. ARENS

E. F. BECKENBACH (1906-1982) F. Wolf

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA UNIVERSITY OF BRITISH COLUMBIA CALIFORNIA INSTITUTE OF TECHNOLOGY UNIVERSITY OF CALIFORNIA MONTANA STATE UNIVERSITY UNIVERSITY OF NEVADA, RENO NEW MEXICO STATE UNIVERSITY OREGON STATE UNIVERSITY UNIVERSITY OF OREGON UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF HAWAII UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

Pacific Journal of Mathematics

Vol. 104, No. 1 May, 1983

Nestor Eugardo Agunera and Eleonor Orena Harboure de Agunera, On
the search for weighted norm inequalities for the Fourier transform 1
Jin Akiyama, Frank Harary and Phillip Arthur Ostrand, A graph and its
complement with specified properties. VI. Chromatic and achromatic
numbers
Bing Ren Li, The perturbation theory for linear operators of discrete type 29
Peter Botta, Stephen J. Pierce and William E. Watkins, Linear
transformations that preserve the nilpotent matrices
Frederick Ronald Cohen, Ralph Cohen, Nicholas J. Kuhn and Joseph
Alvin Neisendorfer, Bundles over configuration spaces
Luther Bush Fuller, Trees and proto-metrizable spaces
Giovanni P. Galdi and Salvatore Rionero, On the best conditions on the
gradient of pressure for uniqueness of viscous flows in the whole space77
John R. Graef, Limit circle type results for sublinear equations
Andrzej Granas, Ronald Bernard Guenther and John Walter Lee,
Topological transversality. II. Applications to the Neumann problem for
y'' = f(t, y, y')
Richard Howard Hudson and Kenneth S. Williams, Extensions of
theorems of Cunningham-Aigner and Hasse-Evans
John Francis Kurtzke, Jr., Centralizers of irregular elements in reductive
algebraic groups
James F. Lawrence, Lopsided sets and orthant-intersection by convex
sets
sets
sets
sets
sets 155 Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and 175 G-spaces 175 Wallace Smith Martindale, III and C. Robert Miers, On the iterates of 179
sets 155 Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and 175 G-spaces 175 Wallace Smith Martindale, III and C. Robert Miers, On the iterates of 179 Thomas H. Pate, Jr, A characterization of a Neuberger type iteration 179
sets 155 Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and 175 Ø-spaces 175 Wallace Smith Martindale, III and C. Robert Miers, On the iterates of 179 Thomas H. Pate, Jr, A characterization of a Neuberger type iteration 179 Thomas H. Pate, Jr, A characterization of classical boundary value 179
sets 155 Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and 175 G-spaces 175 Wallace Smith Martindale, III and C. Robert Miers, On the iterates of 179 Thomas H. Pate, Jr, A characterization of a Neuberger type iteration 179 procedure that leads to solutions of classical boundary value 191
sets155Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and <i>G</i> -spaces175Wallace Smith Martindale, III and C. Robert Miers, On the iterates of derivations of prime rings179Thomas H. Pate, Jr, A characterization of a Neuberger type iteration procedure that leads to solutions of classical boundary value problems191Carl L. Prather and Ken Shaw, Zeros of successive iterates of191
sets155Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and <i>G</i> -spaces175Wallace Smith Martindale, III and C. Robert Miers, On the iterates of derivations of prime rings179Thomas H. Pate, Jr, A characterization of a Neuberger type iteration procedure that leads to solutions of classical boundary value problems191Carl L. Prather and Ken Shaw, Zeros of successive iterates of multiplier-sequence operators205
sets155Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and <i>G</i> -spaces175Wallace Smith Martindale, III and C. Robert Miers, On the iterates of derivations of prime rings179Thomas H. Pate, Jr, A characterization of a Neuberger type iteration procedure that leads to solutions of classical boundary value problems191Carl L. Prather and Ken Shaw, Zeros of successive iterates of multiplier-sequence operators205Billy E. Rhoades, The fine spectra for weighted mean operators219
sets155Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and <i>G</i> -spaces175Wallace Smith Martindale, III and C. Robert Miers, On the iterates of derivations of prime rings179Thomas H. Pate, Jr, A characterization of a Neuberger type iteration procedure that leads to solutions of classical boundary problems191Carl L. Prather and Ken Shaw, Zeros of successive iterates of multiplier-sequence operators205Billy E. Rhoades, The fine spectra for weighted mean operators219Rudolf J. Taschner, A general version of van der Corput's difference155
sets 155 Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and 6-spaces <i>G</i> -spaces 175 Wallace Smith Martindale, III and C. Robert Miers, On the iterates of 179 Thomas H. Pate, Jr, A characterization of a Neuberger type iteration 179 Thomas H. Pate, Jr, A characterization of classical boundary value 191 Carl L. Prather and Ken Shaw, Zeros of successive iterates of 191 Carl L. Prather and Ken Shaw, Zeros of successive iterates of 205 Billy E. Rhoades, The fine spectra for weighted mean operators 219 Rudolf J. Taschner, A general version of van der Corput's difference 231
sets155Åsvald Lima, G. H. Olsen and U. Uttersrud, Intersections of <i>M</i> -ideals and <i>G</i> -spaces175Wallace Smith Martindale, III and C. Robert Miers, On the iterates of derivations of prime rings179Thomas H. Pate, Jr, A characterization of a Neuberger type iteration procedure that leads to solutions of classical boundary value problems191Carl L. Prather and Ken Shaw, Zeros of successive iterates of multiplier-sequence operators205Billy E. Rhoades, The fine spectra for weighted mean operators219Rudolf J. Taschner, A general version of van der Corput's difference theorem231Johannes A. Van Casteren, Operators similar to unitary or selfadjoint155